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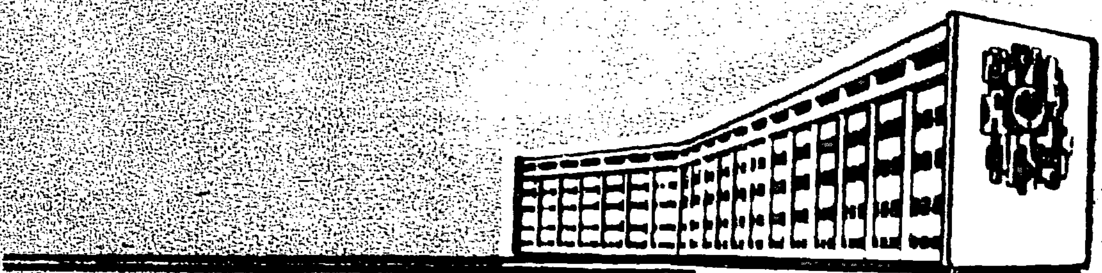
# IDENTITY OF THE SU(3) MODEL PHENOMENOLOGICAL HAMILTONIAN AND THE HAMILTONIAN OF NONAXIAL ROTATOR



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О тождестве феноменологического гамильтониана модели SU(3) и гамильтониана неаксиального ротатора

Интерпретация спектров несферических атомных ядер на основе феноменологических гамильтонианов модели SU (3) обнаружила удовлетворительное согласие модельных расчетов с данными эксперимента. Между тем физический смысл феноменологических гамильтонианов пока не обсуждался. Нами показано, что феноменологические гамильтонианы модели SU(3) сводятся к гамильтониану неаксиального ротатора, но с дополнительными слагаемыми третьей и четвертой степени по оператору углового момента ротатора.

G.F.Filippov, V.I.Avramenko, A.M.Sokolov

Identity of the SU(3) Model Phenomenological Hamiltonian and the Hamiltonian of Nonaxial Rotator

The Draayer and Weeks interpretation of nonspherical nuclei spectra on the basis of the SU(3) model phenomenological hamiltonians reveals a good agreement of model calculations to the experimental date. Physical sense of phenomenological hamiltonians has not yet been discussed. It is shown that the phenomenological hamiltonians of the SU(3) model reduce to the hamiltonian of nonaxial rotator added by terms proportional to the third and the fourth power of the rotator angular momentum operator.



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# IDENTITY OF THE SU(3) MODEL PHENOMENOLOGICAL HAMILTONIAN AND THE HAMILTONIAN OF NONAXIAL ROTATOR

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Kiev - 1985

1. Elliott [1] and then Afanasjev, Rajchev and others [2,3] suggested to use phenomenological hamiltonians composed of scalar combinations of the SU(3) generators for theoretical interpretation of nuclei collective excitations spectra. Complete classification of possible types of such scalar operators is given in [4]. Draayer and Weeks constructed the phenomenological hamiltonians of the pseudo-SU(3) model with the scalar operators containing the second, the third and the fourth powers of the SU(3) generators [5] and showed that these hamiltonians can satisfactorily reproduce the energy of rotational bands and transition probabilities between rotational excitations [6].

Elliott determined relation between the SU(3) generators and the angular momentum projection operators of the particle system onto the principal axes of the irreducible representation high weight vector of this group. The main result of our work is that the hamiltonians, introduced by Draayer and Weeks, reduce to the known hamiltonian of rigid nonaxial rotator added by terms proportional to the third and the fourth power of the angular momentum projection operators. The microscopic version of the SU(3) model leads to a similar hamiltonian provided in a many-particle hamiltonian, obtained by the potential energy averaging of the central exchange nucleon-nucleon interaction between the SU(3) model basis functions, the simplest invariant terms are left [7].

2. The U(3) generators $A_{ik}$ satisfy the following commutational relations [8]:

$$[A_{ik}, A_{pq}] = \delta_{kp} A_{iq} - \delta_{qi} A_{pk} \quad . \tag{1}$$

The simplest realization of the operators $A_{ik}$ can be obtained when we introduce two vectors $\vec{R} = \{x_1, x_2, x_3\}$ and $\vec{R}^* = \{y_1, y_2, y_3\}$ and then define

$$A_{ik} = x_i \frac{\partial}{\partial x_k} + y_i \frac{\partial}{\partial y_k} \quad , \quad k=1,2,3. \tag{2}$$

Then, the commutational relations (1), as is easily verified, will be valid and it is only necessary to point out the basis upon which the generators act. This basis is formed by the components of the tensor

$$T^{(\lambda,\mu)} \equiv \vec{R}^{\lambda} \times [\vec{R}\vec{R}^{*}]^{\mu} \tag{3}$$

obtained by a direct multiplying of the vector $\vec{R}$ $\lambda$ times and the vector product $[\vec{R}\vec{R}^{*}]$ $\mu$ times and the following multiplying of the two direct products. The tensor rank $T^{(\lambda,\mu)}$ is equal to $\lambda + 2\mu$ and this tensor, as it is inherent in the irreducible representation $(\lambda, \mu)$ tensor of the group SU(3), corresponds to a two-row Young table $[\lambda+\mu, \mu]$ of the transposition group of its indices.

