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MAPPINGS WITH CLOSED RANGE AND COMPACTNESS *

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ABSTRACT

The motivation for this note is the result of E.O. Thorp that a normed linear space E is finite dimensional if and only if every continuous linear map for E into any normed linear space has a closed range. Here, a class of Hausdorff topological groups is introduced; called r-compactifiable topological groups, they include compact groups, locally compact Abelian groups and locally convex linear topological spaces. It is proved that a group in this class which is separable, complete metrizable or locally compact, is necessarily compact if its image by a continuous group homomorphism is necessarily closed. It is deduced then that a Hausdorff locally convex is zero if its image by a continuous additive map is necessarily closed.

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A normed linear space E is finite dimensional if and only if $t(E)$ is closed in F whenever t is a continuous linear map from E into any normed linear space F. Thorpe, in [6, Theorem 1] proved this result and then asked [6, p. 213] how it extends to linear topological spaces.

The following is proved in [2, Corollary].

Theorem 1

A Hausdorff locally convex space E is a topological product space R^{Φ} of Φ-copies of the scalar field R if and only if $t(E)$ is closed in F whenever t is a continuous linear map from E into any Hausdorff locally convex space F.

From the proof of Theorem 1 above, we obtain the following:

Corollary

A metrizable locally convex space E is R^I for some countable index set I if and only if $t(E)$ is closed in F whenever t is a continuous linear map from E into a Hausdorff locally convex space F.

In this note, we look at the problem of Thorpe in the context of group homomorphisms of topological groups, and relate it to Theorem 1 above.

If (E,u) is a normed linear space of countably infinite dimension, $(Eu)^{\wedge}$ its completion, and s the discrete topology on E, then the identity map $i: (E,s) \to (E,u)^{\wedge}$ is a continuous group homomorphism from a complete metrizable locally compact topological Abelian group into a complete metrizable topological Abelian additive group (in fact a linear topological space); i does not have a closed range.

We shall say that a Hausdorff topological group G has a <u>compactifiable range</u>, or more shortly, that G is <u>r-compactifiable</u> if there exists a continuous group isomorphism from G into some compact Hausdorff topological group.

Examples of r-compactifiable topological groups

1) Any compact Hausdorff topological group G is r-compactifiable. The identity map $G \to G$ is a group isomorphism.

2) If (G,u) is a Hausdorff locally compact topological Abelian group, then (G,u) is r-compactifiable. For if p is the topology on G of simple convergence on the dual group G' of (G,u), then (G,p) is topologically isomorphic to a subgroup of a compact group – the topological product space $T^{G'}$ of copies of the one-dimensional torus T (see for example [1, p. 167]). Moreover, the topology is coarser than u and thus the identity map $i: (G,u) \to (G,p) \subseteq T^{G'}$, is continuous.

In particular, any Abelian group is r-compactifiable under its discrete topology.

3) Any subgroup G_0 of an r-compactifiable topological group G is r-compactifiable. For if g is a continuous group isomorphism from G into a compact Hausdorff topological group H, the restriction g_0 of g to G_0 is necessarily a continuous group isomorphism from G_0 into H.

4) If (G,u) is an r-compactifiable topological group and v is a group topology on G finer than u, then (G,v) is v-compactifiable. If g is a continuous isomorphism from (G,u) into a compact Hausdorff topological group H, then $g : (G,v) \to H$ is also a continuous group isomorphism.

5) The topological product space $\bigtimes_{\alpha\in\Phi} G_\alpha$ of any family $(G_\alpha)_{\alpha\in\Phi}$ of r-compactifiable topological groups is r-compactifiable. If for each α, f_α is a group isomorphism from G_α into a compact Hausdorff topological group H_α, then $\bigtimes_{\alpha\in\Phi} H_\alpha$ is a compact Hausdorff topological group, and the mapping

$$f : \bigtimes_{\alpha\in\Phi} G_\alpha \to \bigtimes_{\alpha\in\Phi} H_\alpha$$

such that $f((x_\alpha)_{\alpha\in\Phi}) = (f_\alpha(x_\alpha))$ is a continuous group isomorphism. As the real or complex number field R is a locally compact topological Abelian group, the product space R^Φ of Φ copies of R is r-compactifiable.

6) If E is any Hausdorff locally convex space over R, then E is r-compactifiable. For if ω is the weak topology on E, the completion $(E,\omega)^\wedge$ of (E,ω) is topologically isomorphic to some R^Φ for some Φ. Thus by Example (5) above, $(E,\omega)^\wedge$ is r-compactifiable. Therefore, by example (3), (E,ω) is r-compactifiable, and using Example (4), we can see that E is r-compactifiable.

