IC/85/259
INTERNAL REPORT
(Limited distribution)

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON TWO EXAMPLES IN LINEAR TOPOLOGICAL SPACES *

S.O. Iyahen **
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

This note first gives examples of B-complete linear topological spaces, and shows that neither the closed graph theorem nor the open mapping theorem holds for linear mappings from such a space to itself. It then looks at Hausdorff linear topological spaces for which coarser Hausdorff linear topologies can be extended from hyperplanes. For B-complete spaces, those which are barrelled necessarily have countable dimension, and conversely. The paper had been motivated by two questions arising in earlier studies related to the closed graph and open mapping theorems; answers to these questions are contained therein.

MIRAMARE – TRIESTE
November 1985

* To be submitted for publication.

** Permanent address: Department of Mathematics, University of Benin, Benin City, Nigeria.

1. The motivation for this note comes mainly from two questions asked in our earlier papers related to the study of the closed graph and open mapping theorems.

Let (E,u) be a Hausdorff linear topological space. Suppose that (E,u) is the generalized *-inductive limit of a sequence of compact sets. It is asked in [9,p.109, Paragraph 1] if the closed graph theorem holds for linear mappings from ultrabarrelled spaces into (E,u). This question and related matters are discussed in Sec. 3.

Let (F,v) be a Hausdorff linear topological space. Suppose that for every hyperplane F_0 in F, any Hausdorff linear topology on F_0 coarser than the v-induced topology has a Hausdorff linear extension to the whole of F. In (5), such a linear topological space is called an h-extendable space. Every linear topological space in which linear subspaces are closed, is h-extendable. The question is asked in (5) if every linear subspace of an h-extendable space is necessarily closed. This is attended to in Sec. 4. Sec. 2 contains some definitions and examples.

2. Let E be a linear topological space. A closed balanced absorbent subset B_0 of E is called an <u>ultrabarrel</u> if there is a sequence $(B_n)_{n=1}^{\infty}$ of closed balanced absorbent subsets such that $B_{n+1} + B_{n+1} \subseteq B_n$ for all $n = 1,2,...$

A barrel in a linear topological space (that is, a closed absolutely convex absorbent subset) is an ultrabarrel. A locally convex space in which every barrel is a neighbourhood of the origin is called a <u>barrelled space</u>. A linear topological space in which every ultrabarrel is a neighbourhood of the origin is called an <u>ultrabarrelled space</u>. A Frechet space or an inductive limit of a sequence of Frechet spaces has both properties. There are however barrelled spaces which are not ultrabarrelled, and conversely (see [17, Sec. 5] and [7, Sec. 3]).

Let E be a barrelled space (an ultrabarrelled space). Then, any linear map from E into a locally covex space (into a linear topological space) F is nearly continuous, and any linear map from F onto E is nearly open. Further, if there is a continuous linear nearly open map from E onto a linear topological space G, then G is barrelled (ultrabarrelled). (see [7, Sec. 3]).

Let E be a linear space, and suppose that for each γ in an index set Γ, E_γ is a linear topological space, and t_γ is a linear map from E_γ to E. The upper bound n of the set of all linear topologies on E for which all t_γ are continuous, is called the <u>*-inductive limit topology on E induced by $(E;t_\gamma)$</u>, and (E,μ) is referred to as the *-inductive limit of $(E_\gamma;t_\gamma)$. If in this case, $E = \bigcup_{\gamma\in\Gamma} t_\gamma(E_\gamma)$, then (E,n) is called the generalized strict *-inductive limit of $(E_\gamma;t_\gamma)$.

These definitions are in [7, Sec. 2].

Suppose that in the definition of a generalized strict *-inductive limit as above, S_γ is a balanced semiconvex subset of E_γ, f_γ is the restriction of t_γ to S_γ and $E = \bigcup_{\gamma\in\Gamma} f_\gamma(S_\gamma)$, then the finest linear topology on E for which the maps f_γ are continuous, is referred to as the generatized *-inductive limit topology of the semiconvex sets. This topology is discussed in [9]; it is an extension to the situation of general linear topological spaces of the generalized inductive limit topology of convex sets studied by Garling in [3].

