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AN EXPERIMENTAL STUDY OF THE CONNECTION BETWEEN THE HYDRODYNAMIC AND PHASE-TRANSITION DESCRIPTIONS OF THE COUETTE-TAYLOR INSTABILITY

T. Berland, T. Jøssang and J. Feder

Institute of Physics, University of Oslo
Box 1048 Blindern, 0316 Oslo 3, Norway

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Abstract

The *Laser Doppler Velocimetry* technique has been used to measure the radial flow velocity in the Taylor vortex flow at several Taylor numbers close to and above the critical value. The first four harmonics of the flow field have been analyzed using a model described by Davey. Our analysis demonstrates in a unique way that the amplitude of the first harmonic of the super-critical flow field can be regarded as the 'order parameter' of the transition from the laminar Couette flow to the Taylor vortex flow. This transition is described by a generalized Landau theory for classical second order mean–field phase transitions. The analysis of the results of carefully performed experiments not only confirms the findings of earlier experimental work, but in addition we determine all the significant parameters of the full Davey model for this hydrodynamic instability.

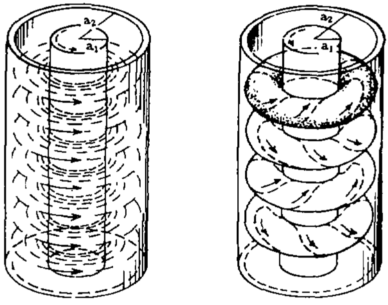

Figure 1: a. Laminar Couette flow. b. Taylor vortex flow.

1 Introduction

The theoretical connection between critical phenomena in statistical physics and flow transitions in hydrodynamics has been extensively studied over the last decades[1,2]. Wesfreid et. al.[3] demonstrated how the appearance of convection rolls in the Rayleigh–Benard instability could be described by Landau theory and they experimentally determined all relevant critical parameters.

Another demonstration of this connection is the flow in the Couette-Taylor geometry with the outer cylinder fixed and the inner one rotating[4,5]. Here the transition from the laminar azimuthal Couette flow to the Taylor vortex flow (fig. 1) seems to follow very closely a generalized Landau equation for ordinary second order phase transitions. The analogy between this hydrodynamic instability and phase transitions was experimentally first investigated by Gollub and Freilich[6,7].

The instability is intimately connected to the velocity field which in turn is governed by the rotation rate Ω of the inner cylinder. The physics of the transition is rooted in the detailed balance between the centrifugal force and the viscous forces on each volume element. This is analogous to the balance between the buoyancy force due to the temperature gradient and the viscous forces in the Rayleigh–Benard problem.

At a critical rotation rate Ω_c this equilibrium is no longer maintained and a new symmetry breaking radial flow field arises. In this non–equilibrium 'phase transition' we expect the classical mean–field behavior observed because of the macroscopic length scales in this problem imposed by the boundary conditions.

The hydrodynamic theory of these instabilities was developed by Taylor[4] and a complete theory was formulated by Davey[8]. The analogy to the Landau theory of phase transitions is, however, absent in this type of work.

The intention of the present work was to verify whether or not the results from very carefully performed experiments satisfy both the phase transition aspects and the detailed structure of the hydrodynamic flow pattern prediction in a consistent way. The results are indeed very encouraging.

In section 2 we describe the results from the hydrodynamic theory and how this can formally be identified with the Landau mean–field theory for second order phase transitions if one assumes that the decay rate of the super-critical perturbation amplitude increases linearly with the dimensionless Taylor number T, which is analogous to the Rayleigh number Ra in the Rayleigh–Benard problem.

In section 3 we describe a carefully constructed experimental test cell and the computerized instrumentation interfaced to it. The accuracy of the single measurements are demonstrated.

In section 4 the data analysis is performed and a procedure allowing the extraction of the maximum possible information from such experiments is described and demonstrated.

Finally, in section 5, we discuss our results and compare them to earlier work.

2 Theory

From earlier work[4,9] we know that a simple solution of the hydrodynamic equations for the radial velocity in the Taylor vortex flow is given by

$$v_r(\epsilon, \mathbf{r}, t) = \sum_p A_p(\epsilon, r, t) \sin(pkz) \ , \tag{1}$$

using cylindrical coordinates where the z denotes the direction of the cylinder axis, and we assume azimuthal invariance of the flow solution. The critical parameter ϵ is defined as

$$\epsilon \stackrel{\text{def}}{=} \begin{cases} 0 & \text{for} \quad T < T_c \\ \dfrac{T - T_c}{T_c} & \text{for} \quad T \geq T_c \end{cases} , \tag{2}$$

where T is the Taylor number defined as

$$T \stackrel{\text{def}}{=} \frac{4(a_1 a_2)^4}{(a_2^2 - a_1^2)^2} \left(\frac{\Omega}{\nu}\right)^2 \tag{3}$$

Here a_1 and a_2 are the inner and outer cylinder radii, Ω is the rotation rate of the inner cylinder and ν is the kinematic viscosity of the fluid in the annulus. T_c is the value of the Taylor number at the flow transition point between the Couette flow and the Taylor flow regime.