The tensor $T^{(\lambda,\mu)}$ is isomorphic to a set of many-particle translationally invariant oscillator basis functions of the SU(3) model [7] and the action of all the physical operators in the space of these basis functions is equivalent to the action of the operators in the space of the tensor $T^{(\lambda,\mu)}$ components. The transition from the complex many-particle functions and many-particle operators to their very simple image - the tensor $T^{(\lambda,\mu)}$ components and operators acting upon them simplifies to a great extent theoretical research of many-nucleon systems within the SU(3) model and enables one to present many results in a compact and obvious form. This transition in practice was shown in [9]. All the stated above accounts for the interest in the tensor properties.

3. Our immediate aim is to express the generators $A_{ik}$ by the SO(3) generators. All the results, we are interested in, will be obtained with the tensor $T^{(\lambda,\mu)}$ and representation (2) for the U(3) generators.

Let us introduce spherical coordinates of the vectors $\vec{R}$ and $\vec{R}^*$

$$\vec{R} = \{ r_1 \sin\theta_1 \cos\varphi_1 ,\ r_1 \sin\theta_1 \sin\varphi_1 ,\ r_1 \cos\theta_1 \} \equiv \{ r_1, \theta_1, \varphi_1 \} ,$$

$$\vec{R}^* = \{ r_2, \theta_2, \varphi_2 \} . \tag{4}$$

Then for the vector $\vec{P} = [\vec{R}\vec{R}^*]$ components in the coordinate system ( $\xi$ , $\eta$ , $\zeta$ ) we shall have

$$P_\xi = r_1 r_2 ( \sin\theta_1 \sin\varphi_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2 \sin\varphi_2 ) ,$$

$$P_\eta = r_1 r_2 ( \cos\theta_1 \sin\theta_2 \cos\varphi_2 - \sin\theta_1 \cos\theta_2 \cos\varphi_2 ), \tag{5}$$

$$P_\zeta = r_1 r_2 \sin\theta_1 \sin\theta_2 \sin(\varphi_2 - \varphi_1) .$$

The vectors $\vec{R}$ and $\vec{P}$ generate three reper vectors ( $\ell_{\tilde{\xi}}$ , $\ell_{\tilde{\eta}}$ , $\ell_{\tilde{\zeta}}$ ) of the coordinate system ( $\tilde{\xi}$ , $\tilde{\eta}$ , $\tilde{\zeta}$ ). Its unit vector $\ell_{\tilde{\zeta}}$ is directed along the vector $\vec{R}$ and the unit vector $\ell_{\tilde{\xi}}$ - along the vector $\vec{P}$ . In the coordinate system ( $\tilde{\xi}$ , $\tilde{\eta}$ , $\tilde{\zeta}$ )

$$\vec{R} = \{ 0, 0, r_1 \} , \qquad \vec{P} = \{ r_1 r_2 \sin\vartheta , 0, 0 \} , \tag{6}$$

where $\vartheta$ represents the angle between the vectors $\vec{R}$ and $\vec{R}^*$ . Let $d = d_{ij}$ be a rotation matrix superposing the reper ( $\ell_\xi$, $\ell_\eta$, $\ell_\zeta$ ) and the reper ( $\ell_{\tilde{\xi}}$ , $\ell_{\tilde{\eta}}$ , $\ell_{\tilde{\zeta}}$ )

$$d_{ij} = \begin{pmatrix} \cos\varphi\cos\theta\cos\Phi - \sin\varphi\sin\Phi & -\cos\varphi\cos\theta\sin\Phi - \sin\varphi\cos\Phi & \sin\theta\cos\varphi \\ \sin\varphi\cos\theta\cos\Phi + \cos\varphi\sin\Phi & -\sin\varphi\cos\theta\sin\Phi + \cos\varphi\cos\Phi & \sin\theta\sin\varphi \\ -\sin\theta\cos\Phi & \sin\theta\sin\Phi & \cos\theta \end{pmatrix} \tag{7}$$

Then it is not difficult to establish a relation between the spherical angles $\theta_1$, $\varphi_1$, $\theta_2$, $\varphi_2$ and the Euler angles $\varphi$, $\theta$, $\Phi$ determining reciprocal orientation of the repers ( $\ell_\xi$, $\ell_\eta$, $\ell_\zeta$ ) and ( $\ell_{\tilde\xi}$, $\ell_{\tilde\eta}$, $\ell_{\tilde\zeta}$ ). Really

$$\sin\theta_1\cos\varphi_1 = d_{13}\,, \quad \sin\theta_1\sin\varphi_1 = d_{23}\,, \quad \cos\theta_1 = d_{33}\,;$$

$$\sin\theta_1\sin\varphi_1\cos\theta_2 - \cos\theta_1\sin\theta_2\sin\varphi_2 = \sin\vartheta\, d_{11}\,,$$

$$\cos\theta_1\sin\theta_2\cos\varphi_2 - \sin\theta_1\cos\theta_2\cos\varphi_1 = \sin\vartheta\, d_{21}\,, \tag{8}$$

$$\sin\theta_1\sin\theta_2\sin(\varphi_2-\varphi_1) = \sin\vartheta\, d_{31}\,.$$