Theorem 2

Let G be either a complete metric or locally compact Hausdorff topological group which is separable. Suppose that G is r-compactifiable. Then G is compact if and only if $t(G)$ is closed in H whenever t is a continuous group homomorphism from G into any r-compactifiable topological group H.

Proof

Suppose that $t(G)$ is closed in H whenever t is a continuous group homomorphism from G into any r-compactifiable topological group H. As G is r-compactifiable, there is a continuous group isomorphism f from G into a compact Hausdorff topological group F. Notice that F being compact and Hausdorff is r-compactifiable. By the hypothesis, $f(G)$ is closed in F, and $f(G)$ is thus a compact Hausdorff topological group under the topology inherited from F. By [4, p.213]

then, the continuous group isomorphism $f : G \to f(G)$ is open, since a compact topological group is of the second category. Thus f is a homeomorphism, and G is compact, since $f(G)$ is compact. The converse also holds, since if g is a continuous map from a compact Hausdorff space X into a Hausdorff topological space Y, then $f(X)$ is compact and is thus closed in Y.

Corollary

A separable Hausdorff locally compact topological Abelian group G is compact if and only if $t(G)$ is closed in H whenever t is a continuous group homomorphism from G into a Hausdorff locally compact topological group.

Theorem 3

A metrizable locally convex space E over R is the singleton set zero if and only if $t(E)$ is closed in H whenever t is a continuous additive group map from E into any r-compactifiable topological group.

Proof

Suppose that $t(E)$ is closed in H whenever t is a continuous additive group map from E into any r-compactifiable topological group. Then since metrizable locally convex spaces are r-compactifiable, any continuous linear map from E into any metrizable locally convex linear topological space has a closed range. By the Corollary to Theorem 1 above, E is the topological product space R^I of *countably many* copies of R. Thus $E = R^I$ is a separable metrizable r-compactifiable topological group which is complete, satisfying the conditions of G in Theorem 2. Therefore E is a compact Hausdorff linear topological space. It must then be the singleton set zero.

c.f. Corollary of Theorem 1. Note that Theorem 3 clearly holds with "metrizable" replaced by "normed".

Theorem 4

A Hausdorff locally convex space E over R is the singleton set zero if and only if $t(E)$ is closed in H whenever t is a continuous additive group map from E into any r-compactifiable topological group H.

Proof

Suppose that $t(E)$ is closed in H whenever t is a continuous additive group map from E into any r-compactifiable topological group. The locally convex space E clearly satisfies the condition of Theorem 1. Therefore $E = R^\Phi$ for some Φ. Let Φ_0 be a countable subset of Φ. Then R^{Φ_0} is a quotient space of $E = R^\Phi$; the quotient map $k : R^\Phi \to R^{\Phi/\Phi_0}$ $(= R^{\Phi_0})$ is continuous. If t is a continuous additive map from $R^{\Phi}0$ onto any r-compactifiable topological group H, then the composition map $t\circ k : E \to H$ is continuous, and by hypothesis, $t\circ k(E)$ $(=t(R^{\Phi}0))$ is closed in H. By Theorem 3, $R^{\Phi}0$ is the singleton set zero. This implies that $E = R^{\Phi}$ is the singleton set zero.

c.f. Theorem 1. Notice that in Theorem 2, from where Theorems 3 and 4 are derived, we could replace G by an Abelian Hausdorff topological group G which is the inductive limit (see (7)) of $(G_n; g_n : n = 1,2,\dots)$ with $G = \bigcup_{n=1}^{\infty} g_n(G_n)$, where each G_n is separable and complete metrizable or locally compact. In this case, in the proof, instead of [4, p. 213], we use the open mapping theorem in [3, Theorems 2.1 and 3.3].

If X is a Hausdorff locally compact non-compact topological space, then the injection map from X into its one point compactification X^c is continuous, but does not have a closed range. Now, a compact Hausdorff space being normal and thus completely regular is homeomorphic to a subspace of a cube [4, p. 141, p. 115, p. 117, p. 145]. Also if t is a continuous map from a compact Hausdorff space X into a Hausdorff space Y, t(X) is compact and therefore closed in Y [4, p. 141]. We therefore have the following:

Theorem 5

A completely regular Hausdorff space X is compact if and only if t(X) is closed in Y whenever t is a continuous map from X into any completely regular Hausdorff space Y.

Now a Hausdorff topological group, (whether r-compactifiable or not) is a completely regular Hausdorff space. Theorem 5 may then be compared with Theorem 2.

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