As shown in [19, Theorem 5], if a Hausdorff barrelled space is the union of an increasing sequence of absolutely convex sets, then it has a generalized inductive limit topology of the convex sets. Similarly from Theorem 3.1 of [4], any Hausdorff ultrabarrelled space which is the union of an increasing sequence of balanced semiconvex sets has a generalized *-inductive limit topology; (the proof given in Theorem 3.1 of [4] still works if k is not fixed).

A Hausdorff linear topological space E is called a B-complete linear topological space if every continuous linear nearly open map t from E onto any Hausdorff linear topological space is open.

If in this definition, we restrict "t" to be a one-to-one linear map, then we say that E is a B_r-complete linear topological space.

The concepts of a B-complete locally convex space and a B_r-complete locally convex space were defined and studied by Ptak in [12] and [13]. If there is a continuous linear nearly open map from a locally convex space onto a linear topological space F, then F is necessarily locally convex. Thus the definition as above, of a B-complete linear topological space, (which is due to Raikov [14]) coincides with that of Ptak in the context of locally convex spaces. Thus most of the examples of B-complete linear topological spaces are B-complete locally convex spaces which examples were obtained through the duality theory of locally convex spaces. Raikov in [14], pointed out however, that any complete metrizable linear topological spaće, is a B-complete linear topological space. He also showed that a closed linear nearly continuous map into a B-complete linear topological space is continuous. In particular, a closed linear map from an ultrabarrelled space into a B-complete linear topological space, is continuous.

Clearly, a B-complete linear topological space is B_r-complete, and a Hausdorff quotient space of a B-complete linear topological space is B-complete. Further, a B_r-complete linear topological space E is B-complete if and only if every Hausdorff quotient spaces of E is B_r-complete. It is also easy to see that a Hausdorff barrelled (ultrabarrelled) space E is B_r-complete if and only if there is no strictly coarser Hausdorff barrelled (ultrabarrelled) topology on E. These observations also yield known examples of B-complete linear topological spaces. We give some of these.

(i) For some index set Φ, let $E = R^\Phi$, the product of Φ-copies of the scalar field R - either the real or complex numbers throughout. Then E is a Hausdorff barrelled space. Also, on E, there is no strictly coarser Hausdorff locally convex topology, and therefore, there is no strictly coarser Hausdorff barrelled topology. Thus E is B_r-complete. Since any Hausdorff quotient of E is also of the form R^Ψ for some index set Ψ [18, p. 111, Exc. 6] we see that E is B-complete.

(ii) If E is a linear space of countably infinite dimension, then under its finest locally convex topology (= the finest linear topology) $\tau(E,E^*)$, E is the strict inductive limit of an increasing sequence of finite dimensional (and therefore Banach) linear subspaces; it is barrelled and ultrabarrelled. If E were to have a strictly coarser Hausdorff barrelled or ultrabarrelled topology defined on it, then, (E,v) would also have the same strict inductive limit topology sequence of the finite dimensional subspaces, by the result of Valdivia [19, Theorem 5] mentioned earlier. In particular then, any linear space E of countable dimension has only one Hausdorff topology for which it is barrelled or ultrabarrelled - its finest locally convex topology. Thus $(E,\tau(E,E^*))$ is B_r-complete. As each quotient space of $(E,\tau(E,E^*))$ has countable dimension and also has its finest locally convex topology, $(E,\tau(E,E^*))$ is B-complete.

(iii) Let (E,u) be a Hausdorff ultrabarrelled space with a fundamental sequence $(A_n)_{n=1}^\infty$ of balanced semiconvex compact sets, then E has a generalized *-inductive limit topology of the compact sets. If v is a Hausdorff coarser ultrabarrelled topology on E, then the sets A_n are v-compact, and the topologies u,v coincide on each A_n. The space (E,v) then also has the same generalized *-inductive limit topology of the compact sets. Thus $u = v$ and (E,u) is B_r-complete. Now, each Hausdorff quotient of (E,u) also has a generalized *-inductive limit topology of compact sets, and we see that (E,u) is B-complete. In particular, if (E,u) is a Hausdorff barrelled space with a fundamental sequence of compact sets, then (E,u) is B-complete; it is of course the strong dual of a Frechet Montel space [see 2, Theorem 4], and [Proposition 2.2].