Davey expresses the Fourier coefficients $A_p(\epsilon, r, t)$ by

$$A_p(\epsilon, r, t) = \sum_{q=0}^{\infty} B_{pq}(r) \left\{M(\epsilon, t)\right\}^{p+2q} \ . \tag{4}$$

The coefficient $M(\epsilon, t)$, depends only on the reduced Taylor number ϵ, and time. For a stationary system we suggest, by analogy to the Landau theory of second order phase transitions, that $M(\epsilon)$ should be considered to be the order parameter describing the transition from the laminar to the convective state. For flow states close to the stationary state the order parameter $M(\epsilon, t)$ then satisfies a time dependent *Landau* equation[10,11]

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \epsilon M - \ell M^3 \ . \tag{5}$$

Here c is the decay rate of the perturbation amplitude from the stationary equilibrium flow pattern and ℓ is the positive Landau constant. For Taylor flow the decay rate increases with increasing ϵ, and according to Davey, we may use $c \propto \epsilon$.

In stationary flow

$$\frac{\mathrm{d}M}{\mathrm{d}t} = 0 \,,$$

and we get

$$M_{stationary} = \left(\frac{c}{\ell}\right)^{\frac{1}{2}} = M_0\,\epsilon^{\frac{1}{2}} \,. \tag{6}$$

The dimension for the order parameter for this problem is velocity. We therefore, for reasons that will become clear in section 5, define M_0 by requiring B_{10} in equation (4) to be dimensionless and equal to 1.

In the Landau mean–field theory of second order phase transitions the relation (6), is obtained by finding the thermal equilibrium value for the order parameter, that is, the minimum value for the thermodynamic potential.

The Landau potential is a power-series in terms of the order parameter[12]

$$G = G_0 + \tfrac{1}{2}cM^2 + \tfrac{1}{4}uM^4 \,, \tag{7}$$

where symmetry considerations ensure that all odd powers of M vanish. The equilibrium value is determined by

$$\frac{\mathrm{d}G}{\mathrm{d}M} = cM + uM^3 = 0 \,, \tag{8}$$

which, except for the sign, is the same as equation (6). However, here $\epsilon \propto (-c) \propto (\Theta_c - \Theta)$, where Θ is the temperature of the system undergoing the analogous phase transition, and u is a positive constant.

In the more general theory of phase transitions the relation between the order parameter and the critical parameter is expressed as

$$M = M_0\,\epsilon^{\beta} \,, \tag{9}$$

and accordingly, we may say that the critical exponent for the Taylor flow transition should be $\beta = 1/2$, which is the exact value for the critical exponent found by solving the mean-field approximation of the Landau equations for ordinary second order phase transitions.

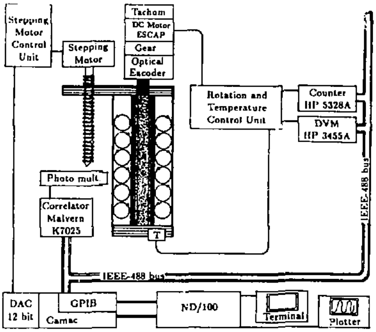

Figure 2: Schematic diagram of the electronic setup.

3 Experimental Conditions

We have measured the transition from Couette flow to Taylor vortex flow in the annulus between a glass cylinder of radius $a_2 = 25.40 \pm 0.03\,mm$ and a blackened aluminum cylinder with radius $a_1 = 15.203 \pm 0.003\,mm$, giving a radius ratio of $\eta = 0.598$. The cylinder length was $h = 83.5\,mm$, giving an aspect ratio of

$$\Gamma = \frac{h}{a_2 - a_1} = 81.9\,.$$

The cylinder is mounted on a vertical Schneeberger slide which allow $150\,mm$ vertical translation, by rotating a positioning screw. This rotation is made with a Superior Electric M061–FD02 stepping motor, having 400 steps per revolution. The step motor was controlled by a special home-built control unit driven by a Kinetic Systems 3112 12-bit DAC. This DAC was in turn controlled by the ND/100 computer via the CAMAC crate (fig. 2). In one revolution of the screw, the cylinder will move $1.5\,mm$, which gives a resolution in the vertical positioning of $3.75\,\mu m$. The cylinder was placed such that the optical measuring volume was half way between the outer glass cylinder and the inner metal cylin-