Therefore, comparing (7) and (8) we get that $\theta=\theta_1$ , $\varphi=\varphi_1$ and moreover,

$$\cos\vartheta = \sin\theta_1\sin\theta_2\cos(\varphi_2-\varphi_1) + \cos\theta_1\cos\theta_2\,,$$

$$\sin\vartheta\cos\Phi = \cos\theta_1\sin\theta_2\cos(\varphi_2-\varphi_1) - \sin\theta_1\cos\theta_2\,, \tag{9}$$

$$\sin\vartheta\sin\Phi = \sin\theta_2\sin(\varphi_2-\varphi_1)$$

Next it is necessary to express the generators through the variables $r_1, r_2, \varphi$, $\theta$, $\Phi$, $\vartheta$ .First, we remind that

$$\frac{\partial}{\partial x_1} = \sin\theta_1\cos\varphi_1\frac{\partial}{\partial r_1} + \frac{\cos\theta_1\cos\varphi_1}{r_1}\frac{\partial}{\partial\theta_1} - \frac{\sin\varphi_1}{r_1\sin\theta_1}\frac{\partial}{\partial\varphi_1}\,,$$

$$\frac{\partial}{\partial x_2} = \sin\theta_1\sin\varphi_1\frac{\partial}{\partial r_1} + \frac{\cos\theta_1\sin\varphi_1}{r_1}\frac{\partial}{\partial\theta_1} + \frac{\cos\varphi_1}{r_1\sin\theta_1}\frac{\partial}{\partial\varphi_1}\,, \tag{10}$$

$$\frac{\partial}{\partial x_3} = \cos\theta_1\frac{\partial}{\partial r_1} - \frac{\sin\theta_1}{r_1}\frac{\partial}{\partial\theta}\,.$$

Similar formulas (where only $r_1$, $\theta_1$, $\varphi_1$ are substituted for $r_2, \theta_2, \varphi_2$ ) exist for the derivatives $\frac{\partial}{\partial y_1}$, $\frac{\partial}{\partial y_2}$, $\frac{\partial}{\partial y_3}$. After the transition from the spherical angles $\theta_1, \varphi_1, \theta_2, \varphi_2$ to new variables $\vartheta, \varphi, \theta, \Phi$ we obtain

$$\frac{\partial}{\partial x_1} = \sin\theta\cos\varphi\frac{\partial}{\partial r_1} + \frac{\cos\theta\cos\varphi}{r_1}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r_1\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\sin\vartheta}{r_1}\operatorname{ctg}\theta\frac{\partial}{\partial\Phi} - \frac{d_{11}}{r_1}\frac{\partial}{\partial\vartheta} - \frac{d_{12}}{r_1}\operatorname{ctg}\vartheta\frac{\partial}{\partial\Phi},$$

$$\frac{\partial}{\partial x_2} = \sin\theta\sin\varphi\frac{\partial}{\partial r_1} + \frac{\cos\theta\sin\varphi}{r_1}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r_1\sin\theta}\frac{\partial}{\partial\varphi} - \frac{\cos\vartheta}{r_1}\operatorname{ctg}\theta\frac{\partial}{\partial\Phi} - \frac{d_{21}}{r_1}\frac{\partial}{\partial\vartheta} - \frac{d_{22}}{r_1}\operatorname{ctg}\vartheta\frac{\partial}{\partial\Phi},$$

$$\frac{\partial}{\partial x_3} = \cos\theta\frac{\partial}{\partial r_1} - \frac{\sin\theta}{r_1}\frac{\partial}{\partial\theta} - \frac{d_{31}}{r_1}\frac{\partial}{\partial\vartheta} - \frac{d_{32}}{r_1}\operatorname{ctg}\vartheta\frac{\partial}{\partial\Phi}, \qquad (11)$$

$$\frac{\partial}{\partial y_k} = (\sin\vartheta\, d_{k1} + \cos\vartheta\, d_{k3})\frac{\partial}{\partial r_2} + \frac{1}{r_2}(\cos\vartheta\, d_{k1} - \sin\vartheta\, d_{k3})\frac{\partial}{\partial\vartheta} +$$

$$+ \frac{d_{k2}}{r_2\sin\vartheta}\frac{\partial}{\partial\Phi},$$

$k = 1, 2, 3.$

Now, using formulas (8),(9),(10),(11), we can find an explicit expression for the generators $A_{ik}$ by the variables $r_1, r_2, \vartheta, \varphi, \theta, \Phi$. However, it is convenient to introduce first one more set of operators $-\{\tilde{A}_{\alpha\beta}\}$, having defined $\tilde{A}_{\alpha\beta}$ by the relations

$$A_{ik} = \sum_{\alpha,\beta} d_{i\alpha} d_{k\beta} \tilde{A}_{\alpha\beta}, \qquad (12)$$

$$\tilde{A}_{\alpha\beta} = \sum_{i,k} d_{i\alpha} d_{k\beta} A_{ik}. \qquad (12a)$$

Formula (12) shows that the generators $A_{ik}$ are expressed by $\tilde{A}_{\alpha\beta}$. It also follows from (12) that similarly to $A_{ik}$ the operators $\tilde{A}_{\alpha\beta}$ are the generators of the group U(3), but only in the system of principal axes of the tensor $T^{(\lambda,\mu)}$, where the only component of this tensor which is not equal to zero, becomes the high weight vector of the group SU(3) irreducible representation ( $\lambda$ , $\mu$ ).