(iv) Suppose that (E,v) is a Hausdorff barrelled space with a fundamental sequence (A_n) of absolutely convex weakly compact sets. If ω is a coarser Hausdorff barrelled topology on E, then (E,v) is the generalized inductive limit of (A_n,v), and (E,ω) is the generalized inductive limit of (A_n,ω). Moreover, since each A_n is a weakly compact subset of (E,v) and $\omega < v$, then on each A_n, the weak topologies relative to v and ω coincide, and from this, one sees that on each A_n, v, ω coincide and thus $v = \omega$. Thus (E,v) is B_r-complete. As in (i), (ii) and (iii), (E,v) is in fact B-complete. In particular, the strong dual of a reflexive Frechet space is B-complete (see [15, p. 9]).

3. Theorem 3.1. Let (E,u) be the generalized $*$-inductive limit of an increasing sequence (A_n) of compact sets. Then (E,u) is B_r-complete.

Proof

We show that (E,u) is B_r-complete. And since any quotient space of (E,u) is also the generalized $*$-inductive limit of compact sets, it will follow that (E,u) is B-complete.

If v is a Hausdorff linear topology on E coarser than u, let $v(u)$ be the linear topology on E coarser than u, let $v(n)$ be the linear topology on E with a base of neighbourhoods consisting of the v-closures of members of a base for (E,u). Thus $v \leq v(u) \leq u$. The identity map $i : (E,u) \to (E,v)$ is nearly open if $v = v(u)$, being open if $v = u$.

To show that (E,u) is B_r-complete, we assume that $v = v(u)$. Let (E,ω) be the generalized $*$-inductive limit of (A_n,v). The sets A_n are also compact in (E,ω), and we see that $\omega = u$. By [20, p. 86, (4)], we then see that (E,u) has a base of neighbourhoods consisting of v-closed sets, since (E,ω) necessarily has, and $\omega = u$. Since $v = v(u)$, we must then have that $v = v(u) = u$, and the identity map $i : (E,u) \to (E,v)$ is open.

In particular, we have the following answer to the question raised in Paragraph 1, page 109 of (9).

Corollary

Let F be a Hausdorff ultrabarrelled space, and E a Hausdorff linear topological space, the generalized $*$-inductive limit of a sequence of compact sets. Then, any closed linear map from F into E is continuous.

The following uses a modification of the technique of proof of Theorem 3.1 of (8).

Theorem 3.2

If a Hausdorff linear topological space (E,u) is the generalized $*$-inductive limit of an increasing sequence of balanced semiconvex compact sets, and f is a linear map $(E,u) \to (E,u)$, with a closed graph, then f is continuous.

Proof

Let E_n be the linear span of A_n, and u_n the Hausdorff locally bounded linear topology on E_n with the sequence $\left(\frac{1}{m} A_n\right)_{m=1}^{\infty}$ as a base of neighbourhoods; (E_n,u_n) is complete. The graph of the restriction map of f to E_n is closed in $(E_n,u_n) \times (E,u)$, and therefore, since (E,u) is B_r-complete by Theorem 3.1, and (E_n,u_n) is ultrabarrelled, each restriction map of f to $(E_n,u_n) \to (E,u)$ is continuous. Thus each $f(A_n)$ is bounded in (E,u). Now by [20, p. 87 (6)] and [6, Prop. 2], $(A_n)_{n=1}^{\infty}$ is a fundamental sequence of bounded sets in (E,u). Therefore, for each n, there is a positive integer $m(n)$ such that $f(A_n) \subseteq A_{m(n)}$. If f_n is the restriction of f to A_n, then the graph G_n of $f_n : (A_n,u) \to (A_{m(n)},u)$ is closed. That is, G_n is a closed subset of the compact Hausdorff space $(A_n,u) \times (A_{m(n)},u)$. Thus G_n is compact under its induced topology.

