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QUANTIZED FIELDS AND OPERATORS ON A PARTIAL INNER PRODUCT SPACE *

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ABSTRACT

We investigate the connection between the space OpV of all operators on a partial inner product space V and the weak sequential completion of the * algebra $L^+(V^\#)$ of all operators X such that $V^\# \subset D(X) \cap D(X^*)$ and both X and its adjoint X^* leave $V^\#$ invariant. This connection gives a mathematical description of quantized fields in terms of elements of OpV.

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1. INTRODUCTION

The fundamental concept of Wightman axiomatics is the concept of quantized field A(x) at point x, which is usually defined [1] as an operator-valued distribution on some space of test functions (x is the four-dimensional coordinate of space-time).

Let D be a dense linear manifold of a Hilbert space $\mathcal{H}$, and denote by $L^+(D)$ the *-algebra of all operators X such that $D \subset D(X) \cap D(X^*)$, and both X and its adjoint X^* leave D invariant. It has been first proposed by Haag [2] that a quantized field $A(x)$ at any point x should be described in terms of sesquilinear forms on $D \times D$, corresponding to the heuristically defined mapping $(f,g) \mapsto (A(x)f,g)$. This idea has been particularized by Ascoli, Epifanio and Restivo [3] in such a way that these sesquilinear forms may be considered as elements of the weak sequential completion $\widetilde{L^+(D)}^w$ of $L^+(D)$.

On the other hand, it is well known that if V is an arbitrary partial inner product (PIP) space [4], which is quasi complete in its canonical Mackey topology $\tau(V,V^\#)$, then the space $L^+(V^\#)$ is isomorphic to the *-algebra Reg V of all regular operators on V [5].

In this note, after a brief recall in Sec. 2 of basic facts on PIP spaces and operators on them [4-7], we investigate in Sec. 3 the connection between the space OpV of all operators on a PIP space V, and the weak sequential completion of $L^+(V^\#)$. In particular, we show that if V is an arbitrary PIP space, and $\langle V^\#, V\rangle$ is a reflexive dual pair, then $\widetilde{L^+(V^\#)}^w$ is isomorphic to the set of operators on V, which are defined on the whole space V; in other words a quantized field may be described in terms of operators on a PIP space.

2. PIP-SPACES AND OPERATORS ON THEM [4-7]

A PIP-space V is a complex vector space with the following structure:

(i) $\mathcal{J} = \{V_r, r \in I\}$ is a collection of vector subspaces of V which covers V and is an involutive lattice with respect to set intersection, vector sum and lattice involution: $V_r \leftrightarrow V_{\bar r}$.

Besides elements of $\mathcal{J}$, we consider the extreme spaces:

$$V^\# \equiv \bigcap_{r \in I} V_r \qquad \text{and} \qquad V \equiv \bigcup_{r \in I} V_r$$

(ii) A nondegenerate hermitian form $\langle \cdot | \cdot \rangle$ (the partial inner product) is defined on $\bigcup_{r \in I} V_r \times V_{\bar r}$.

(iii) There exists a unique element $0 = \bar{0}$ in I such that $V_0 = V_{\bar{0}} \equiv \mathcal{H}$ is a Hilbert space with respect to $\langle\cdot|\cdot\rangle$.

The nondegeneracy assumption $(V^{\#})^{\perp} = \{0\}$, implies that every pair $\langle V_r, V_{\bar{r}}\rangle$, as well as $\langle V^{\#}, V\rangle$ is a dual pair with respect to the form $\langle\cdot|\cdot\rangle$. We may therefore equip each V_r with its Mackey topology $\tau(V_r, V_{\bar{r}})$ and similarly for $V^{\#}, V$.

An operator A on a PIP space V is a map $D(A) \to V$, where $D(A)$ is the largest union of V_r's such that the restriction of A to any of them is linear and continuous into V.

