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BIFURCATION OF STEADY TEARING STATES

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Abstract

We apply the bifurcation theory for compact operators to the problem of the nonlinear solutions of the 3-dimensional incompressible visco-resistive MHD equations. For the plane plasma slab model we compute branches of nonlinear tearing modes, which are stationary for the range of parameters investigated up to now.

1. INTRODUCTION

In laboratory experiments on plasmas or on fluids, the system often reaches a time-asymptotic state, which may be stationary, periodic in time, or even turbulent. In order to describe such states one has to investigate the time-asymptotic solutions of nonlinear equations. A systematic investigation of the structure of the set of nonlinear solutions is possible in the framework of bifurcation theory.

In the present paper we give general results on bifurcation for the incompressible visco-resistive magnetohydrodynamic equations (Section 2) and particular numerical solutions for a plane plasma slab model (Section 3).

2. THEORETICAL RESULTS ON BIFURCATION FOR 3-DIMENSIONAL INCOMPRESSIBLE MHD.

Let Ω be a torus with smooth boundary $\partial\Omega$, n the outward normal to that boundary, Σ a section of the torus by a meridian plane and σ the normal to Σ (see Fig.1).

Let be given a static (zero velocity) equilibrium of the plasma characterized by a magnetic field B_{eq} and a pressure P_{eq} which we assume to be smooth solutions of the usual incompressible MHD equations with viscosity ν and resistivity η. For simplicity we assume ν and η to be constant.

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Denoting the scales of length, magnetic field and mass density by d, B_0 and ρ_0, respectively, we define as usual the resistive time scale τ_R, the hydromagnetic time scale τ_H and the corresponding characteristic velocities by

$$\tau_R = d^2/\eta \quad , \quad \tau_H = d\sqrt{\rho_0}/B_0 \quad V_R = d/\tau_R \quad , \quad V_A = d/\tau_H \tag{1}$$

We normalize time and velocity with respect to τ_R and V_R, respectively, and introduce as usual the Lundquist number $S = \tau_R/\tau_H$ and the magnetic Prandtl number $P_R = \nu/\eta$.

Making the change of unknowns

$$B = B_{eq} + \hat{B} \quad , \quad v = S\hat{v} \tag{2}$$

we obtain the following set of equations, where we have omitted the notation $\hat{}$:

$$\frac{\partial v}{\partial t} - P_R \Delta v + \nabla P = -S\left[v.\nabla v - B_{eq}.\nabla B - B.\nabla B_{eq} - B.\nabla B\right] \tag{3}$$

$$div\, v = 0 \tag{4}$$

$$\frac{\partial B}{\partial t} + \nabla\times\nabla\times B = S\,\nabla\times\left[v\times B + v\times B_{eq}\right] \tag{5}$$

$$div\, B = 0 \tag{6}$$

For the unknowns of eqs. (3)-(6) we choose the following boundary conditions:

$$\left.\begin{array}{l} v = 0 \quad \text{on } \partial\Omega \\ B.n = 0 \quad \text{and} \quad (\nabla\times B)\times n = 0 \quad \text{on } \partial\Omega \end{array}\right\} \tag{7}$$

Thus, $\partial\Omega$ is considered as a conducting wall. We also require

$$\int_\Sigma B.\sigma \; d\Sigma = 0 \tag{8}$$

and we choose appropriate initial conditions for the unknowns B, v.

We define an operator T by the following prescription: With each $u = (B, v)$ we associate the right hand sides of eqs. (3),(5) for $S=1$, denoting these expressions by

$f_1(u)$ and $f_2(u)$, respectively. We then define $u' = (B', v')$ by

$$-P_R \Delta v' + \nabla P' = f_1(u) \quad , \quad \mathrm{div}\, v' = 0 \; ,$$
$$\nabla \times \nabla \times B' = f_2(u) \quad , \quad \mathrm{div}\, B' = 0 \; .$$

with the boundary conditions (7),(8) for u'. The operator T is then defined by $u' = T(u)$.