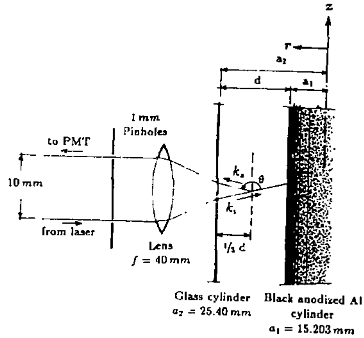

Figure 3: The scattering geometry.

der, and the measurements were done in the range $250 - 400\,mm$ from the upper cylinder end.

In order to measure the radial velocity in the vortex flow we have used reference beam Laser Doppler Velocimetry[13] with the same Michelson-like optical configuration as described in ref.[6]. This involves near–backscattering geometry (fig. 3), where the Doppler shift is related to the velocity by

$$\omega_D = \mathbf{q} \cdot \mathbf{v} \; . \tag{10}$$

In our geometry the scattering vector $\mathbf{q}$ points radially outwards, hence

$$\omega_D = \|\mathbf{q}\|\, v_r = q\, v_r \; . \tag{11}$$

The magnitude of the scattering vector is

$$q = \|\mathbf{k}_s - \mathbf{k}_i\| = \left(\frac{4\pi n}{\lambda_0}\right) \sin\left(\frac{\theta}{2}\right) \; , \tag{12}$$

where n is the refractive index of the liquid, λ_0 is the He-Ne laser wavelength, and θ is the scattering angle. A maximum value of q is obtained for backscattering where $\theta = 180°$ and

$$q = \frac{4\pi n}{\lambda_0} \; .$$

The resolution in the velocity measurements then is at its maximum.

In these experiment we have measured the Doppler shift of light scattered at an angle of $\theta = 169.3°$ from an incident $35\,mW$ He-Ne Spectra-Physics laser beam with a vacuum wavelength of $\lambda_0 = 632.8\,nm$. The scattering agents were $1.00\,\mu m$ DYNO latex spheres uniformly dispersed in a solution of SDS (Sodium Dodecyl Sulphate) in pure water. The concentration of SDS in the solution was $1.5\,mg/ml$ and a concentration of $2\,ppm$ latex spheres in this solution proved sufficient to produce a strong backscattering signal. The experiment was performed in a temperature controlled laboratory, and the temperature was kept constant to $t = 20.0 \pm 0.5°C$. The temperature was recorded continuously during the whole experiment series by a National Semiconductor LX5600 transistor thermometer placed at the bottom end of the cylinder cell. The output voltage from this thermometer was measured by a Hewlett–Packard 3455A digital multimeter. Before starting the experiments, the LX5600 was calibrated against a HP quartz thermometer[14], to ensure correct values for the measured temperatures and to find the uncertainties in our calculated viscosity values.

The temperature dependence of the kinematic viscosity was assumed to be proportional to the temperature dependence of the kinematic viscosity for pure water, that is

$$\nu_{solution}(t) = \alpha\, \nu_{water}(t)\,. \tag{13}$$

The proportionality constant α was assumed independent of the temperature t within the actual temperature range and was measured with a standard Ubbelodhe capillary viscometer.

The light signals from the scattering volume and the reference beam were mixed on the cathode of a Malvern RF313 photomultiplier tube. This assembly includes a built-in amplifier/discriminator unit, and the resulting electronic pulse train was analyzed with a Malvern K7027 4–bit 128–channel full–multiplicative digital correlator. The Doppler shift was obtained by finding the main frequency component of the autocorrelation function. This was done by transferring the autocorrelation data to a ND/100 computer, using a CAMAC–GPIB data acquisition system. On this computer a damped cosine function was fitted to the measured autocorrelation function, using a special version of the general FORTRAN scientific fitting routine VA02A. Using equation (11), we obtained the radial velocity v_r of the fluid in the scattering volume. This technique proved extremely sensitive for the determination of very low velocities. In our experiments we obtained absolute values for the flow velocity down to $15.0 \pm 0.3\,\mu m/s$, which correspond to an optical Doppler shift of $65 \pm 1\,Hz$.