If P_n, $P_{m(n)}$ are the projection maps of $(A_n,u) \times (A_{m(u)},u)$ onto (A_n,u), $(A_{m(n)}u)$ respectively, then the restriction P_{G_n} of P_n to G_n is continuous one-to-one, and is thus a homeomorphism, since G_n is compact. As $f_n = P_{m(n)} P_{Gn}^{-1}$, each f_n is continuous. This implies that $f : (E,u) \to (E,u)$ is continuous.

Corollary 1

Let E be a Frechet space with dual E'. Let c denote the topology on E' of compact convergence, and $\sigma(E',E)$, $\tau(E',E)$ respectively the associated weak and Mackey topologies on E'. Then, under each of these topologies, any closed linear map $E \to E$ is continuous.

Proof

That (E',c) has the property follows immediately from Theorem 3.2.

A linear map $f : (E', \tau(E',E)) \to (E',\tau(E',E))$ is closed if and only if $f : (E',c) \to (E',c)$ is closed and this is true if and only if $f : (E', \sigma(E',E)) \to (E', \sigma(E',E))$ is closed. By Theorem 3.2, $f : (E',c) \to (E',c)$ is continuous if closed. This implies that $f : (E', \sigma(E',E)) \to (E', \sigma(E',E))$

is continuous (see for example [16, p. 39, Prop. 13]). And similarly $f : (E', \tau(E',E)) \to (E', \tau(E',E))$ is continuous (see for example [16, p. 62, Prop. 14]).

Let E be a Frechet space with dual E' such that the Mackey topology $\tau(E',E)$ is different from the topology c on E' of compact convergence.

Let $G = (E',c) \times (E',\tau(E', E))$, and let $f : G \to G$ be such that $f(x,y) = (y,x)$. Because $c \leq \tau(E',E)$, the map $f : (E',c) \times (E', \tau(E',E)) \to (E',c) \times (E',c)$ is continuous, and hence closed, since its range is Hausdorff. Therefore, $f : (E',c) \times (E',\tau(E',E)) \to (E',c) \times (E', \tau(E',E))$ is closed.

Since c is strictly coarser that $\tau(E',E)$, there is a net (x_α) which converges to some point x in (E',c), but (x_α) does not converge in $(E',\tau(E',E))$. Let y be a fixed point of E, and for each α as above, let $y_\alpha = y$. Then the net (x_α,y_α) converges to (x,y) in $(E',c) \times (E',\tau(E',E))$, but $f(x_\alpha,y_\alpha)) = ((y_\alpha,x_\alpha))$ does not converge in $(E',c) \times (E',\tau(E',E))$. Thus f is not continuous.

Notice that $(E',c) \times (E',\tau(E',E))$ is the dual of a Frechet space $E \oplus E$ under some topology ω, say between the topology of compact convergence and the Mackey topology. By [15, p. 9) then, $(E',c) \times (E',\tau(E',E))$ is B-complete. We have therefore proved the following, since f is one-to-one and onto.

<u>Corollary 2</u>

A closed linear one-to-one map from a B-complete linear topological space onto itself need not be continuous or open.

4. Let $\mathcal{F}$ be a class of Hausdorff linear topological spaces. We say that a Hausdorff linear topological space (E,n) is <u>$\mathcal{F}$-extendable</u> if for each hyperplane E_0 in E, and Hausdorff topology v_0 on E_0 such that $v_0 \leq u_0$ and $(E_0,v_0) \in \mathcal{F}$, there is a Hausdorff linear extension v of v_0 to the whole of E such that $v \leq u$.

Let $\mathcal{F}$ be the class of all Hausdorff linear topological spaces. Let $(E,u) \in \mathcal{F}$, and let E_0 be a closed hyperplane in (E,u). If v is a Hausdorff linear topology on E_0 coarser than the u-induced topology, then $(E_0,v) \times R$ is Hausdorff, and its topology is coarser than u. Thus in this case, any linear topological space in which all hyperplanes are closed, is $\mathcal{F}$-extendable.