In terms of the operator T, $u = (B, v)$ is a stationary solution of eqs. (3)-(8) if and only if it verifies

$$u - S\,T(u) = 0 \tag{9}$$

Using the Sobolev imbedding theorems we can prove [1] that T is a compact operator on a functional space E, if we choose for instance

$$E = \left\{ B \in (H^2(\Omega))^3 ;\; \mathrm{div} B = 0,\; \int_\Sigma B.\sigma\, d\Sigma = 0,\; B.n = 0 \;\; \text{on } \partial\Omega \right\} \times \left\{ v \in (H^2(\Omega))^3 ;\; \mathrm{div}\, v = 0,\;\; v = 0 \;\; \text{on } \partial\Omega \right\} \tag{10}$$

where $H^2(\Omega)$ is the classical Sobolev space, that is

$$\left\{ u \in L^2(\Omega) ;\; \frac{\partial u}{\partial x_i} \text{ and } \frac{\partial^2}{\partial x_i \partial x_j} \in L^2(\Omega) ,\;\; \forall\; 1 \leq i,j \leq 3 \right\}$$

We have now to solve the functional equation (9) for $u = (B, v)$ belonging to the space E of eq.(10), where T is compact on E.

Using the results of ref. 2-4 we can prove [1] the following

<u>PROPOSITION</u>

There exists $S^* > 0$ such that the solution $u = 0$ is the only stationary solution of eq.(9) for $0 \leq S < S^*$. Furthermore, let A be the Frechet derivative of T at 0, and let S_1 be a simple characteristic value of A which verifies condition d) of theorem 1.7 of ref.2 (this condition means that the equilibrium solution $u = 0$ looses its stability when S crosses S_1). Then a non-linear solution branch of eq.(9) bifurcates from the equilibrium solution at $S = S_1$ with exchange of stability between the two branches.

If the equilibrium depends only on one coordinate, the bifurcating branch obtained according to the above proposition is generally of parabolic type due to the periodicity of the domain Ω, as is shown in ref.1 .

For some particular types of equilibria, as for instance that studied in Section 3, we can prove as in ref.4 that the first strictly positive characteristic value of the linearized operator is simple, so that one of the hypotheses of the above proposition is verified. Furthermore, we can directly apply global results of ref.3 (theorem 1.3) to the different branches of nonlinear solutions bifurcating from characteristic values of odd multiplicity.

The results of this section are graphically represented in Fig.2 .

The technics described here are also applicable to

- other boundary conditions (such as those of Section 3);
- equilibria with nonzero velocity v_{eq} ;
- other bifurcation parameters.

Finally, for the evolution problem defined by eqs.(1)-(8) with given initial conditions, all the mathematical results known about Navier-Stokes equations and Maxwell's equations can be generalized to the coupled system (1)-(8), like for instance in ref.5 .

3. BIFURCATION OF STATIONARY SOLUTIONS IN THE PLANE PLASMA SLAB MODEL

3.1. Physical model

We now consider a plane plasma layer of (normalized) thickness 1 in the x-direction and of length L in the y-direction (Fig. 3). The z-direction can be ignored. The equilibrium field B_{eq} is chosen as follows (see Fig.3):

$$B_{eq,y}(x) = \tanh(\alpha x) \qquad (11)$$

The parameter α represents the ratio of layer thickness to equilibrium scale length.

In the following, we shall use either S or α as a bifurcation parameter. Varying S may be interpreted as varying the resistivity, whereas varying α corresponds physically to changing the position of conducting walls (the boundaries $x = \pm 1/2$) while keeping the profile of the equilibrium magnetic field fixed in space.

