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ON SOME SOLVABLE MODELS IN NON-RELATIVISTIC QUANTUM MECHANICS *

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ABSTRACT

The theory of self-adjoint extensions is employed to generalize some previous results in non-relativistic quantum interactions. In particular, the Hamiltonian $H = -\Delta + V$, where Δ is the Laplacian and the potential V consists of a strongly singular interaction, a Coulomb and a δ-shell interaction is studied. The spectral properties are discussed and phase shifts as well as low energy parameters are obtained.

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1. INTRODUCTION

In recent years there has been a lot of interest in studying point interactions and δ-shell interactions in non-relativistic quantum mechanics; see for example [1], [2] and references therein. The motivation for considering such interactions is basically due to the fact that point interactions as well as δ-shell interactions have been employed to obtain mathematically tractable models for many physical problems; see for example [3] and [4], and there has therefore been a need to provide a rigorous mathematical framework for such interactions.

The Hamiltonians describing point and δ-shell interactions are exactly solvable in the sense that all questions about bound states, resonances, scattering, etc., may be answered by explicit formulas. In this paper we study the formal quantum Hamiltonian:

$$H = -\Delta + \frac{\gamma}{|x|} + \frac{\beta^2 - \frac{1}{4}}{|x|^2} + \alpha\,\delta(|x| - R) + \eta\,\delta(x)\,;$$
$$x \in \mathbb{R}^n - \{0\}\,;\quad 0 < \beta < 1\,;\quad \alpha, \gamma, \eta \in \mathbb{R}\,;\quad n \geq 2\,. \tag{1.1}$$

The presence of the strongly singular interaction $V(x) = \frac{\beta^2 - 1/4}{|x|^2}$ in (1.1) enables us to generalize some of the results of Refs. [1] and [2].

It is instructive to examine the different cases corresponding to values of α, β and η.

Case 1: $\beta = \frac{1}{2}$, $\alpha = 0$, $\eta \neq 0$.
This corresponds to a system of charged particles with a zero-range interaction and has been discussed in Refs.[1] and [5].

Case 2: $\beta \neq \frac{1}{2}$, $\alpha = 0$, $\eta \neq 0$.
This has been studied in [6]. It corresponds to a system of charged particles with the strongly singular interaction $V(x) = \frac{\beta^2 - 1/4}{|x|^2}$ plus an additional zero range interaction.

Cases 3 and 4: $\beta = \frac{1}{2}$, $\alpha \neq 0$, $\eta = 0$ (resp $\beta = \frac{1}{2}$, $\alpha \neq 0$, $\eta \neq 0$).
These correspond to a system of charged particles with a δ-shell interaction (resp. with a δ-shell interaction plus an additional zero-range interaction) and have been discussed in [2].

The paper is organized as follows: In Section 2 we give the precise mathematical meaning to the formal expression (1.1) in terms of self-adjoint extensions of densely defined symmetric operators. In Section 3 we derive the resolvent of (1.1) and discuss its spectral properties. Finally, in Section 4 we obtain phase shifts and low energy parameters corresponding to (1.1)

2. SELF-ADJOINT EXTENSIONS

We consider in arbitrary dimensions $n \geq 2$, a quantum-mechanical system of charged particles moving under the influence of a strongly singular potential $V(x) = \frac{4\beta^2 - (n-2)^2}{4|x|^2}$, $x \in \mathbb{R}^n - \{0\}$, $0 < \beta < 1$, and an additional δ-potential with support on a shell $K(Y,R)$ of center $Y \in \mathbb{R}^n$ and radius R.

The quantum Hamiltonian describing such a system is formally given by:

$$H = H_c + V(x) + \alpha\,\delta(|x| - R), \tag{2.1}$$

where H_c is the Coulomb Hamiltonian:

$$H_c = -\Delta + \frac{\gamma}{|x|} \; ; \quad \mathcal{D}(H_c) = \mathcal{D}(-\Delta) = H^{2,2}(\mathbb{R}^n), \tag{2.2}$$

and $H^{2,2}(\mathbb{R}^n)$ denotes the Sobolev space of indices (2,2) [7].

We will obtain the precise mathematical formulation of the formal expression (2.1) by constructing self-adjoint extensions of the closure of the minimal operator:

$$\dot{H} = H_c + V(x) \tag{2.3}$$

$$\mathcal{D}(\dot{H}) = C_0^\infty\left(\mathbb{R}^n \setminus \partial\overline{K(Y,R)}\right). \tag{2.4}$$

We will restrict our analysis to the case $n = 3$. The generalization to arbitrary dimensions $n \geq 2$, may be obtained for example following [2].