The rotation of the inner cylinder is controlled by an ESCAP 26PL11-213 DC motor connected to a gear box of fixed reduction ratio of 1:156. To ensure stability in the rotation rate, a feedback loop has been constructed. A tachometer mounted on the motor axis produces a voltage depending on the rotation rate of the motor. This voltage is compared with a manually adjustable reference voltage, using an operational amplifier. The input DC level to the motor is then set by this amplifier. The rotation rate Ω was measured using a stroboscope disc mounted on the cylinder axis, with an Hewlett–Packard HEDS-5000 optical encoder connected. This system produces 500 electrical square pulses for each revolution of the inner cylinder. By measuring their frequency with $0.01\,Hz$ resolution ($100\,s$ gating period), using a Hewlett–Packard 5328 counter, the Taylor number can be determined to better than 1%. The necessary analog electronics for the control of the rotation rate and the temperature has been put into a separate control unit (fig. 2).

The vertical movement of the cylinder and the reading of the correlator was completely controlled in real time by the ND/100 computer, and fully automatic logging of temperature and rotation rate data was also performed with the same computer.

The radial velocity $v_r(z)$ was measured at ~ 60 different vertical positions with a mutual distance of $0.75\,mm$. This was done by moving the whole cylinder up or down between each velocity measurement at a speed of approximately $30\,\mu m/s$, which was regarded as sufficiently slow in order not to disturb the flow. Such vertical scan was made for 12 different values of Ω. The first scan was made after a sudden start to a rotation rate high above the transition point ($\Omega \approx 2.5\,\Omega_c$) and the following scans were made after gradually slowing down the inner cylinder. In order to assure stationarity in the velocity field, each rotation rate Ω was kept constant for at least 2.5 hours before starting the measurements as function of height.

4 Data analysis

For each value of Ω, the velocity profile

$$v(z) = v_t + \sum_{p=1}^{4} A_p \sin[pk(z - z_0)] \,, \tag{14}$$

was fitted to the set of measured radial velocities. This was done using an APL general function fitting program on an IBM XT computer. Due to non-perfect optical alignment a small contribution v_t from the transverse (azimuthal) velocity component was detected. Fig. 4 shows examples of Fourier series with four terms fitted to the measured velocity data. The value z_0 was found to be weakly dependent on p. The reason for this vertical displacement of the higher harmonics is not clear.

Having determined the values for v_t, k, and z_0 for all data sets and all harmonics, the complete set of velocity data was used to find the ϵ–dependence of the Fourier amplitudes. The full Davey model (4) was fitted to the complete set of experimental results by fitting the function

$$v_r(T, z) = v_t + \sum_{p=1}^{4} \left\{ \sum_{q=0}^{m(p)} B_{pq} \left[M_0 \left(\frac{T - T_c}{T_c} \right)^{\beta} \right]^{p+2q} \sin[pk(z - z_0)] \right\} \,, \tag{15}$$

to the measured velocities. For each harmonic p, the value $m(p)$ was increased until the fitted value for B_{pq} with the largest q became comparable to the standard deviation in the fitted value, and therefore experimentally insignificant. The large number of velocity data (356) together with the large number of parameters to be fitted ($T_c, \beta, M_0, B_{pq}; p = 1, 2, 3, 4; q = 0, \ldots, m$), made it necessary to use a larger

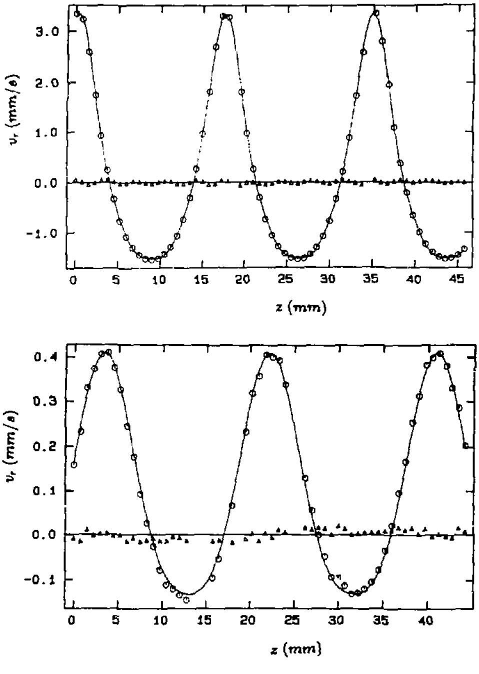

Figure 4: Measured velocity profiles for $\epsilon = 3.72$ (above) and for $\epsilon = 0.063$ (below). The solid curves represent the four–term Fourier fit to the measured data. The small triangles show the deviation between the measured results and the hydrodynamic model for each data point.

computer. The fit was therefore done using the FORTRAN routine MINUITS (in 64 bits precision) on a ND100/500 computer. This program also gives estimates for the standard deviations in the fitted parameter values.