If an l.t.s. E is $\mathcal{F}$-extendable for arbitrary $\mathcal{F}$, then we say that E is <u>extendable</u>. We shall see from Theorem 4.1 below that this definition coincides with that of extendable spaces as in (5). Theorem 4.1 also answers the question asked in (5), whether every linear subspace of an h-extendable linear topological space is necessarily closed.

We need some Lemmas.

<u>Lemma 4.1</u>

[10, Lemma 3]. Let t be a linear map from an l.t.s. (E,u) to an l.t.s. (F,v). Let ω be the linear topology of F with base $(t(u) + v : U \in \mathcal{U}, V \in \nu$, where $\mathcal{U}, \nu$ are bases for the topologies u, v respectively. Then t is closed if and only if (F,ω) is Hausdorff.

<u>Lemma 4.2</u>

(see [20, p. 55]). Let (F,n) be a Hausdorff linear topological space, which is a dense proper linear subspace of a Hausdorff linear topological space (F,v). Let $\mathcal{U}$ be the base of all neighbourhoods of the origin in (E,u). Let $x_0 \in F\backslash E$. Then the family of sets: $((Rx_0 + \bar{U}^v) \cap E : U \in \mathcal{U})$ is a base of neighbourhoods for a Hausdorff linear topology ω on E strictly coarser than u, and such that the identity map $(E,u) \to (E,\omega)$ is nearly open.

<u>Lemma 4.3</u>

[5, Theorem 4.1]. Let (E,u) be a Hausdorff linear topological space. Then for a linear subspace E_0 of E, the following conditions are equivalent:

(i) Any Hausdorff linear topology on E_0 coarser than the u-induced topology is the restriction to E_0 of a Hausdorff linear topology on E coarser than u.

(ii) The graph of a closed linear map from a Hausdorff l.t.s. (F,v) into (E_0,u_0) is necessarily closed in $(F,v) \times (E,u)$.

<u>Proof</u>

(i) $\to$ (ii). Assume (i). Let t be a closed linear map from a Hausdorff l.t.s. (F,v) into (E_0,u_0). Then, by Lemma 4.1, there is a Hausdorff linear topology ω say, on E_0 coarser than the u-induced topology u_0, such that $t : (F,v) \to (F_0,\omega)$ is continuous. By (i), there is a Hausdorff linear topology z on E coarser than u such that ω is the restriction of z to E_0. Clearly, the map $t : (F,v) \to (E,z)$ is continuous. As (E,z) is Hausdorff, the graph G of t is closed in $(F,v) \times (E,z)$. Since z is coarser than u, G is closed in $(F,v) \times (E,u)$.

(ii) → (i). Assume (ii). Let ω be a Hausdorff linear topology on E_0 coarser than u_0. Thus the identity map $i : (E_0,\omega) \to (E_0,u_0)$ is closed, its inverse being continuous.

By (ii) the graph of i is closed in $(F_0,\omega) \times (E,u)$. This implies by Lemma 4.1 that for some Hausdorff linear topology z on E coarser than u, the identity map $i : (E_0,\omega) \to (E,z)$ is continuous.

Clearly, the z-induced topology z_0 on E_0 is coarser than ω. We now show that $\omega \leq z_0$, and (i) will follow.

Let W be a neighbourhood of the origin in (E_0,ω). Then there is a neighbourhood W_1 in (E_0,ω) such that $W_1 + W_1 \subseteq W$. As $\omega \leq u_0$, there is a neighbourhood U in (E,u) such that $U \cap E_0 \subseteq W_1$.

Thus $(U + W_1) \cap E_0 \subseteq U \cap E_0 + W_1 \subseteq W_1 + W_1 \subseteq W$. By Lemma 4.1, $U + W_1$ is a z-neighbourhood, $\omega \leq z_0$, and the proof is complete.