We decompose $L^2(\mathbb{R}^3)$ with respect to angular momenta and introduce the unitary transformation

$$U : L^2((0,\infty); r^2 dr) \longrightarrow L^2((0,\infty), dr);$$
$$(Uf)(r) = r f(r). \tag{2.5}$$

Then, the closure $\bar{\dot{H}}$ of $\dot{H}$ reads [1],[2],[8],[9]:

$$\bar{\dot{H}} = \bigoplus_{\ell=0}^{\infty} U^{-1} \bar{\dot{h}}_\ell \, U \otimes 1 \tag{2.6}$$

$$\bar{\dot{h}}_\ell = -\frac{d^2}{dr^2} + \frac{\gamma}{r} + \frac{\beta^2 + \ell(\ell+1) - \frac{1}{4}}{r^2} \; ; \quad r > 0, \quad \ell \in \mathbb{N}_0 \tag{2.7}$$

$$\mathcal{D}(\bar{\dot{h}}_0) = \Big\{ f \in L^2((0,\infty)) \,\Big|\, f, f' \in AC_{loc}((0,\infty)); \; f(0_+) = f'(0_+) = 0, \; f(R_\pm) = 0; \; -f'' + \frac{\gamma}{r} f + \frac{\beta^2 - \frac{1}{4}}{r^2} f \in L^2((0,\infty)) \Big\}, \tag{2.8}$$

$$\mathcal{D}(\bar{\dot{h}}_\ell) = \Big\{ f \in L^2((0,\infty)) \,\Big|\, f, f' \in AC_{loc}((0,\infty)); \; f(R_\pm) = 0, \; -f'' + \frac{\gamma}{r} f' + \frac{\beta^2 + \ell(\ell+1)}{r^2} f \in L^2((0,\infty)) \Big\}; \quad \ell \geq 1, \tag{2.9}$$

where $AC_{loc}(a,b)$ is the set of locally absolutely continuous functions on the interval (a,b). The adjoint $\dot{h}_\ell^*$ of $\bar{\dot{h}}_\ell$ reads:

$$\dot{h}_\ell^* = -\frac{d^2}{dr^2} + \frac{\gamma}{r} + \frac{\beta^2 + \ell(\ell+1) - \frac{1}{4}}{r^2} \; ; \tag{2.10}$$

$$\mathcal{D}(\dot{h}_\ell^*) = \Big\{ f \in L^2((0,\infty)) \,\Big|\, f, f' \in AC_{loc}((0,\infty) \setminus \{R\}); \; f(R_-) = f(R_+) = f(R); \; -f'' + \frac{\gamma}{r} f + \frac{\beta^2 + \ell(\ell+1) - \frac{1}{4}}{r^2} f \in L^2((0,\infty)) \Big\}, \quad \ell \geq 0. \tag{2.11}$$

A direct computation shows that the differential equation:

$$\left(\dot{h}_\ell^{*} - k^2\right)\phi_\ell(k,\gamma) = 0\,; \quad \mathrm{Im}\,k > 0 \,, \tag{2.12}$$

admits (up to multiplicative constants) the unique solution: [2], [10], [11]

$$\phi_\ell(k,\gamma,r) = \begin{cases} G_\ell^{(0)}(k,\gamma,R)\,F_\ell^{(0)}(k,\gamma,r)\,; & r \le R \\ F_\ell^{(0)}(k,\gamma,R)\,G_\ell^{(0)}(k,\gamma,r)\,; & r \geq R \end{cases} \tag{2.13}$$

where

$$F_\ell^{(0)}(k,\gamma,r) = r^\lambda e^{ikr}\, {}_1F_1\left(\lambda + \frac{i\gamma}{2k}\,;\, 2\lambda\,;\, -2ikr\right), \tag{2.14}$$

$$G_\ell^{(0)}(k,\gamma,r) = \Gamma(2\lambda)^{-1}\,\Gamma\left(\lambda+\frac{i\gamma}{2k}\right)(-2ikr)^{2\lambda-1}\, r^\lambda e^{ikr}$$
$$U\left(\lambda+\frac{i\gamma}{2k}\,;\,2\lambda\,;\,-2ikr\right), \tag{2.15}$$

$$\lambda = \nu + \frac{1}{2}\,; \quad \nu = \left[\beta^2 + \ell(\ell+1)\right]^{1/2}. \tag{2.16}$$