We introduce stream functions ψ, ψ_{eq} and ϕ by putting

$$\left.\begin{aligned} &B = e_z \times \nabla\psi(x,y) \quad , \quad B_{eq} = e_z \times \nabla\psi_{eq}(x) \ , \\ &v = e_z \times \nabla\phi(x,y) \quad . \end{aligned}\right\} \qquad (12)$$

(Note that B and v represent the modified unknowns defined in eq.(2); the notation $\hat{}$ is omitted). The equations of Section 2 may then be written as follows.

$$\frac{\partial}{\partial t}\Delta\phi - \frac{P}{R}\Delta^2\phi = S\left[\frac{\partial(\Delta\phi,\phi)}{\partial(x,y)} + \frac{\partial(\psi,\Delta\psi)}{\partial(x,y)} + \frac{d\psi_{eq}}{dx}\frac{\partial\Delta\psi}{\partial y} - \frac{d^3\psi_{eq}}{dx^3}\frac{\partial\psi}{\partial y}\right] \qquad (13)$$

$$\frac{\partial\psi}{\partial t} - \Delta\psi = S\left[\frac{\partial(\psi,\phi)}{\partial(x,y)} + \frac{d\psi_{eq}}{dx}\frac{\partial\phi}{\partial y}\right] \qquad (14)$$

We use the following boundary conditions

$$\left.\begin{aligned} &\phi = \Delta\phi = 0 \ ; \quad \psi = \Delta\psi = 0 \quad \text{at } x = \pm 1/2 \\ &\text{periodicity in } y \text{ (period L)} \end{aligned}\right\} \qquad (15)$$

Conditions (15) at $x = \pm 1/2$ correspond to free-slip boundary conditions for the velocity; they are different from the conditions (7) for v assumed in Section 2. In the numerical calculations we sometimes also impose symmetry in the x- and y-directions.

In order to have a consistent equilibrium solution of the resistive MHD equations, we should either use a nonzero equilibrium velocity $v_{eq}(x)$ (if η = const.) or assume an appropriate space variation of the resistivity, $\eta(x)$. In the first case, the equations for the stream functions ϕ and ψ would contain v_{eq}. In the present calculations we have assumed η = const. and arbitrarily put $v_{eq} = 0$ in the equations for ϕ and ψ, thus obtaining eqs.(13),(14) .

3.2. Eigenvalues of the linearized equations

Linearizing eqs.(13),(14) and assuming ϕ and ψ to have the form $f(x)\exp(iky + \gamma t)$, we obtain for each wave number k

and given $\mathcal{P}_R$, S and α an eigenvalue problem for γ. Since we wish to know for what values of S or α (our bifurcation parameters) the equilibrium solution looses its stability, we look for the conditions of marginal stability of the equilibrium by putting $\gamma = 0$. If we fix $\mathcal{P}_R$ and α we thus obtain for each k an eigenvalue S(k) ; similarly, fixing $\mathcal{P}_R$ and S we obtain for each k an eigenvalue $\alpha(k)$. In Fig. 4 we show typical curves S(k) and $\alpha(k)$, which have been obtained by numerically solving the linearized eqs.(13),(14) with $\gamma = 0$.

3.3. Bifurcation of stable stationary solutions

Numerical examples of parabolic solution branches (see Fig.2) have been presented in a preceding paper [6] for the case $\alpha = \mathcal{P}_R = 10$, $k = 2.5$. In Fig.5 we show for the same parameters the variation of the vorticity density and of the width of the magnetic island along the nonlinear branch. We also show the form of the stream functions near the bifurcation point (Fig.6) and far from that point (Fig.7).

3.4. A stationary solution for small viscosity

In Fig.8 we show a case where $\mathcal{P}_R \ll 1$. Choosing $\alpha = .5$, $S = 10^4$, $\mathcal{P}_R = 0.2$ we obtain a stable stationary nonlinear solution whose velocity field is strongly concentrated near the separatrix of the magnetic island. As a consequence, the vorticity density Ω_z is highly localized about the separatrix.

4. CONCLUSION

We have shown that in the case of the incompressible visco-resistive MHD equations we can apply bifurcation technics for compact operators to obtain a number of general theoretical results.

For the case of a tearing-unstable plane plasma slab, choosing either the Lundquist number S or the wall distance parameter α as bifurcation parameter, we have obtained numerical examples of nonlinear solution branches representing stationary tearing states of the plasma.

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FIGURE CAPTIONS

Fig. 1. Toroidal domain Ω with boundary $\partial\Omega$; n is the outward normal to $\partial\Omega$, Σ is a meridian section with normal σ .