The explicit form of the generators $\tilde{A}_{\alpha\beta}$ is as follows:

$$\tilde{A}_{11} = r_2 \sin^2\vartheta \frac{\partial}{\partial r_2} + \sin\vartheta\cos\vartheta \frac{\partial}{\partial\vartheta}, \quad \tilde{A}_{12} = \frac{\partial}{\partial\Phi},$$

$$\tilde{A}_{13} = r_2 \sin\vartheta\cos\vartheta \frac{\partial}{\partial r_2} - \sin^2\vartheta \frac{\partial}{\partial\vartheta}, \quad \tilde{A}_{21} = \tilde{A}_{22} = \tilde{A}_{23} = 0,$$

$$\tilde{A}_{31} = r_2 \sin\vartheta\cos\vartheta \frac{\partial}{\partial r_2} - \sin^2\vartheta \frac{\partial}{\partial\vartheta} - \cos\Phi \frac{\partial}{\partial\theta} + \frac{\sin\Phi}{\sin\theta}\frac{\partial}{\partial\varphi} - \sin\Phi \operatorname{ctg}\theta \frac{\partial}{\partial\Phi}, \tag{13}$$

$$\tilde{A}_{32} = -\sin\Phi \frac{\partial}{\partial\Phi} + \frac{\cos\Phi}{\sin\theta}\frac{\partial}{\partial\varphi} - \cos\Phi \operatorname{ctg}\theta \frac{\partial}{\partial\Phi},$$

$$\tilde{A}_{33} = r_3 \frac{\partial}{\partial r_3}.$$

Further simplification of the generators $\tilde{A}_{\alpha\beta}$ is due to a simple structure of the tensor $T^{(\lambda,\mu)}$ in the coordinate system ( $\xi$ , $\eta$ , $\zeta$ ), where the only nonzero component of this tensor is expressed only by $r_1$ , $r_2$ and $\sin\vartheta$ :

$$\tilde{R}_\zeta^\lambda \tilde{R}_\zeta^\mu = r_1^{\lambda+\mu}(r_2 \sin\vartheta)^\mu. \tag{14}$$

The rotation operation with the Euler angles $\varphi$ , $\theta$ , $\Phi$ retains a simple structure of the tensor $T^{(\lambda,\mu)}$ components in the system ( $\xi$ , $\eta$ , $\zeta$ ) as well, where each of them takes the form of a product of two multipliers. The first multiplier is $r_1^{\lambda+\mu}(r_2 \sin\vartheta)^\mu$ and the second is constructed from $\lambda$ matrix elements $d_{3i}$ and $\mu$ matrix elements $d_{1K}$ of the matrix ( $d$ ). The matrix elements $d_{3i}$ which are included in this multiplier can differ in value of the second index. The same refers to the matrix elements $d_{1K}$ . It depends on indices of the tensor $T^{(\lambda,\mu)}$ component in the coordinate system ( $\xi$ , $\eta$ , $\zeta$ ) which index is to be chosen $i$ or $K$ . Since $r_2$ and the angle $\vartheta$ constitute only the first multiplier being combined as $r_2 \sin\vartheta$ , it is reasonable to introduce one variable $r_3$ instead of $r_2$ and $\vartheta$ , having defined

$$r_3 = r_2 \sin\vartheta$$

and to incorporate a derivative in $r_3$ instead of derivatives in $r_2$ and $\vartheta$ in the generators $\tilde{A}_{iK}$ and $\tilde{A}_{\alpha\beta}$ acting upon the tensor $T^{(\lambda,\mu)}$ . Moreover, it is suitable to express the generators $\tilde{A}_{\alpha\beta}$ via the generators $M_{\alpha\beta}$ , having written usual differential expressions for the SU(3) generators

$$\tilde{M}_{32} = M_{\breve{\xi}} = -\sin\Phi\frac{\partial}{\partial\theta} + \frac{\cos\Phi}{\sin\theta}\frac{\partial}{\partial\varphi} - \cos\Phi\, ctg\theta\frac{\partial}{\partial\Phi} ,$$

$$\tilde{M}_{31} = -M_{\breve{\eta}} = \cos\Phi\frac{\partial}{\partial\theta} + \frac{\sin\Phi}{\sin\theta}\frac{\partial}{\partial\varphi} - \sin\Phi\, ctg\theta\frac{\partial}{\partial\Phi}, \tag{14}$$

$$\tilde{M}_{12} = -M_{\breve{\zeta}} = \frac{\partial}{\partial\Phi} .$$

in projections onto the principal axes of the system ( $\breve{\xi}$ , $\breve{\eta}$ , $\breve{\zeta}$ ) [10] .