Here ${}_1F_1(a;\, b;\, z)$ and $U(a;\, b;\, z)$ denote the confluent hypergeometric functions and $\Gamma(z)$ is the gamma function [10]. In the short range case ($\gamma = 0$), the solution (2.13) becomes:

$$\phi_\ell(k,r) = \phi_\ell(k,0,r) = \begin{cases} \frac{i\pi}{2} R^{1/2} H_\nu^{(1)}(kr)\, r^{1/2} J_\nu(kr)\,; & r \le R\,, \\ \frac{i\pi}{2} R^{1/2} J_\nu(kR)\, r^{1/2} H_\nu^{(1)}(kr)\,; & r \geq R\,, \end{cases} \tag{2.17}$$

where $J_\nu(z)$ is the Bessel function of order ν and $H_\nu^{(1)}(z)$ is the Hankel function of first kind, of order ν [10].

It follows from the above discussion that the operator $\bar{\dot{h}}_\ell$ has deficiency indices (1,1). Therefore, all self-adjoint extensions of $\bar{\dot{h}}_\ell$ are given by a one parameter family $h_{\ell,\theta}$ of self-adjoint operators defined by [8],[12] and [13]:

$$D(h_{\ell,\theta}) = \left\{ g + c\,\psi_+ + c\,e^{i\theta}\psi_- \;\middle|\; g \in D(\bar{\dot{h}}_\ell)\,;\; c \in \mathbb{C} \right\} \tag{2.18}$$

$$h_{\ell,\theta}\left(g + c\,\psi_+ + c\,e^{i\theta}\psi_-\right) = \bar{\dot{h}}_\ell\, g + i\,c\,\psi_+ - i\,c\,e^{i\theta}\psi_-\,; \quad \theta \in [0,2\pi] \tag{2.19}$$

where

$$\psi_\pm(r) = \phi_\ell\left(\sqrt{\pm i},\, \gamma,\, r\right); \quad \mathrm{Im}\sqrt{\pm i} > 0\,. \tag{2.20}$$

A straightforward computation shows that:

$$\left(g + c\,\psi_+ + c\,e^{i\theta}\psi_-\right)'(R_+) - \left(g + c\,\psi_+ + c\,e^{i\theta}\psi_-\right)'(R_-) = -c\left(1+e^{i\theta}\right) \tag{2.21}$$

If we impose this jump to be equal to

$$\alpha_\ell\left[g(R) + c\,\psi_+(R) + c\,e^{i\theta}\psi_-(R)\right] \tag{2.22}$$

where

$$\alpha_\ell = -\frac{1+e^{i\theta}}{\psi_+(R) + e^{i\theta}\psi_-(R)} \tag{2.23}$$

then we obtain the following:

Proposition

All self-adjoint extensions of $\bar{\dot{H}}$ are given by:

$$H_{\ell,\alpha_\ell,\beta,\gamma} = \bigoplus_{\ell=0}^{\infty} U^{-1} h_{\ell,\alpha_\ell,\beta,\gamma}\, U \otimes 1\,, \tag{2.24}$$

where

$$h_{\ell,\alpha_\ell,\beta,\gamma} = -\frac{d^2}{dr^2} + \frac{\gamma}{r} + \frac{\beta^2 + \ell(\ell+1) - \frac{1}{4}}{r^2}\,; \quad r \neq R,\; -\infty < \alpha_\ell \leq \infty \tag{2.25}$$

$$\mathcal{D}(h_{0,\alpha_0,\beta,\gamma}) = \{ g \in L^2((0,\infty)) \mid g, g' \in AC_{loc}((0,\infty)\setminus\{R\});$$
$$g(R_+) = g(R_-) = g(R); \quad g(0_+) = 0;$$
$$g'(R_+) - g'(R_-) = \alpha_0 \, g(R);$$
$$-g'' + \frac{\gamma}{r} g + \frac{\beta^2 - \frac{1}{4}}{r^2} g \in L^2((0,\infty)) \}, \tag{2.26}$$

$$\mathcal{D}(h_{\ell,\alpha_\ell,\beta,\gamma}) = \{ g \in L^2((0,\infty)) \mid g, g' \in AC_{loc}((0,\infty)\setminus\{R\});$$
$$g(R_+) = g(R_-); \quad g'(R_+) - g'(R_-) = \alpha_\ell \, g(R);$$
$$-g'' + \frac{\gamma}{r} g' + \frac{\beta^2 + \ell(\ell+1) - \frac{1}{4}}{r^2} g \in L^2((0,\infty)) \};$$
$$\ell \geq 1 . \tag{2.27}$$

