IC/85/160

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

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INTERNATIONAL
ATOMIC ENERGY
AGENCY

UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION

1985 MIRAMARE-TRIESTE

IC/85/160

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
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ABSTRACT

We explored the θ-structures in bosonic string theories which are similar to those in gauge field theories. The θ-structure of string is due to the multiply connected spatial compact subspace of space-time. The work of this paper shows that there is an energy band $E(\theta)$ in the string theory and one may move the tachyon out in theory by choosing some proper θ parameters.

MIRAMARE – TRIESTE
September 1985

* To be submitted for publication.

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Isham et al. [1] pointed out that there may exist a set of inequivalent quantizations in quantum theories providing the topologies of space M and field configuration space Q are non-trivial. If we put q as a field collection, then the propagator from q to q' is

$$K_\theta(q',q) = \sum_{[p]} \theta([p])\, K_{[p]}(q',q) \tag{1}$$

where [p] denotes the collection of paths connecting q and q' which are homotopic to a particular path p. $\theta([p])$ is a complex factor depending on [p]. Mathematically rather than physically, one can distinguish the [p]s by choosing a base point q_0 in Q and joining it to q and q' by ω and ω', then $[\omega'^{-1}p\omega]$ forms a loop passing through q_0. We write $[\gamma] = [\omega'^{-1}p\omega]$, thus Eq.(1) can be written as

$$K_\theta(q',q) = \sum_{[\gamma]} \theta(\gamma)\, K_{[\gamma]}(q',q) \tag{2}$$

$\theta(\gamma)$ can be determined by demanding K_θ changing a complex factor when ω and ω' are changed, so $\theta(\gamma)$ is an isomorphic map $\pi_1(Q) \to U(1)$. Hence, the set of the inequivalent quantization is

$$\mathrm{Hom}\,[\,\pi_1(Q),\, U(1)\,] \tag{3}$$

In bosonic string theories, there is only a consistent theory in which the dimension of space-time is 26, i.e. Veneziano model [2),3)]. A more realistic theory is the 10-d superstring model of Green and Schwarz [4)]. To obtain a realistic correspondence in nature, all these theories have to be compactified. The space-time may be of the form of $M_4 \times M_{d-4}$, where M_4 is Minkowskian and M_{d-4} is a compact spatial space. We suppose that the internal space be multiply connected, $\pi_1(M_{d-4}) \neq e$. String is parametrized by $X^\mu(\sigma,\tau)$, where σ and τ are spatial and time-like parameters, respectively. The propagator between two configurations $X^\mu(\sigma)$ and $X^{\mu'}(\sigma)$ is $K(X^{\mu'}(\sigma), X^\mu(\sigma))$, the integral in Eq.(1) must be summed over all paths $X^\mu(\sigma,\tau)$ joining $X^\mu(\sigma)$ and $X^{\mu'}(\sigma)$. Paths are homotopic to each other only if the trajectories of $\sigma = 0$ of them are homotopic to each other (due to continuity), therefore all paths joining $X^\mu(\sigma)$ and $X^{\mu'}(\sigma)$ can be classified by the fundamental group of M_{d-4}, i.e. $\pi_1(M_{d-4})$. We find finally that the set of the quantizations of string is

$$\mathrm{Hom}[\pi_1(M_{d-4}),\ U(1)] \tag{4}$$

For definitiveness, we consider the VM string and suppose that the space-time takes the form of $M_{D-d} \times T^d$, where M_{D-d} is the D-d dimensional Minkowskian space-time, T^d is d-dimensional torus, hence $\pi_1(T^d) = Z^d$ and

$$\mathrm{Hom}[\pi_1(T^d),\ U(1)] = U(1)^d \tag{5}$$

We can regain Eq.(5) as the following: take the vacuum state as $|X^\mu(\sigma)\rangle$, we separate co-ordinates into two parts, X^i, $i = 1 \sim D\text{-}d$ and X^a, $a = D\text{-}d\text{+}1$ — D, then $|X^\mu(\sigma)\rangle = |X^i, X^a\rangle$. The radii of X^as are R_a, because of periodicity, $|X^i, X^a + 2\pi R_a\rangle$ must be different from $|X^i, X^a\rangle$ by a complex factor, that is