Fig. 2. Schematic bifurcation diagram for the bifurcation parameter S . S^* and S_1 are defined in the text; at $S = S_1$ a nonlinear solution branch bifurcates from the equilibrium solution.

Fig. 3. The plasma slab model.

Fig. 4. (a) Marginal eigenvalues $S(k)$ for $\mathcal{P}_R = 10$, $\alpha = 4$.
(b) Marginal eigenvalues $\alpha(k)$ for several values of $S/\sqrt{\mathcal{P}_R}$.

Fig. 5. (a) Maximum of the vorticity density as a function of S .
(b) Width of magnetic island as a function of S .

Fig. 6. Level curves of the stream functions $\psi_{tot} = \psi_{eq} + \psi$, ϕ , and of the perturbed current density $j_z = \Delta\psi$ and the vorticity density $\Omega_z = \Delta\phi$ for a value of S near the bifurcation point S_1 . The curves are represented in the domain $0 < x < 1/2$, $0 < y < L/2$. ($S = 200$, $\mathcal{P}_R = 10$, $\alpha = 10$, $L = 0.8\pi$)

Fig. 7. Level curves as in Fig.6 for a value of S far from the bifurcation point S_1 . ($S = 10^4$, $\mathcal{P}_R = 10$, $\alpha = 10$, $L = 0.8\pi$)

Fig. 8. Level curves as in Fig.6 for small viscosity ($\mathcal{P}_R = 0.2$, $S = 10^4$, $\alpha = 5$, $L = 0.8\pi$).