The case $\alpha_\ell = 0$ leads to the operator:

$$H_{\ell,0,\beta,\gamma} = \bigoplus_{\ell=0}^{\infty} U^{-1} h_{\ell,0,\beta,\gamma} \, U \otimes 1 \tag{2.28}$$

$$h_{\ell,0,\beta,\gamma} = -\frac{d^2}{dr^2} + \frac{\gamma}{r} + \frac{\beta^2 + \ell(\ell+1) - \frac{1}{4}}{r^2} \tag{2.29}$$

$$\mathcal{D}(h_{0,0,\beta,\gamma}) = \{ g \in L^2((0,\infty)) \mid g, g' \in AC_{loc}((0,\infty)); \; g(0_+) = 0;$$
$$-g'' + \frac{\gamma}{r} g + \frac{\beta^2 - \frac{1}{4}}{r^2} g \in L^2((0,\infty)) \} \tag{2.30}$$

$$\mathcal{D}(h_{\ell,0,\beta,\gamma}) = \{ g \in L^2((0,\infty)) \mid g, g' \in AC_{loc}((0,\infty));$$
$$-g'' + \frac{\gamma}{r} g + \frac{\beta^2 + \ell(\ell+1) - \frac{1}{4}}{r^2} g \in L^2((0,\infty)) \}; \quad \ell \geq 1 . \tag{2.31}$$

Actually $h_{0,0,\beta,\gamma}$ is the Friedrichs extension of $\bar{\dot{h}}_0$. If $\alpha_\ell = \infty$, then $h_{\ell,\infty,\beta,\gamma}$ is the self-adjoint extension of $\bar{\dot{h}}_\ell$ with a Dirichlet boundary condition at $r = R$ (i.e. $g(R_\pm) = 0$).

If $\beta > 1$, then $\bar{\dot{h}}_\ell$ is already self adjoint; see e.g. [14], and hence no additional δ-shell interaction exists. If $\beta \leq 0$ then the self-adjoint extensions of $\bar{\dot{h}}_\ell$ are not bounded from below, and therefore they do not correspond to physical situations.

If $|\alpha_\ell| < \infty$, then $H_{\ell,\alpha_\ell,\beta,\gamma}$ is the precise mathematical formulation of the formal expression (2.1).

In order to show this, one considers for each fixed angular momentum ℓ, the radial Schrödinger equation:

$$\left[-\frac{d^2}{dr^2} + \frac{\gamma}{r} + \frac{\beta^2 + \ell(\ell+1) - \frac{1}{4}}{r^2} + \alpha_\ell \, \delta(r-R) \right] \Psi_\ell(E,\gamma) = E \, \Psi_\ell(E,\gamma) \tag{2.32}$$

Integrating this equation formally from $R - \varepsilon$ to $R + \varepsilon$ and taking the limit $\varepsilon \to 0$, we get precisely the boundary condition (2.27).

The whole analysis performed so far may be extended to include an additional zero-range interaction; in other words one can use the techniques of self-adjoint extensions of densely defined symmetric operators in order to give a precise mathematical meaning to the formal expression:

$$H = -\Delta + \frac{\gamma}{|x|} + \frac{\beta^2 - \frac{1}{4}}{|x|^2} + \alpha \, \delta(|x| - R) + \eta \, \delta(x);$$
$$x \in \mathbb{R}^3 - \{0\} . \tag{2.33}$$

Indeed, following e.g. [1], [2], [15], one can show that after decomposing $L^2(\mathbb{R}^3)$ into angular momenta and introducing the unitary transformation from $L^2((0,\infty), r^2 dr)$ onto $L^2((0,\infty), dr)$, the mathematical interpretation of the formal expression (2.33) is given by the following family of self-adjoint operators:

$$H_{\ell,\alpha_\ell,\beta,\gamma,\eta_\ell} = \bigoplus_{\ell=0}^{\infty} U^{-1} h_{\ell,\alpha_\ell,\beta,\gamma,\eta_\ell} \, U \otimes 1 \tag{2.34}$$

where:

$$h_{\ell,\alpha_\ell,\beta,\gamma,\eta_\ell} = -\frac{d^2}{dr^2} + \frac{\gamma}{r} + \frac{\beta^2+\ell(\ell+1)-\frac{1}{4}}{r^2}\ ; \quad r \neq R\ ,$$