After all the transformations mentioned above we obtain

$$\tilde{A}_{11} = r_1\frac{\partial}{\partial r_1} , \quad \tilde{A}_{12} = \tilde{M}_{12} , \quad \tilde{A}_{13} = 0 , \tag{15}$$

$$\tilde{A}_{21} = \tilde{A}_{22} = \tilde{A}_{23} = 0 ,$$

$$\tilde{A}_{31} = \tilde{M}_{31} , \quad \tilde{A}_{32} = \tilde{M}_{32} , \quad \tilde{A}_{33} = r_3 \frac{\partial}{\partial r_3} .$$

And, finally, there comes the last simplification. All the tensor $T^{(\lambda,\mu)}$ components are proportional to $r_1^{\lambda+\mu} r_3^{\mu}$ , therefore, the action of the operator $r_1 \frac{\partial}{\partial r_1}$ on the tensor $T^{(\lambda,\mu)}$ is reduced to multiplying $T^{(\lambda,\mu)}$ by $\lambda+\mu$ and the action of the operator $r_3 \frac{\partial}{\partial r_3}$ - to multiplying by $\mu$ , i.e.

$$\tilde{A}_{11} = \lambda + \mu \quad , \qquad \tilde{A}_{33} = \mu \ . \tag{16}$$

It should be noted that according to (1) the generators $\check{M}_{\alpha\beta}$ satisfy the following commutational relations:

$$[\tilde{M}_{12}, \tilde{M}_{32}] = - \tilde{M}_{31} ,$$

$$[\tilde{M}_{31}, \tilde{M}_{12}] = - \tilde{M}_{32}, \tag{17}$$

$$[\tilde{M}_{32}, \tilde{M}_{31}] = - \tilde{M}_{12} .$$

The SU(3) generators $B_{ik}$ are related to the U(3) generators $A_{ik}$ by

$$B_{ik} = A_{ik} - \delta_{ik} \frac{1}{3} ( A_{11} + A_{22} + A_{33} ) . \tag{18}$$

Similar relation between the SU(3) and U(3) generators exists in the coordinate system ( $\tilde{\xi}$ , $\tilde{\eta}$ , $\tilde{\zeta}$ ):

$$\tilde{B}_{\alpha\beta} = \tilde{A}_{\alpha\beta} - \delta_{\alpha\beta} \frac{1}{3} ( \tilde{A}_{11} + \tilde{A}_{22} + \tilde{A}_{33} ) . \tag{18a}$$

It follows from (18a) that

$$\tilde{B}_{11} = \frac{1}{3}(\mu - \lambda)\ ,\ \tilde{B}_{22} = -\frac{1}{3}(\lambda + 2\mu)\ ,\ \tilde{B}_{33} = \frac{1}{3}(2\lambda + \mu)\ . \tag{19}$$

We should constantly keep in mind that the object under action of the components $B_{ik}$ and $\tilde{B}_{\alpha\beta}$ is the tensor $T^{(\lambda,\mu)}$.

Thus, the SU(3) generators, determined in the space of the tensor $T^{(\lambda,\mu)}$ components, are expressed by the SO(3) generators in projections onto the principal axes of the tensor $T^{(\lambda,\mu)}$. Since the Draayer-Weeks phenomenological hamiltonians represent the linear superposition of the scalar combinations of the SU(3) generators, the relations (15),(16),(18),(18a) enable one to express these hamiltonians by the SO(3) generators as well, or, by the orbital momentum projection operators of the nucleon system onto the principal axes of the tensor $T^{(\lambda,\mu)}$ which, in essence, is the same.

4. The simplest scalar combination of the SU(3) generators proves to be the second order Casimir operator . This operator is determined by

$$C_2 = \sum_{i,k} B_{ik} B_{ki} = \sum_{i,k} \sum_{\alpha\beta} \sum_{\gamma\delta} d_{i\alpha} d_{k\beta} \tilde{B}_{\alpha\beta} d_{k\gamma} d_{i\delta} \tilde{B}_{\gamma\delta}\ . \tag{20}$$

Though it was not used in [5] for the construction of phenomenological hamiltonians, it is a good example of the transformations of the generator scalar combinations at the transition from the coordinate system ($\xi$, $\eta$, $\zeta$) to the system ($\tilde{\xi}$, $\tilde{\eta}$, $\tilde{\zeta}$). The action of the operator $C_2$ upon the tensor $T^{(\lambda,\mu)}$ is reduced to multiplying of its components by the operator $C_2$ eigenvalue. This eigenvalue can be easily obtained as a result of the transfer of $C_2$ into the system ($\tilde{\xi}$, $\tilde{\eta}$, $\tilde{\zeta}$).