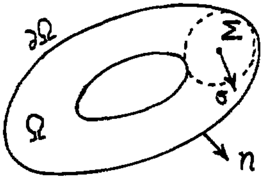


Fig. 1

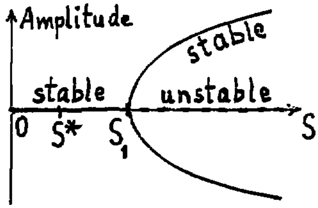


Fig. 2

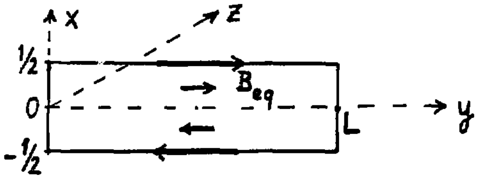


Fig. 3

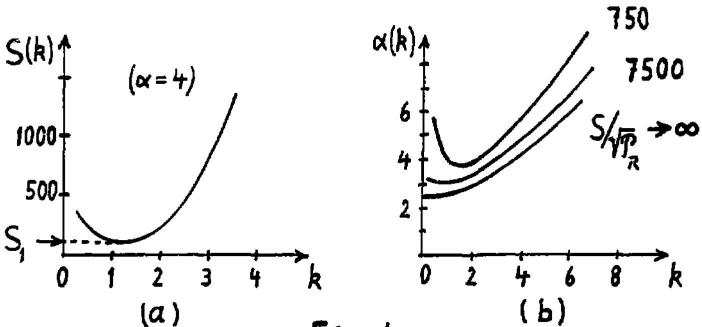


Fig. 4

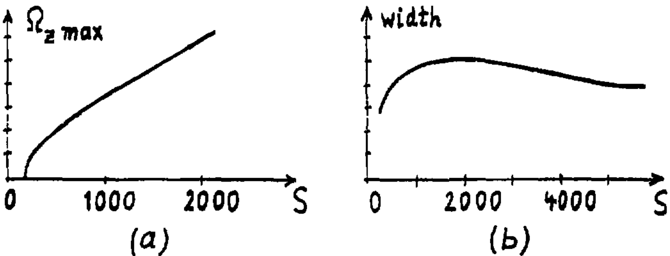


Fig. 5

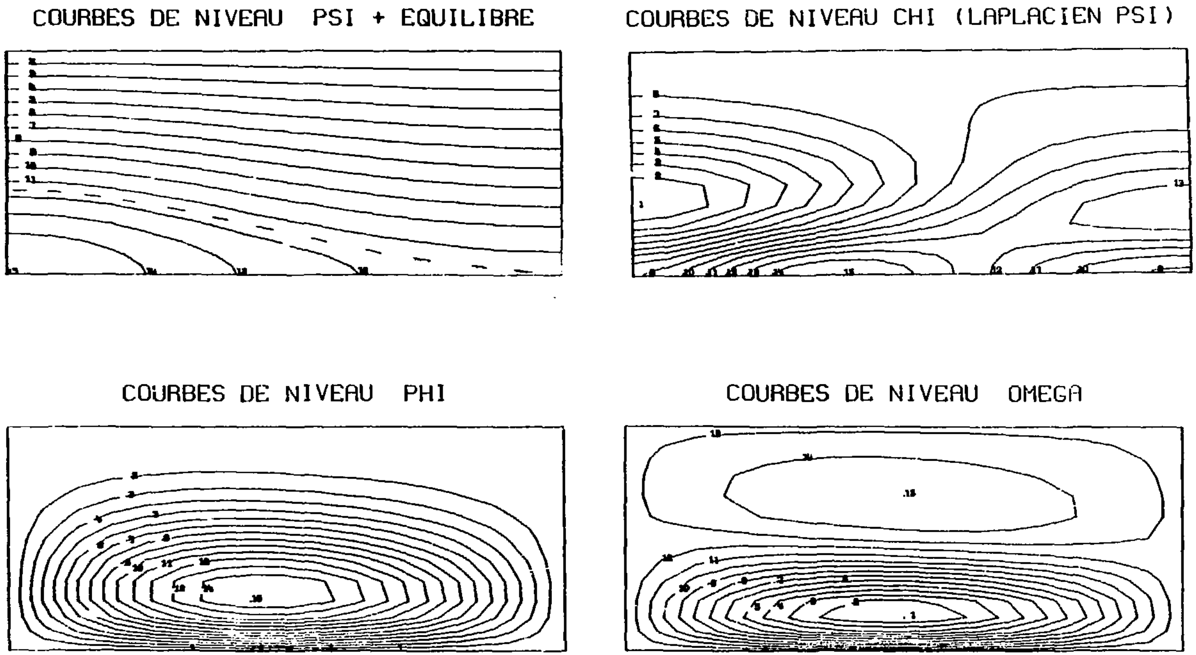
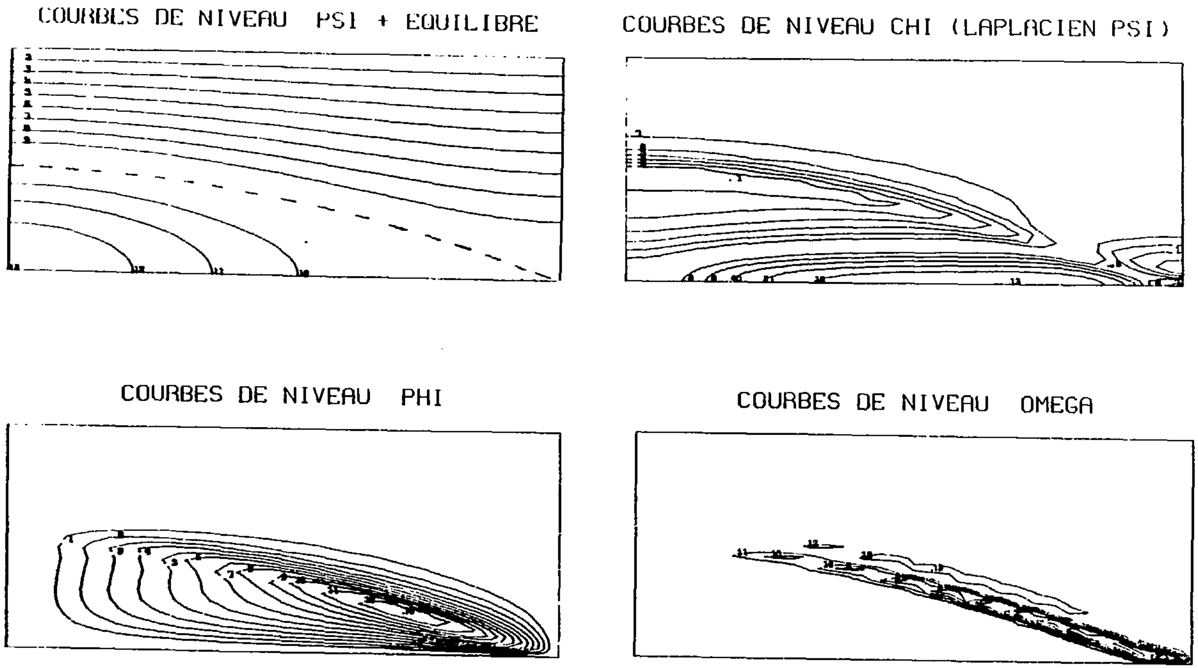