$$|X^i,\ X^a + 2\pi R_a\rangle = e^{i\theta_a}\,|X^i,\ X^a\rangle \tag{6}$$

It is obvious that θ_a are parameters of $U(1)^d$. We can define vacuum states $|n_a\rangle$ just like that in gauge theories, by demanding that the end of string $X^a(0)$ takes the values in $[2\pi n_a R_a,\ 2\pi(n_a+1)R_a]$. Define the translation operators T_a, T_a : $X^a \to X^a + 2\pi R_a$, then

$$T_a\,|n_a\rangle = |n_a + 1\rangle \tag{7}$$

The vacuum state in Eq.(6) can be defined as

$$|\theta\rangle = \sum_{[n_a]} e^{i\sum_a n_a\theta_a} \prod_a |n_a\rangle \tag{8}$$

and Eq.(6) can be rewritten as

$$T_a\,|\theta\rangle = e^{-in_a\theta_a}\,|\theta\rangle \tag{9}$$

The amplitudes between vacuum states $|[n_a]\rangle$ are

$$\langle [m_a]|\,e^{-H\tau}\,|[n_a]\rangle \underset{\tau\to\infty}{=} \int d[X^i]\,d[X^a]_{[m_a-n_a]}\, e^{-S}$$
$$|[n_a]\rangle = \prod_a |n_a\rangle \tag{10}$$

where S is Euclidean action of string, and $d[X^a]_{[m_a-n_a]}$ denotes the measure of all configurations of $X^a(\sigma,\tau)$ in which the trajectories $X^a(\sigma,\tau)$ of fixed σ in the interval of $\tau = [-\infty,+\infty]$ form loops with winding number $m_a - n_a$ in the subspace X^a. Using Eq.(10), we find again the formula similar to Eq.(2)

$$\langle\theta'|\,e^{-H\tau}\,|\theta\rangle \underset{\tau\to\infty}{=} \delta(\theta'-\theta)\, I(\theta)$$
$$I(\theta) = \sum_{[n_a]} \exp(-i\textstyle\sum n_a\theta_a) \displaystyle\int d[X^i]\,d[X^a]_{[n_a]}\, e^{-S} \tag{11}$$

There is a unique integral expression of $n_a\theta_a$

$$n_a\theta_a = \frac{1}{2\pi^2 R_a}\int d\tau\, d\sigma\, \partial_\tau X^a(\sigma,\tau)_{[n_a]} \tag{12}$$

where the integral is total divergence, just like $\frac{1}{32\pi^2}\, {}^*F^a_{\mu\nu}F^{a\mu\nu}$ in gauge theories [5]. Therefore, the amplitudes between θ vacuum can be written as:

$$\langle\theta'|\,e^{-H\tau}\,|\theta\rangle \underset{\tau\to\infty}{=} \int d[X^\mu]\, e^{-S'}$$
$$S' = S + i\sum_a \frac{\theta_a}{2\pi^2 R_a}\int d\tau\, d\sigma\, \dot{X}^a \tag{13}$$

The Lorentzian action derived from Eq.(13) is

$$S_L' = S_L - \sum_a \frac{\theta_a}{2\pi^2 R_a}\int d\tau\, d\sigma\, \dot{X}^a \tag{14}$$

If we take the orthogonal gauge, then

$$S_L = \frac{1}{4\pi\alpha'}\int [\dot{X}^\mu \dot{X}_\mu - X^{\mu\prime} X'_\mu]\, d\tau\, d\sigma - \sum \frac{\theta_a}{2\pi^2 R_a}\int \dot{X}^a\, d\tau\, d\sigma$$

$$\dot{X}^\mu X'_\mu = 0$$

$$\dot{X}^\mu \dot{X}_\mu + X^{\mu\prime} X'_\mu = 0 \tag{15}$$

In Eq.(15) the addition destroys the explicit Lorentz invariance of M_D, the residual invariance is $O(D-d-1) \times U(1)^d$. In the orthogonal gauge, the momentums which are conjugate to X^μ are

$$p^i = \frac{1}{2\pi\alpha'}\dot{X}^i$$

$$p^a = \frac{1}{2\pi\alpha'}\dot{X}^a - \frac{\theta_a}{2\pi^2 R_a} \tag{16}$$

In canonical quantization formalism, the commutators between X^μ and P^ν are

$$[X^\mu(\sigma,\tau), P^\nu(\sigma',\tau')] = i\eta^{\mu\nu}\delta(\sigma-\sigma')\delta(\tau-\tau') \tag{17}$$