$$-\infty < \alpha_\ell, \eta_\ell < \infty\ , \quad 0 < \beta < 1 \tag{2.35}$$

$$\begin{aligned} D(h_{0,\alpha_0,\beta,\gamma,\eta_0}) = \Big\{ f \in L^2((0,\infty)) \,\Big|\, & f, f' \in AC_{loc}((0,\infty)\setminus\{R\})\,; \\ & f(R_+) = f(R_-) = f(R)\,; \ -4\pi\eta_0 f_0 + f_1 = 0\,, \\ & f'(R_+) - f'(R_-) = \alpha_0 f(R)\,; \ -f'' + \frac{\gamma}{r} f + \frac{\beta^2 - \frac{1}{4}}{r^2} f \in L^2((0,\infty)) \Big\} \end{aligned} \tag{2.36}$$

$$\begin{aligned} D(h_{\ell,\alpha_\ell,\beta,\gamma,\eta_\ell}) = \Big\{ f \in L^2((0,\infty)) \,\Big|\, & f, f' \in AC_{loc}((0,\infty)\setminus\{R\})\,; \\ & f(R_+) = f(R_-)\,; \ f'(R_+) - f'(R_-) = \alpha_\ell f(R)\,; \\ & -g'' + \frac{\gamma}{r} g + \frac{\beta^2+\ell(\ell+1)-\frac{1}{4}}{r^2} g \in L^2((0,\infty)) \Big\}\,; \quad \ell \geq 1\,, \end{aligned} \tag{2.37}$$

where f_0 and f_1 are defined by

$$f_0 = \lim_{r\to 0_+} r^{\beta-\frac{1}{2}} f(r)\ , \tag{2.38}$$

$$\begin{aligned} f_1 = \lim_{r\to 0_+} r^{-\beta-\frac{1}{2}} \Big\{ f(r) - f_0 \Big[& r^{-\beta+\frac{1}{2}} + \gamma^{2\beta}\left(\frac{\Gamma(-2\beta)}{\Gamma(2\beta)} + 1 - 2\mathcal{C}\right) r^{\beta+\frac{1}{2}} \\ & - \gamma(2\beta-1)^{-1} r^{-\beta+\frac{3}{2}} \Big] \Big\}\ , \quad \text{if } \beta \neq \tfrac{1}{2}\ ; \end{aligned} \tag{2.39}$$

(where $\mathcal{C}$ denotes Euler's constant [10]), and by

$$f_0 = f(0_+)\ , \tag{2.40}$$

$$f_1 = \lim_{r\to 0_+} r^{-1}\left\{ f(r) - f(0_+)\left[1 + \gamma r \ln(\gamma r)\right]\right\}\ , \quad \text{if } \beta = \tfrac{1}{2}\ . \tag{2.41}$$

We note that for $\ell \geq 1$, $h_{\ell,\alpha_\ell,\beta,\gamma,\eta_\ell} = h_{\ell,\alpha_\ell,\beta,\gamma}$.

This illustrates the fact that point interactions exist only in the s-wave, i.e. in the subspace of angular momentum zero; see e.g. [16]. Further results concerning the Hamiltonian $H_{\ell,\alpha_\ell,\beta,\gamma,\eta_\ell}$, as well as nonlocal interactions corresponding to the formal expression (2.33), will be reported elsewhere [17]

3. RESOLVENT AND SPECTRAL PROPERTIES OF $h_{\ell,\alpha_\ell,\beta,\gamma}$

Proposition 3.1

The resolvent of $h_{\ell,\alpha_\ell,\beta,\gamma}$ is given by

$$\left(h_{\ell,\alpha_\ell,\beta,\gamma} - k^2\right)^{-1} = \left(h_{\ell,0,\beta,\gamma} - k^2\right)^{-1} - \frac{\alpha_\ell}{1 + \alpha_\ell\, g_{\ell,0,\beta,\gamma}(k,R,R)} \left|\phi_\ell(k,\gamma)\right\rangle\left\langle\phi_\ell(-\bar{k},\gamma)\right|\ ;$$

$$k^2 \in \rho\left(h_{\ell,\alpha_\ell,\beta,\gamma}\right)\ ; \quad \mathrm{Im}\, k > 0\ , \quad -\infty < \alpha_\ell \leq \infty\ , \tag{3.1}$$

where the function $\phi_\ell(k,\gamma,r)$, defined by (2.13)-(2.16) has been chosen in such a way that it coincides with $g_{\ell,0,\beta,\gamma}(k,R,r)$, the Green function corresponding to $h_{\ell,0,\beta,\gamma}$.