The set of all operators on V, denoted by OpV is isomorphic to $\mathcal{L}(V^{\#}, V) = \{$linear continuous maps $V^{\#} \to V\}$. Equivalently OpV is isomorphic to $B(V^{\#}, V^{\#}) = \{$separately continuous sesquilinear forms on $V^{\#} \times V^{\#}\}$. Thus, OpV is a vector space. Moreover, OpV carries an involution $A \leftrightarrow A^{x}$ (adjoint of A), but it is not an algebra since the multiplication is not always defined. Such sets are called partial-x-algebras [8].

An operator A on a PIP space V is called regular [5] if $D(A) = D(A^*) = V$. Equivalently, a regular operator is a linear continuous map of $V^{\#}$ into itself, which maps V into itself continuously. The set of all regular operators on V, denoted by Reg V is a *-algebra.

We assume that V is quasi complete in its Mackey topology. Then Reg V is isomorphic to the *-algebra $L^+(V^{\#})$ of all closable operators on $\mathcal{H}$ which, together with their (Hilbertian) adjoint, leave $V^{\#}$ invariant. Actually almost all PIP spaces are quasi complete in their canonical Mackey topologies, the only known exceptions being quite pathological [9].

We will endow OpV with the weak topology defined by the following family of seminorms:

$$A \mapsto |\langle Af, g\rangle|;\ f, g \in V^{\#} \ .$$

On Reg $V = L^+(V^{\#})$ we will consider the weak topology inherited from OpV.

3. OpV AND THE WEAK SEQUENTIAL COMPLETION OF $L^+(V^{\#})$

Following [3] we denote by $S_{V^{\#}}$ the space of all sesquilinear forms on $V^{\#} \times V^{\#}$. It has been proved in [3] that the space $S_{V^{\#}}$ endowed with the topology of pointwise convergence given by the set of seminorms: $\{F \mapsto |F(f,g)|;\ f, g \in V^{\#}\}$

is isomorphic to the weak completion of $L^+(V^{\#})$, i.e. in notations of [3]:

$$S_{V^{\#}} = \widehat{L^+(V^{\#})}^{w} \ .$$

On the other hand, it is clear that $S_{V^{\#}}$ contains the space OpV which is isomorphic to the space $B(V^{\#}[\tau], V^{\#}[\tau])$ of all Mackey separately continuous sesquilinear forms on $V^{\#} \times V^{\#}$.

In what follows, we want to answer the following question: Given a PIP space V, when is OpV isomorphic to the weak sequential completion $\widetilde{L^+(V^{\#})}^{w}$ of $L^+(V^{\#})$? If this isomorphism exists, then the sesquilinear forms which describe quantized fields may be considered as elements of OpV equipped with the weak topology.

In general, for a given PIP space V, whenever OpV is weakly sequentially complete, we have the following relation between OpV and $\widetilde{L^+(V^{\#})}^{w}$:

$$\widetilde{L^+(V^{\#})}^{w} \subseteq \mathrm{OpV} \subseteq \widehat{L^+(V^{\#})}^{w} = S_{V^{\#}}$$

We will show that this relation holds if in particular $\langle V^{\#}, V\rangle$ is a reflexive dual pair. Indeed we have the following:

<u>Proposition 1</u>

Let V be a PIP space. If $\langle V^{\#}, V\rangle$ is a reflexive dual pair, then OpV and $\mathrm{Op}_I V = \{A \in \mathrm{OpV} | D(A) = V\}$ are weakly sequentially complete.

<u>Proof</u>

Let $\{T_n\}$ be a weak Cauchy sequence in OpV, i.e. $\forall f \in V^{\#}$, $\{T_n f\}$ is a weak Cauchy sequence in V. Since $\langle V^{\#}, V\rangle$ is reflexive, $V^{\#}$ and V are quasi complete (i.e. closed bounded sets are complete) with respect to the weak topology and therefore $V^{\#}$ and V are weakly sequentially complete, i.e. $\text{w-}\lim_{n\to\infty} T_n f = Tf \in V$. This shows that T is a map from $V^{\#}$ into V.