The expansions of X^μ and P^ν are

$$X^i = x_0^i + 2\alpha' p^i \tau + \sqrt{2\alpha'}\sum_n \frac{1}{\sqrt{n}}\cos n\sigma\,(a_n^i e^{-in\tau} + h.c.)$$

$$X^a = X_0^a + 2\alpha'\left(p^a + \frac{\theta_a}{2\pi R_a}\right)\tau + \sqrt{2\alpha'}\sum_n \frac{1}{\sqrt{n}}\cos n\sigma\,(a_n^a e^{-in\tau} + h.c.)$$

$$P^\mu = p^\mu/\pi + \sqrt{2\alpha'}\sum_n \sqrt{n}\cos n\sigma\,(-i a_n^\mu e^{-in\tau} + h.c.)$$

$$[x_0^\mu, p^\nu] = i\eta^{\mu\nu}$$

$$[a_n^\mu, a_m^{\nu+}] = \eta^{\mu\nu}\delta_{nm} \tag{18}$$

From the well-known constraint integral, we have

$$H = L_0 = \alpha' p' p' + \sum_n n a_n^+ a_n$$

$$L_n = -i\sqrt{2n\alpha'}\, p' a_n + \sum_{m>0}\sqrt{(m+n)m}\, a_m^+ a_{m+n}$$

$$p' = \begin{cases} p^i \\ p^a + \dfrac{\theta_a}{2\pi R_a} \end{cases} \tag{19}$$

The appearance of corrected P' is due to the fact that not $\frac{\dot{X}^a}{2\pi\alpha'}$ but $\frac{1}{2\pi\alpha'} X^a - \frac{1}{2\pi^2 R_a}\theta^a$ are conjugate momentums. One can check the commutators of L_m, they are just the same as the original ones:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3-m)\delta_{m,-n} \tag{20}$$

In the light cone gauge, we can get the relations between generators of Lorentz transformation $O(D-d-1,1)$

$$[M^{-i}, M^{-j}] = -2i\left(L_0^{tr} - \frac{D^{tr}}{24}\right)M^{ij} - i\left(\frac{D^{tr}}{24} - 1\right)\sum n\,(a_n^{i+}a_n^j - a_n^{j+}a_n^i) \tag{21}$$

where all quantities P^μ in original formulas must be replaced by $P^{\mu\prime}$. In order to get a consistent theory, the dimension of space time must be 26 and all physical states $|\psi\rangle$ must satisfy

$$L_0|\psi\rangle = |\psi\rangle$$

$$L_m|\psi\rangle = 0 \quad (m>0) \tag{22}$$

We obtain the spectrum formula from Eqs.(19) and (22)

$$-\alpha' p^i p_i = \alpha'(\text{mass})^2 = \alpha'\left(p^a + \frac{\theta_a}{2\pi R_a}\right)^2 + \sum_1^\infty n a_n^+ a_n - 1 \tag{23}$$

Noticing that p^a can only take the values M_a/R_a, M_a = integer, we then get the spectrum formula

$$\alpha'(\text{mass})^2 = \alpha'\left(M_a + \frac{\theta_a}{2\pi}\right)^2 / R_a^2 + \sum_1^\infty n a_n^+ a_n - 1 \tag{24}$$

this formula shows that there exists an energy band due to the θ-structure of string theories. The minimal value of mass square M^2_{min} satisfies

$$\alpha' M^2_{min} = \sum_a \alpha' \min M_a'^2 / R_a^2 - 1 \tag{25}$$

where

$$\min M_a'^2 = \min\left(M_a + \theta_a/2\pi\right)^2 = \min\left(\frac{\theta_a^2}{4\pi^2}, \left(1 - \frac{\theta_a}{2\pi}\right)^2\right) \tag{26}$$

If

$$\frac{1}{\alpha'} = \sum \min M_a'^2 / R_a^2 \tag{27}$$

then we move the tachyon state out. If $\alpha' M^2_{min}$ is not exactly zero or -1, then a zero mass state does not exist, this is an interesting result of the θ structure of quantum strings.

ACKNOWLEDGMENTS

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

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