Proof

From Krein's formula [12], it follows that:

$$\left(h_{\ell,\alpha_\ell,\beta,\gamma} - k^2\right)^{-1} - \left(h_{\ell,0,\beta,\gamma} - k^2\right)^{-1} = \mu_{\ell,\beta,\gamma}(k) \left|\phi_\ell(k,\gamma)\right\rangle\left\langle\phi_\ell(-\bar{k},\gamma)\right|\ .$$

In order to determine the factor $\mu_{\ell,\beta,\gamma}$ we proceed as follows. Let $f \in L^2((0,\infty))$ and define;

$$g_{\ell,\beta,\gamma}(r) = \left(\left(h_{\ell,\alpha_\ell,\beta,\gamma} - k^2\right)^{-1} f_\ell\right)(r)$$

$$= \int_0^\infty dr'\, g_{\ell,0,\beta,\gamma}(k,r,r')\, f_\ell(r')$$

$$+ \mu_{\ell,\beta,\gamma}(k)\, \phi_\ell(k,\gamma,r) \int_0^\infty dr'\, \phi_\ell(k,\gamma,r')\, f_\ell(r') . \qquad (3.2)$$

Since $g_{\ell,\beta,\gamma}$ is contained in $D(h_{\ell,\alpha_\ell,\beta,\gamma})$, it follows from the definition of $h_{\ell,\alpha_\ell,\beta,\gamma}$ that $g_{\ell,\beta,\gamma}$ should satisfy the boundary conditions:

(i) $$g_{\ell,\beta,\gamma} \in AC_{loc}((0,\infty)) \qquad (3.3)$$

(ii) $$g'_{\ell,\beta,\gamma}(R_+) - g'_{\ell,\beta,\gamma}(R_-) = \alpha_\ell\, g_{\ell,\beta,\gamma}(R) \qquad (3.4)$$

(iii) $$\left(h_{\ell,\alpha_\ell,\beta,\gamma} - k^2\right) g_{\ell,\beta,\gamma}(r) = f_\ell(r) \quad ; \quad r>0,\ r \neq R . \qquad (3.5)$$

Imposing the boundary conditions (3.3)-(3.5) we obtain $\mu_{\ell,\beta,\gamma}$.

Remark 3.2

Following for example [18] one may define bound states (resp resonances) of $h_{\ell,\alpha_\ell,\beta,\gamma}$ as zeros of the equation:

$$1 + \alpha_\ell\, g_{\ell,0,\beta,\gamma}(k,R,R) = 0 \qquad (3.6)$$

lying on the positive part of the imaginary axis (resp in Im $k \leqslant 0$).

Equivalently, bound states of $h_{\ell,\alpha_\ell,\beta,\gamma}$ may be obtained by looking for square integrable solutions of the Schrödinger equation:

$$\left(h_{\ell,\alpha_\ell,\beta,\gamma} - E\right)\Psi_\ell(E,\gamma=0) ; \quad E<0 ,$$
$$\Psi_\ell(E,\gamma) \in D\left(h_{\ell,\alpha_\ell,\beta,\gamma}\right) . \qquad (3.7)$$

For $r \neq R$, regular and irregular solutions of (3.7) are respectively given by [11]:

$$F_\ell^{(0)}(E,\gamma,r) = r^\lambda \exp(-\sqrt{-E}\, r)\, {}_1F_1\left(\lambda + \frac{\gamma}{2\sqrt{-E}} ; 2\lambda ; 2\sqrt{-E}\, r\right) \qquad (3.8)$$

$$G_\ell^{(0)}(E,\gamma,r) = \Gamma(2\lambda)^{-1}\, \Gamma\left(\lambda + \frac{\gamma}{2\sqrt{-E}}\right)(-4E)^{\lambda-\frac{1}{2}}\, r^\lambda \exp(-\sqrt{-E}\, r)$$

$$U\left(\lambda + \frac{\gamma}{2\sqrt{-E}} ; 2\lambda ; 2\sqrt{-E}\, r\right) ; \qquad (3.9)$$

$$\lambda = \nu + \frac{1}{2} , \quad \nu = \left[\beta^2 + \ell(\ell+1)\right]^{1/2} . \qquad (3.10)$$