The transposition of the nonzero operators $\tilde{B}_{\alpha\beta}$ and the matrix elements $d_{ik}$ obeys the following rule

$$\tilde{B}_{\alpha\beta} d_{ik} - d_{ik}\tilde{B}_{\alpha\beta} = \delta_{\alpha k} d_{i\beta} - \delta_{\beta k} d_{i\alpha} \qquad (21)$$

simplifying significantly the calculation procedure of the operator $C_2$ eigenvalue. Hence the known result [8] is obtained:

$$C_2 = \sum_{\alpha\beta} \tilde{B}_{\alpha\beta}\tilde{B}_{\beta\alpha} + 2(B_{33} - B_{22}) = \frac{2}{3}\left(\lambda^2 + \lambda\mu + \mu^2 + 3\lambda + 3\mu\right).$$

The expression for $C_3$, the third order Casimir operator, also coinciding with the operator eigenvalue in the system ( $\xi$ , $\eta$ , $\zeta$ ) is calculated similarly

$$C_3 = \sum B_{ij} B_{jk} B_{ki} =$$

$$= \sum_{\alpha,\beta,\gamma} \tilde{B}_{\alpha\beta}\tilde{B}_{\beta\gamma}\tilde{B}_{\gamma\alpha} - \frac{1}{2}\sum_{\alpha,\beta} \tilde{B}_{\alpha\alpha}\tilde{B}_{\beta\beta} + 3\left(\tilde{B}_{33}^2 - \tilde{B}_{22}^2\right) - 3\tilde{B}_{11} + \frac{3}{2} C_2 = \qquad (22)$$

$$= \frac{1}{9}(\lambda - \mu)\left[(2\lambda + \mu)(\lambda + 2\mu) + 9(\lambda + \mu + 1)\right] + \frac{3}{2} C_2 .$$

Five linear combinations $Q_\mu$ , $\mu = 0, \pm 1, \pm 2$ which are transformed at the rotation of the coordinate axes as the second rank irreducible tensor components [8], can be constructed from the generators $B_{ik}$ . In the coordinate system ( $\xi$ , $\eta$ , $\zeta$ ) these components have the following form:

$$\tilde{Q}_0 = (2\lambda + \mu) , \quad \tilde{Q}_{\pm 1} = \pm\sqrt{\tfrac{3}{2}}\,\tilde{L}_{\pm} , \quad \tilde{Q}_{\pm 2} = \sqrt{3}\left(\mu \pm \tilde{L}_0\right),$$

$$\tilde{L}_{\pm} = -\frac{1}{\sqrt{2}}\left(\tilde{B}_{31} \pm i\tilde{B}_{32}\right) = -\frac{1}{\sqrt{2}}\left(\tilde{M}_{31} \pm \tilde{M}_{32}\right) = -\frac{i}{\sqrt{2}}\left(\mp M_{\xi} - iM_{\eta}\right), \qquad (23)$$

$$\tilde{L}_0 = -i\tilde{B}_{12} = -i\tilde{M}_{12} = -iM_{\zeta} .$$

Elliott [1] considered scalar convolution of the operators $Q_\mu$ as the SU(3) model phenomenological hamiltonian. The convolution is usually denoted $QQ$. In the system ( $\tilde{\xi}$, $\tilde{\eta}$, $\tilde{\zeta}$ )

$$QQ = 4(\lambda^2 + \lambda\mu + \mu^2 + 3\lambda + 3\mu) - 3(M_{\tilde{\xi}}^2 + M_{\tilde{\eta}}^2 + M_{\tilde{\zeta}}^2) = 6C_2 - 3M^2, \tag{24}$$

$$M^2 = M_{\tilde{\xi}}^2 + M_{\tilde{\eta}}^2 + M_{\tilde{\zeta}}^2 .$$

Instead of $B_{ik}$ the generators

$$T_{ik} = \frac{1}{2}(B_{ik} + B_{ki}) \tag{25}$$

obtained by the symmetrization of the generators $B_{ik}$ are often used. Having at our disposal $T_{ij}$ and $M_k$ we can construct the following scalar operators

$$MTM = \sum_{i,k} M_i T_{ik} M_k \quad , \quad MTTM . \tag{26}$$

The first of them differs from the well-known Bargman-Moshinsky operator [11] only by its multiplier. Both scalar operators and the operators $QQ$ and $M^4$ have been used by Draayer and Weeks [5] to construct phenomenológical hamiltonians of the pseudo- SU(3) model. Let us find an explicit form of these operators in the system ( $\tilde{\xi}$, $\tilde{\eta}$, $\tilde{\zeta}$ ).

$$MTM = \sum d_{i\alpha} \tilde{M}_\alpha d_{i\beta} d_{k\gamma} \tilde{T}_{\beta\gamma} d_{k\delta} \tilde{M}_\delta =$$

$$= \sum \tilde{M}_\alpha \tilde{T}_{\alpha\beta} \tilde{M}_\beta + M_{\tilde{\zeta}}^2 - M_{\tilde{\eta}}^2 = \tag{27}$$

$$= \frac{1}{3}\lambda(3M_{\tilde{\zeta}}^2 - M^2) + \frac{1}{3}\mu(M^2 - 3M_{\tilde{\eta}}^2) +$$

$$+\frac{1}{2}\left(3M_{\tilde\xi}^{2}-M^{2}-M_{\tilde\xi}M_{\tilde\eta}M_{\tilde\zeta}-M_{\tilde\eta}M_{\tilde\xi}M_{\tilde\zeta}\right),$$

$$MTTM=\sum\tilde M_{\alpha}\tilde T_{\alpha\beta}\tilde T_{\beta\gamma}\tilde M_{\gamma}+\sum\tilde M_{\alpha}(\tilde T_{\alpha 3}\tilde M_{3}-\tilde T_{\alpha 2}\tilde M_{2})+$$