Because of functional orthogonality of the different harmonics, the coupling between the different parameters B_{pq} is very weak, and it was possible to fit up to a total of 16 parameters in a single run. We could therefore obtain higher-order ($q > 0$) terms even for the fourth harmonic before the B-values became insignificant. The final results are tabulated below, where the standard deviation estimated are given as subscripts to the last significant digits.:

$$\begin{aligned} T_c &= 1.03_4 \cdot 10^5 \\ \beta &= 0.51_8 \\ M_0 &= 1.13_8 \, mm/s \end{aligned}$$

B_{pq}	$q=0$	$q=1$	$q=2$	$q=3$
$p=1$	1	-0.00_1	0.000_2	
$p=2$	0.28_3	-0.056_4	0.0053_9	0.0000_1
$p=3$	0.04_1	-0.009_2	0.0008_8	0.00000_8
$p=4$	0.007_4	-0.0007_7	0.0000_1	

Table I. Numerical results for the Davey model.

The units of the B-values become a little complicated by our definition of M_0. Here B_{pq} has dimension $(mm/s)^{1-(p+3q)}$.

5 Discussion

The obtained value for β is in excellent agreement with the expected value of $\beta = 1/2$ from the Landau theory. On the other hand, the value for T_c is somewhat smaller than the value $T_c = 1.10 \cdot 10^5$ tabulated by Swinney[15] for a radius ratio ($\eta = 0.6$), which is very close to our value of $\eta = 0.598$.

Referring to the table I, we find that all the *higher–order terms* in the expression for the first harmonic A_1 ($p = 1$) vanish. With our definition of M_0, we therefore have the very simple result

$$A_1(\epsilon) = M(\epsilon) = 1.13_6\, \epsilon^{0.51_5}\ . \tag{16}$$

That is, we have found it experimentally consistent to use the first harmonic of the velocity flow field directly as the 'order parameter' in the Landau type theory.

Gollub and Freilich[7] found, on the other hand, that the second term for the first harmonic was non–vanishing. This discrepancy can be explained by noting that they chose a different definition for ϵ, namely

$$\hat{\epsilon} \stackrel{\text{def}}{=} \frac{\Omega - \Omega_c}{\Omega_c} \tag{17}$$

The result obtained by Gollub and Freilich for the first harmonic is

$$A_1 = 1.58_8\, \hat{\epsilon}^{0.50_2} + 0.29_5\, \hat{\epsilon}^{1.43_5}. \tag{18}$$

According to equations (2),(3) and (17), ϵ and $\hat{\epsilon}$ are related by

$$\epsilon = \hat{\epsilon}(\hat{\epsilon} + 2)\ , \tag{19}$$

in the case of constant viscosity.

Using equation (16) and the transformation equation (19), we obtain by an expansion in $\hat{\epsilon}$, a power series for our result:

$$A_1 = 1.13\epsilon^{0.5} = 1.13[\hat{\epsilon}(\hat{\epsilon} + 2)]^{0.5} = 1.60\hat{\epsilon}^{0.5} + 0.40\hat{\epsilon}^{1.5} - 0.05\hat{\epsilon}^{2.5} + \dots .$$

Taking into account that the *series* in (18) only contains two terms, we may conclude that the results of Gollub and Freilich are, within experimental errors, consistent with our result (16). Their non–vanishing higher–order terms are therefore a consequence of their definition of the critical parameter. However, by introducing exponents which are not an integer number times the critical exponent β, and also by cutting the power series after two terms only, their analysis does not follow rigorously the correct Davey model. Our approach using the Taylor number and the full Davey theory in the analysis, gives a more consistent picture and a simpler presentation of the results.

The values for B_{pf} for the higher harmonics have not been determined earlier, and therefore no comparison with other work is possible. The analysis of Gollub and Freilich for the higher harmonics based on two-term series expansion with free exponents, and also independent analysis of each Fourier component, is not comparable to a Davey model analysis. But our results indicates that any universal behavior of this system is present in the first harmonic only.

6 Conclusions

The full Davey model has been used to analyze the radial flow velocity of the Taylor vortex flow. The result for the critical exponent is consistent with the predicted value of $\beta = 1/2$ obtained from the generalized Landau theory for mean–field phase transitions. The analysis of the first harmonic of the flow field is consistent with earlier work, but we have also shown that the use of the *Taylor number* to describe the flow makes possible a consistent description and simplifies the results substantially. We have found here, for the first time, that the 'order parameter' for this transition simply is the first harmonic of the radial velocity field. We have also obtained results for the higher harmonics, where several higher–order terms are necessary to make the full Davey model fit the measured velocity data set.

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