The behaviour of these solutions at the point $r = R$ is obtained using the boundary conditions:

$$\Psi_\ell(E,\gamma,R_-) = \Psi_\ell(E,\gamma,R_+) = \Psi_\ell(E,\gamma,R) \qquad (3.11)$$

$$\Psi'_\ell(E,\gamma,R_+) - \Psi'_\ell(E,\gamma,R_-) = \alpha_\ell\, \Psi_\ell(E,\gamma,R) \qquad (3.12)$$

A straightforward calculation gives the bound state equation for $h_{\ell,\alpha_\ell,\beta,\gamma}$. For the short range case ($\gamma = 0$) we get:

$$\frac{1}{R} + \alpha_\ell\, K_\nu(\sqrt{-E}\, R)\, I_\nu(\sqrt{-E}\, R) = 0 . \qquad (3.13)$$

This equation may be solved graphically and we obtain the following result: if $|\alpha_\ell| < \infty$, then for each fixed angular momentum ℓ, $h_{\ell,\alpha_\ell,\beta,0}$ has exactly one bound state if and only if $\alpha_\ell < -\frac{2\nu}{R}$. If $\alpha_\ell = \infty$ then obviously equation (3.13) has no solution. For the pure δ-shell interaction (i.e. $\beta = \frac{1}{2}$), a similar condition has been obtained in Ref. 2.

The spectrum of $h_{\ell,\alpha_\ell,\beta,\gamma}$ is given by the following: (here we discuss the short range case, i.e. $\gamma \equiv 0$)

<u>Proposition</u>

Let $-\infty < \alpha_\ell \leq \infty$, $\ell \in \mathbb{N}_0$, $0 < \beta < 1$. Then, the essential spectrum $\sigma_{ess}(h_{\ell,\alpha_\ell,\beta,0})$ is purely absolutely continuous and covers the non-negative real axis:

$$\sigma_{ess}\left(h_{\ell,\alpha_\ell,\beta,0}\right) = \sigma_{ac}\left(h_{\ell,\alpha_\ell,\beta,0}\right) = [0,\infty) \ . \tag{3.14}$$

If $\alpha_\ell < -\frac{2\nu}{R}$, then the point spectrum $\sigma_p(h_{\ell,\alpha_\ell,\beta,0})$ contains exactly one point.
If $\alpha_\ell \geq -\frac{2\nu}{R}$ or $\alpha_\ell = \infty$, then $\sigma_p(h_{\ell,\alpha_\ell,\beta,0}) = \emptyset$.

<u>Proof</u>

The relation (3.14) follows from Weyl's theorem [19, p. 112], since the difference

$$\left(h_{\ell,\alpha_\ell,\beta,0} - k^2\right)^{-1} - \left(h_{\ell,0,\beta,0} - k^2\right)^{-1} ; \quad k^2 \in \rho\left(h_{\ell,\alpha_\ell,\beta,0}\right), \ \mathrm{Im}\, k > 0 \ ,$$

$0 < \beta < 1$, is a rank one operator. The structure of the point spectrum of $h_{\ell,\alpha_\ell,\beta,0}$ is determined by (3.13).

4. PHASE SHIFTS AND LOW ENERGY PARAMETERS

Phase shifts of $h_{\ell,\alpha_\ell,\beta,\gamma}$ are obtained by solving the equation:

$$h_{\ell,\alpha_\ell,\beta,\gamma}\ \psi_\ell(k,r) = k^2\ \psi_\ell(k,r) \ ; \quad k, r > 0 \ , \tag{4.1}$$

and studying the asymptotic behaviour of the solution $\psi_\ell(k,\gamma,r)$ for large values of r. For simplicity we restrict ourselves to the short range case (the generalization to $\gamma \neq 0$ is straightforward). In this case, the solution $\psi_\ell(k,r)$ of (4.1) may be written in the form:

$$\psi_\ell(k,r) = \begin{cases} r^{1/2} J_\nu(kr) \ ; & 0 < r \leq R \\ C_1(k,R,\alpha_\ell,\beta)\, r^{1/2} J_\nu(kr) & \\ \quad + C_2(k,R,\alpha_\ell,\beta)\, r^{1/2} Y_\nu(kr) \ ; & r \geq R \ , \end{cases} \tag{4.2}$$

where $J_\nu(z)$ and $Y_\nu(z)$ are, respectively Bessel and Neumann functions [10]. The constants $C_1(k,R,\alpha_\ell,\beta)$ and $C_2(k,R,\alpha_\ell,\beta)$ are obtained from the boundary conditions:

$$\psi_\ell(k,R_-) = \psi_\ell(k,R_+) = \psi_\ell(k,R) \tag{4.3}$$

$$\psi_\ell'(k,R_+) - \psi_\ell'(k,R_-) = \alpha_\ell\, \psi_\ell(k,R) \tag{4.4}$$

since $\psi_\ell(k,r)$ is contained in the domain $D(h_{\ell,\alpha_\ell,\beta,0})$.