$$+\sum(\tilde M_{3}\tilde T_{3\beta}-\tilde M_{2}\tilde T_{2\beta})\tilde M_{\beta}+M_{\tilde\xi}^{2}+M_{\tilde\eta}^{2}=$$

$$=\frac{1}{18}\left(2\lambda^{2}-4\lambda\mu+2\mu^{2}+6\lambda-6\mu+\frac{9}{2}\right)M_{\tilde\xi}^{2}+$$

$$+\frac{1}{18}\left(2\lambda^{2}+8\lambda\mu+8\mu^{2}+6\lambda+30\mu+\frac{9}{2}\right)M_{\tilde\eta}^{2}+ \tag{28}$$

$$+\frac{1}{18}\left(8\lambda^{2}+8\lambda\mu+2\mu^{2}+30\lambda+6\mu+\frac{81}{2}\right)M_{\tilde\zeta}^{2}-$$

$$-\frac{2}{3}(\lambda-\mu+3)M_{\tilde\xi}M_{\tilde\eta}M_{\tilde\zeta}+M_{\tilde\eta}^{2}M_{\tilde\xi}^{2},$$

$$M_{1}=M_{\tilde\xi},\quad M_{2}=M_{\tilde\eta},\quad M_{3}=M_{\tilde\zeta}.$$

The operators MTM and MTTM reveal interesting qualities of symmetry. At the simultaneous transposition of quantum numbers $\lambda$ and $\mu$ and the operators $M_{\tilde\xi}$ and $M_{\tilde\eta}$ the first of them changes its sign and the second one remains unchanged. It follows from (27) and (28) that the problem of searching for the eigenvalues of the scalar operators MTM and MTTM is similar to the problem of the diagonalization of the nonaxial rotator hamiltonian. We note also that the known scalar operator of the third and the fourth order

$$MQM, \quad QQQ, \quad ([MQ]_1[QM]_1) \quad \text{and so on,}$$

are easily expressed via the MTM and MTTM.

$$\begin{aligned}
MQM &= -3\sqrt{\tfrac{2}{5}}\, MTM, \\
([MQ]_1[QM]_1) &= \tfrac{18}{5}\, MTTM, \\
QQQ &= 4\sqrt{\tfrac{2}{7}}\, MTM + 9\sqrt{\tfrac{2}{7}}\, \mathcal{Y}_3, \qquad (29)\\
\mathcal{Y}_3 &= -(\lambda-\mu)\left[(2\lambda+\mu)(\lambda+2\mu)+9(\lambda+\mu+1)\right].
\end{aligned}$$

The phenomenological hamiltonian of the H(3) Draayer-Weeks model has the following form

$$H(3) = \kappa_2 QQ + \kappa_3 QQQ + \kappa_4 ([MQ]_1[QM]_1). \qquad (30)$$

In comparison with H(3) the second hamiltonian is added by a term which is proportional to $M^4$. It immediately follows from (27),(28) that in the system ( $\tilde{\xi}$ , $\tilde{\eta}$ , $\tilde{\zeta}$ ) the hamiltonian H(3) reads as

$$\begin{aligned}
H(3) &= 4C_2\kappa_2 + 4\sqrt{\tfrac{2}{7}}\, \mathcal{Y}_3\kappa_3 + \\
&+ M_{\tilde{\xi}}^2\left[3\kappa_2 + \tfrac{9}{2}\sqrt{\tfrac{2}{7}}\,\kappa_3(2\mu-2\lambda-3) + \tfrac{1}{10}\kappa_4(4\lambda^2-8\lambda\mu+4\mu^2+12\lambda-12\mu+9)\right] + \\
&+ M_{\tilde{\eta}}^2\left[3\kappa_2 - \tfrac{9}{2}\sqrt{\tfrac{2}{7}}\,\kappa_3(2\lambda+4\mu+3) + \tfrac{1}{10}\kappa_4(4\lambda^2+16\lambda\mu+16\mu^2+12\lambda+60\mu+9)\right] + \\
&+ M_{\tilde{\zeta}}^2\left[3\kappa_2 + \tfrac{9}{2}\sqrt{\tfrac{2}{7}}\,\kappa_3(4\lambda+2\mu+3) + \tfrac{1}{10}\kappa_4(16\lambda^2+16\lambda\mu+4\mu^2+60\lambda+12\mu+81)\right] + \\
&+ M_{\tilde{\xi}}M_{\tilde{\eta}}M_{\tilde{\zeta}}\left[27\sqrt{\tfrac{2}{7}}\,\kappa_3 - \tfrac{12}{5}\kappa_4(\lambda-\mu+3)\right] + M_{\tilde{\eta}}^2 M_{\tilde{\zeta}}^2\, \tfrac{18}{5}\kappa_4 .
\end{aligned}$$

Simple relations exist between the operators $M_{\tilde{\xi}}$, $M_{\tilde{\eta}}$, $M_{\tilde{\zeta}}$ (they are often called shift operators [12] ) and the angular momentum projection operators $L_{\tilde{\xi}}$ , $L_{\tilde{\eta}}$ , $L_{\tilde{\zeta}}$ of the system onto the coordinate axes $\tilde{\xi}$ , $\tilde{\eta}$ , $\tilde{\zeta}$ :

$$L_{\tilde{\xi}} = -iM_{\tilde{\xi}} \ , \quad L_{\tilde{\eta}} = -iM_{\tilde{\eta}} \quad , \quad L_{\tilde{\zeta}} = -iM_{\tilde{\zeta}} \ .$$