Theorem 4.1

A Hausdorff linear topological space is extendable if and only if every linear subspace is closed.

Proof

Suppose that a Hausdorff l.t.s. (E,u) is extendable. We claim that every hyperplane in (E,u) is closed. This would imply that every linear subspace is closed.

If (E,u) has a dense hyperplane E_0, then by Lemma 4.2, on E_0, there is a Hausdorff linear topology ω say, which is strictly coarser than the u-induced topology u_0. The identity map $i : (E_0,\omega) \to (E_0,u_0)$ is closed, since its inverse is continuous. Since (E,u) is $\mathcal{F}$-extendable for arbitrary $\mathcal{F}$, it follows from Lemma 4.3 that the graph of i is closed in $(E_0,\omega) \times (E_0,u)$. Therefore, the continuous map $i^{-1} : (E_0,u_0) \to (E_0,\omega)$ has a graph which is closed in $(E,u) \times (E_0,\omega)$. Its domain must then be closed. Thus E_0 is closed in E.

The converse has been proved above.

Corollary

A Hausdorff linear topological space is extendable if and only if each hyperplane is closed.

Theorem 4.2

Let $\mathcal{F}$ be a class of Hausdorff linear topological spaces such that

(i) If $E \in \mathcal{F}$, and there is a continuous nearly open linear map from E onto a Hausdorff linear topological space F, then $F \in \mathcal{F}$,

(ii) If $E \in \mathcal{F}$, and E_0 is a hyperplane in E, then under the induced topology, $E_0 \in \mathcal{F}$.

A Hausdorff linear topological space E in $\mathcal{F}$ is $\mathcal{F}$-extendable if and only if E is extendable.

Proof

Let (E,u) be $\mathcal{F}$-extendable. Let E_0 be a hyperplane in E, and let u_0 denote the u-induced topology on E_0. We show that E_0 is closed in (E,u).

Notice that by the hypothesis, $(E_0,u_0) \in \mathcal{F}$.

Suppose that E_0 is dense in (E,u). Then by Lemma 4.2, there is a Hausdorff linear topology v_0 on E_0 strictly coarser than u_0 and such that the identity map $i : (E_0,u_0) \to (E_0,v_0)$ is nearly open. By hypothesis then, $(E_0,v_0) \in \mathcal{F}$, since $(E_0,u_0) \in \mathcal{F}$.

The inverse map $i^{-1} : (E_0,v_0) \to (E_0,v_0)$ is closed. Since (E_0,u_0) is $\mathcal{F}$-extendable, v_0 has a Hausdorff linear extension to E, and it follows from Lemma 4.3 ((i) → (ii)) that $i^{-1} : (E_0,v_0 \to (E,u))$ is closed. We can then show as in Therem 4.1, that E_0 is closed in (E,u), and that (E,u) is extendable by Theorem 4.1.

Such $\mathcal{F}$ as in Theorem 3.2 could be the class of all metrizable linear topological spaces or all Hausdorff barrelled or ultrabarrelled spaces. In view of [11, p. 53, Problem H(a)], we have the following

Corollary

Let $\mathcal{F}$ be the class of all metrizable linear topological spaces. Then a metrizable linear topological space is $\mathcal{F}$-extendable if and only if it is finite dimensional.

If we choose $\mathcal{F}$ to be the class of all Hausdorff barrelled spaces or ultrabarrelled spaces, then a Hausdorff linear topological space is extendable if and only if every linear subspace is closed. Further, we have the following, the proof of which depends on the continuum hypothesis.

Theorem 4.3

A Hausdorff barrelled space is extendable and B-complete if and only if it has countable dimension.

Proof

Let (E,u) be a Hausdorff barrelled space. If it has countable dimension, then there is no other Hausdorff topology making E barrelled, and the topology u is the finest locally convex topology on E. Every linear subspace of such (E,u) is closed, and also (E,u) is B-complete, as pointed out in Sec. 2.

Now, let the Hausdorff barrelled space E be extendable and B-complete.