COURBES DE NIVEAU PSI + EQUILIBRE
COURBES DE NIVEAU CHI (LAPLACIEN PSI)
COURBES DE NIVEAU PHI
COURBES DE NIVEAU OMEGA

Fig.6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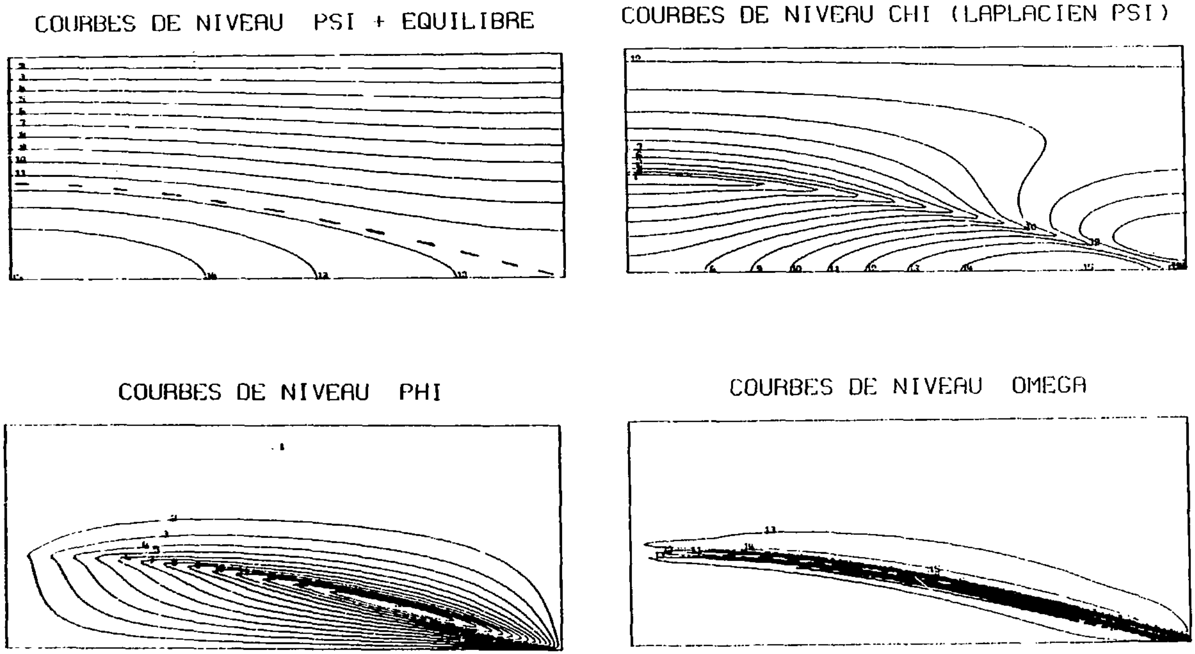
COURBES DE NIVEAU PHI
COURBES DE NIVEAU OMEGA

Fig. 7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COURBES DE NIVEAU CHI (LAPLACIEN PSI)
COURBES DE NIVEAU PHI
COURBES DE NIVEAU OMEGA

Fig. 8