In order to determine the asymptotic behaviour of $\psi_\ell(k,r)$ when $r \to \infty$, one uses the asymptotic expansions of Bessel and Neumann functions. After a straightforward calculation we get that the phase shift $\delta_\ell(k,R,\alpha_\ell,\beta)$ of $h_{\ell,\alpha_\ell,\beta,0}$ is given by:

$$\delta_\ell(k,R,\alpha_\ell,\beta) = \mathrm{arctg}\ \frac{-\frac{\pi}{2} R\alpha_\ell J_\nu^2(kR)}{1 - \frac{\pi}{2} R\alpha_\ell Y_\nu(kR) J_\nu(kR)} \ , \tag{4.5}$$

The phase shift $\delta_\ell(k,R,\alpha_\ell,\beta)$ is related to the effective range function (ERF) $K_{\ell,\alpha_\ell,\beta}(k^2)$ by:

$$K_{\ell,\alpha_\ell,\beta}(k^2) = k^{2\ell+1} \cot g\ \delta_\ell(k,R,\alpha_\ell,\beta) \ . \tag{4.6}$$

The coefficients of the expansion of the ERF:

$$K_{\ell,\alpha_\ell,\beta}(k^2) = -\frac{1}{a_{\ell,\alpha_\ell,\beta}} + \frac{1}{2} r_{\ell,\alpha_\ell,\beta}\, k^2 - P_{\ell,\alpha_\ell,\beta}\, r^3_{\ell,\alpha_\ell,\beta}\, k^4$$
$$+ Q_{\ell,\alpha_\ell,\beta}\, r^5_{\ell,\alpha_\ell,\beta}\, k^6 - \dots \tag{4.7}$$

are related to the low energy-scattering parameters: $a_{\ell,\alpha_\ell,\beta}$ is the scattering length, $r_{\ell,\alpha_\ell,\beta}$ the effective range, and $P_{\ell,\alpha_\ell,\beta}$ and $Q_{\ell,\alpha_\ell,\beta}$ are the shape parameters.

For the strongly singular interaction $V(r) = \frac{\beta^2-1/4}{r^2}$ plus a δ-shell interaction, we obtain:

$$K_{l,\alpha_l,\beta}(k^2) = \frac{-k^{2l+1}\left[1-\frac{\pi}{2}R\alpha_l Y_\nu(kR)J_\nu(kR)\right]}{\frac{\pi}{2}R\alpha_l J_\nu^2(kR)}\,. \tag{4.8}$$

Expanding the right-hand side of (4.8) (we use the expansions of Bessel and Neumann functions) and comparing with (4.7) we obtain:

$$\begin{aligned}
K_{l,\alpha_l,\beta}(k^2) = &-\left[(2\nu)!!\right]^2 R^{-2\nu}\left(\frac{1}{\alpha_l R}+\frac{1}{2\nu}\right) \\
&-\left[(2\nu)!!\right]^2 R^{-2(\nu-1)}\left(\frac{1}{\alpha_l R}+\frac{1}{2(\nu-1)}\right)\frac{1}{2(\nu+1)}k^2 \\
&-\left[(2\nu)!!\right]^2 R^{-2(\nu-2)}\left(\frac{1}{\alpha_l R}+\frac{1}{2(\nu-2)}\right)\frac{\nu+\frac{5}{2}}{[2(\nu+1)]^2[2(\nu+2)]}k^4 \\
&-\left[(2\nu)!!\right]^2 R^{-2(\nu-3)}\left(\frac{1}{\alpha_l R}+\frac{1}{2(\nu-3)}\right)\frac{2\nu^2+13\nu+23}{3[2(\nu+1)]^3[2(\nu+1)][2(\nu+3)]}k^6 \\
&+ \cdots
\end{aligned}$$

For the pure δ-shell interaction (i.e. $\beta = \frac{1}{2}$), similar relations have been derived in Ref. 20.

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