As E is extendable, every linear subspace is closed. Thus every linear functional is continuous. As E is barrelled, it then must have the Mackey topology $\tau(E,E^*)$ - the finest locally convex topology on E.

If E has uncountable dimension c, give E a topology ω such that (E,u) is topologically isomorphic to the Banach space ℓ^2. Thus ω is strictly coarser than $\tau(E,E^*)$, implying that $(E,\tau(E,E^*))$ is not even B_r-complete. Further, if the dimension of E exceeds c, then $(E,\tau(E,E^*))$ would have as a closed linear subspace, some $(F,\tau(F,E^*))$ of dimension c. Since by the above, such $(F_1,\tau(F,F^*))$ is not B_r-complete, and a closed linear subspace of a B_r-complete space is B_r-complete, we deduce that $(E,\tau(E_x E^*)$ is not B_r-complete.

Notice that this is not true if "B-complete" is replaced by "complete", since any linear space E is barrelled, extendable and complete under $\tau(E,E^*)$.

Corollary

A linear space has countable dimension if and only if it admits of a Hausdorff topology under which it is barrelled, extendable and B_r-complete.

ACKNOWLEDGMENTS

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

[1] J. Dieudonne, "Denumerability conditions in locally convex vector spaces", Proc. Amer. Math. Soc. 8 (1957) 367-372.

[2] D.J.H. Garling, "Locally convex spaces with denumerable systems of weakly compact subsets", Proc. Cambridge.Philos. Soc. 60 (1964), 813-815.

[3] D.J.H. Garling, "A generalized inductive limit topology for vector spaces", Proc. London Math. Soc. (3) 14 (1964) 1-28.

[4] S..O. Iyahen, "A completeness theorem", Rev. Roum Math. Pures et. Appl. 18 (5) (1973) 681-687.

[5] S.O. Iyahen, "Linear mappings with closed range and finite dimensional linear spaces", ICTP, preprint IC/84/167 (1984), to appear in the Journal of the Nigerian Mathematical Society, Vol. 3.

[6] S.O. Oyahen, "Linear topological space with fundamental sequences of compact sets", Math. Ann. 200 (1973) 179-183.

[7] S.O. Iyahen, "On certain classes of linear topological spaces", Proc. London Math. Soc. 18 (3) (1968) 285-307.

[8] S.O. Iyahen, "On the closed graph theorem", Israel J. of Maths, 10 (1) (1971) 96-105.

[9] S.O. Iyahen and J.O. Popoola, "A generalized inductive limit topology for linear spaces", Glasgow Math. J. 14 (1973) 105-110.

[10] J.L. Kelley, "Hypercomplete linear topological spaces", Michigan Math. J. 5 (1958) 235-246.

[11] J.L. Kelley, I. Namioka and co-authors, Linear Topological Spaces (Van-Nostrand, New York, 1963).

[12] V. Ptak, "Completeness and the open mapping theorem", Bull. Soc. Math. Fr. 86 (1958) 41-74.

[13] V. Ptak, "On complete topological linear spaces", Czechoslovak Math. J. 3 (78) (1953) 301-364.

[14] D.A. Raikov, "Closed graph theorems and completeness in linear topological spaces", Proc. 4th All Union Math. Congress Leningrad (1961).

[15] A.P. Robertson and W.J. Robertson, "On the closed graph theorem", Proc. Glasgow Math. Assoc. 3 (1956) 9-12.

[16] A.P. Robertson and W.J. Robertson, Topological Vector Spaces (Cambridge University, Tract, 1964).

[17] W. Robertson, "Completions of topological vector spaces", Proc. London Math. Soc. 8 (1958) 242-257.

[18] H.H. Schaefer, Topological Vector Spaces (Springer Verlag, New York, 1964).

[19] M. Valdivia, "Absolutely convex sets in barrelled spaces", Ann. Inst. Fourier, Grenoble, 21 (1971) 3-13.

[20] N. Adasch, B. Ernst and D. Klem, Topological Vector Spaces; The Theory Without Convexity Conditions. (Springer Verlag, Berlin, 1978).