As a result of the transition from the shift operators to the angular momentum operators, the hamiltonian H(3) becomes adequate to the nonaxial rotator hamiltonian though added by nonlinear, non-Hermitian terms, that are proportional to

$$L_{\tilde{\xi}} L_{\tilde{\eta}} L_{\tilde{\zeta}} \quad , \quad L^2_{\tilde{\eta}} L^2_{\tilde{\zeta}} \ .$$

The hamiltonian H(3) nonhermicity is caused by the nonorthogonality of the basis, on which it is determined.

So far, when we mentioned the basis, on which the operators $A_{ik}$ , $\tilde{A}_{\alpha\beta}$ , $\tilde{B}_{\alpha\beta}$ and also other operators constructed from $\tilde{B}_{\alpha\beta}$ are determined, we mean a set of Cartesian components of the tensor $T^{(\lambda,\mu)}$ (more exactly, a set of its linearly independent Cartesian components). After the Euler angles $\psi$ , $\theta$ , $\Phi$ and the moduli $\tau_1$ , $\tau_3$ had been introduced, each of the tensor components proved to be a product of the multiplier $\tau_1^{\lambda+\mu} \tau_3^{\mu}$ and functions depending only on the Euler angles. However, as soon as the derivatives in $\tau_1$ and $\tau_3$ , included in the operators $\tilde{A}_{\alpha\beta}$ and $\tilde{B}_{\alpha\beta}$ , were changed for the combinations of the quanta numbers $\lambda$ and $\mu$ , it became possible to omit the multiplier $\tau_1^{\lambda+\mu} \tau_3^{\mu}$ and to consider the Cartan components $\mathcal{D}^{L}_{KM}(\psi,\theta,\phi)$ of the tensor $T^{(\lambda,\mu)}$ ,i.e. spherical Wigner functions, instead of its Cartesian components. Wigner functions result from the reduction of the group SU(3) onto the group SO(3). The indices of the Wigner functions mean: $L$ - a state angular momentum, $M$ - an angular momentum projection onto the axis $\zeta$ of the coordinate system ( $\xi$ , $\eta$ , $\zeta$ ) and $K$ - an angular momentum projection onto the axis $\tilde{\zeta}$ of the coordinate system ( $\tilde{\xi}$ , $\tilde{\eta}$ , $\tilde{\zeta}$ ). Each of the functions $\mathcal{D}^L_{KM}$ is a component superposition of the tensor $T^{(\lambda,\mu)}$ arranged in the proper way. Possible values of the quantum numbers $L$ and $K$ are limited by Elliott's [1] conditions, and the quantum number $M$ takes all the whole number values from $-L$ to $L$ .

As it was mentioned above, in the system ( $\tilde{\xi}$ , $\tilde{\eta}$ , $\tilde{\zeta}$ ) only one tensor $T^{(\lambda,\mu)}$ component is nonzero. It means that in the system ( $\xi$ , $\eta$ , $\zeta$ ) the tensor Cartesian components include functions $\mathcal{D}^L_{KM}$ in the form of quite definite linear combinations over $K$ when $L$ and $M$ are fixed. The action of the scalar operators MTM, MTTM and similar ones upon the tensor $T^{(\lambda,\mu)}$ is reduced to the tensor rotations in respect to the axes of the system ( $\tilde{\xi}$ , $\tilde{\eta}$ , $\tilde{\zeta}$ ). The tensor orientation in respect to the axes of the system ( $\xi$ , $\eta$ , $\zeta$ ) is preserved. It manifests itself in the fact that the scalar operators fail to change the indices $L$ and $M$ of the Wigner functions, but change the index $K$ in general case. Therefore, the hamiltonians H(3) and H(4), constructed from the operators MTM and MTTM, have no axial symmetry and the angular momentum projection onto the axis $\tilde{\zeta}$ is not an integral of motion for them. Hence, it is necessary to look for the eigenfunctions of these hamiltonians in the form of a state superposition with different $K$ , though taking into account Elliott's conditions and limitations, caused by the symmetry of the hamiltonians H(3) and H(4) in respect to the transformation group of the axes of the coordinate system ( $\tilde{\xi}$ , $\tilde{\eta}$ , $\tilde{\zeta}$ ). First of all, Elliott's conditions are reduced to the choice of the $\mathcal{D}^L_{KM}$ functions only with such $K$ , which are included in the tensor $T^{(\lambda,\mu)}$ components.

Even Elliott noted that the basis states with different $K$ are not orthogonal and it manifests itself finally in the nonhermicity of the hamiltonians H(3) and H(4).

The phenomenological hamiltonians H(3) and H(4) can be added by the terms proportional to the scalar operators of higher degree of the generators $\tilde{B}_{\alpha\beta}$ than the fourth is. However, in this case they will be reduced to the nonaxial rotator hamiltonian with nonlinear corrections.

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