To prove that T is continuous one uses the dual mapping theory [10]. ■

<u>Remarks</u>

(i) The same proof applies to $\mathrm{Op}_I V$ and here one can easily show that the limit operator T is defined on the whole space V.
(ii) For Reg $V = L^+(V^{\#})$ one can also try to perform the same proof as in Proposition 1, but in general we do not have that $D(T) = D(T^*) = V$. So, in general $L^+(V^{\#})$ is not weakly sequentially complete. Actually this fits with results of [3] where it is shown that $\widetilde{L^+(V^{\#})}^{w}$ may contain elements which are not operators.

The condition of reflexivity of the dual pair $\langle V^{\#}, V\rangle$ is weak enough to cover most spaces of practical interest, in particular, all spaces of distributions. Typical instances when the dual pair $\langle V^{\#}, V\rangle$ is reflexive are [7]:

- $V^{\#}$ is a Hilbert space; then so is V.
- $V^{\#}$ is a reflexive Banach space; then so is V.
- $V^{\#}$ is a reflexive Frechet space; V is then a reflexive complete (DF)-space [10].
- $V^{\#}$ is a Montel space; then so is V.

Now, given a PIP space V, when is OpV contained in $\widetilde{L^{+}(V^{\#})}^{w}$?

Let $A \in OpV$, $V^{\#}$ separable, e_ν an orthonormal basis in $V^{\#}$ and $P_\nu = |e_\nu\rangle\langle e_\nu|$ the orthogonal projection on e_ν. In the terminology of [5], P_ν is a very regular operator.

Consider $P_\nu A : V^{\#} \to V \to V^{\#}$. This operator leaves $V^{\#}$ invariant, but in general it is not defined on the whole space V. For the adjoint we have: $D((P_\nu A)^{x}) = V$. So, $P_\nu A$ is a regular operator if and only if $D(A) = V$. Assume that $D(A) = V$ and let B_n be the sequence in $L^{+}(V^{\#})$ defined by: $B_n = \sum_{\nu=1}^{n} P_\nu A$. Since $\{e_\nu\}$ is an orthonormal basis, we have: $\forall f \in V^{\#}$, $\lim_{n\to\infty} \sum_{\nu=1}^{n} P_\nu f = f$, and obviously this implies that: $\lim_{n\to\infty} \left\langle \sum_{\nu=1}^{n} P_\nu f, g \right\rangle = \langle f,g\rangle$.

Consequently:

$$\lim_{n\to\infty} \langle B_n f, g\rangle = \lim_{n\to\infty} \left\langle \sum_{\nu=1}^{n} P_\nu A f, g \right\rangle = \lim_{n\to\infty} \sum_{\nu=1}^{n} \langle Af, P_\nu g\rangle$$

$$= \lim_{n\to\infty} \left\langle Af, \sum_{\nu=1}^{n} P_\nu g \right\rangle = \langle Af, g\rangle .$$

Thus, if $D(A) = V$, the arbitrary element $A \in OpV$ is the weak limit of a weakly convergent (hence weak Cauchy) sequence of $L^{+}(V^{\#})$, i.e. $A \in \widetilde{L^{+}(V^{\#})}^{w}$.

We summarize this analysis in the following:

<u>Proposition 2</u>

If V is a PIP space, then $Op_I V \subset \widetilde{L^{+}(V^{\#})}^{w}$. ∎

Now, putting together Propositions 1 and 2 we can state our main result (which shows that a quantized field at a point may be described in terms of operators on a PIP space).

<u>Proposition 3</u>

Let V be an arbitrary PIP space. If $\langle V^{\#}, V\rangle$ is a reflexive dual pair, then $Op_I V$ is isomorphic to $\widetilde{L^{+}(V^{\#})}^{w}$. ∎

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