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EXACT SOLITON-LIKE SOLUTIONS OF PERTURBED φ^4-EQUATION *

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ABSTRACT

Exact soliton-like solutions of damped, driven φ^4-equation are found. The exact expressions for the velocities of solitons are given. It is non-perturbatively proved that the perturbed φ^4-equation has stable kink-like solutions of a new type.

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1. Recently, the interest in the perturbed sine-Gordon and φ^4-equations has increased, due to the possible application of these equations in field theory and condensed matter physics, specially in phase transition theory.

A review of former investigations on these equations can be found in Refs.[1,2]. In other articles, for example [3-8] these equations are studied in the presence of dissipation and external fields. The majority of these articles are perturbative or numerical investigations of sine-Gordon equation, except for the work of Puri [7] in which the sine-Gordon equation with dissipation and driven term is investigated by means of a non-perturbative method.

In the present paper, we investigate the φ^4-equation with dissipation and constant external field with the help of phase-plane analysis. We show the shape of different solutions for different range of parameters. Some of our results resemble those of Puri for the sine-Gordon equation (as for example the existence of stable kink-soliton solutions travelling with unique velocity). But we go further to find the exact soliton solutions and the exact necessary conditions for their existence. Then we show the existence of other soliton solutions not analyzed in [7].

2. The equation we consider is

$$-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - R\frac{\partial\varphi}{\partial t} + \frac{\partial^2\varphi}{\partial x^2} + \alpha'\varphi + \beta'\varphi^3 + h' = 0 \qquad (1)$$

where $\alpha', \beta', h', R, C$ are constants, $\alpha' > 0$, $\beta' < 0$. We look for solutions in the form of travelling waves. Making $\xi = x - vt$ we obtain

$$\frac{d^2\varphi}{d\xi^2} + \gamma\frac{d\varphi}{d\xi} + \alpha\varphi + \beta\varphi^3 = h \qquad (2)$$

where $\gamma = \dfrac{RV}{1-\dfrac{v^2}{c^2}}$, $\alpha = \dfrac{\alpha'}{1-\dfrac{v^2}{c^2}}$, $\beta = \dfrac{\beta'}{1-\dfrac{v^2}{c^2}}$, $h = -\dfrac{h'}{1-\dfrac{v^2}{c^2}}$.

Defining $\frac{d\varphi}{d\xi} = z$ this last equation becomes a system of two first-order equations

$$\frac{d\varphi}{d\xi} = z$$

$$\frac{dz}{d\xi} = -\gamma z - \alpha\varphi - \beta\varphi^3 + h \qquad (3)$$

The fixed critical points of the system (3) in the phase plane can be found from

$$z = 0, \quad \alpha\varphi + \beta\varphi^3 = h \quad . \tag{4}$$

The roots of the cubic equation (4) are

$$y_1 = \frac{1}{3}(U + V)$$

$$y_2 = -\frac{1}{6}(U + V) - \frac{\sqrt{3}}{6}(U - V)\,i$$

$$y_3 = -\frac{1}{6}(U + V) + \frac{\sqrt{3}}{6}(U - V)\,i \tag{5}$$

where

$$U = \sqrt[3]{\frac{27}{2}\frac{h}{\beta} + \frac{3\sqrt{3}}{2\beta}\sqrt{27h^2 + 4\frac{\alpha^3}{\beta}}}$$

$$V = \sqrt[3]{\frac{27}{2}\frac{h}{\beta} - \frac{3\sqrt{3}}{2\beta}\sqrt{27h^2 + 4\frac{\alpha^3}{\beta}}}$$

The necessary condition for the existence of three fixed points (i.e. three real roots of the cubic equation (4)) is

$$h^2 < -\frac{4}{27}\frac{\alpha^3}{\beta} \quad . \tag{6}$$

We consider that this condition holds. In the opposite case the system has only one fixed point and soliton solutions cannot exist. The eigenvalues of the system are given by

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4(\alpha + 3\beta y^2)}}{2} \tag{7}$$

where y takes one of the values (5) in concordance with the fixed point we are analyzing.

In the simplest well-known situation $\gamma = h = 0$ we have one centre and two saddle points. The separatrix joining the saddle points corresponds to well-known kink-soliton.

Let us now analyze the situation

$$\gamma = 0 \quad , \quad 0 < h^2 < -\frac{4}{27}\frac{\alpha^3}{\beta} \quad .$$

The qualitative analysis shows again one centre and two saddle points. However, the latter are not at the same distance of the centre. Then the phase trajectory that starts from one saddle point (that closer to the centre) does not reach the other but comes back to the starting point. This separatrix corresponds to a Bell-soliton (represented in Figs.1).

If $h < 0$ the picture of the Bell-soliton with respect to the z axis appears inverted in Figs.1, then in order to be specific we always consider $h \geqslant 0$. Let us suppose that $0 < \gamma^2 < 4\ (\alpha + 3\beta y_1^{\,2})$. Then the centre becomes a focus and the phase trajectories are spirals in its neighbourhood. For a certain value of h (γ fixed) there exists a separatrix joining the saddle points. This solution has a kink-like form (see Figs.2).

Other possible solutions are "semi-soliton"-like and correspond to connections of saddle points and the focus. They look like partial solitons (antisolitons) with wiggly tails. When $\gamma < 0$ the focus is unstable and the separatrix shown in Fig.2 is an antikink. It moves with negative velocity.

Finally, when $\gamma^2 \geqslant 4(\alpha + 3\beta y_1^{\,3})$ the focus becomes a node in which the trajectories starting from the sadle points end. These trajectories correspond to new kink-like solutions (see Fig.3).

3. If $\gamma = 0$ Eq.(2) can be easily integrated if first we multiply it by $\frac{d\varphi}{d\xi}$ [9]

$$\frac{1}{2}\left(\frac{d\varphi}{d\xi}\right)^2 + \frac{\alpha\varphi^2}{2} + \frac{\beta\varphi^4}{4} = h\varphi + \text{const.} \tag{8}$$

The soliton-like solutions are expressed by the formula

$$\varphi = \frac{2\sqrt{2\rho}\,(\rho - 3y^2)}{(\rho - y^2)\,e^{K} + \rho\, e^{-K} + 2y\sqrt{2\rho}} + y \tag{9}$$

where $\rho = -\frac{\alpha}{\beta}$, $K = \sqrt{\beta(\rho - 3y^2)}\,(\xi - \xi_0)$, $\xi_0 = \text{const}$. The magnitude y is one of the roots of the cubic equation (4), when $h > 0$, $y = y_2$; when $h < 0$, $y = y_3$. If $h \neq 0$ the solution describes a Bell-soliton (Figs.1).

The minimum of the function shown in Figs.1 is

$$\varphi_m = \sqrt{2(\rho - y_2^2)} - y_2 \quad . \tag{10}$$

Let $h = 0$ then $y = -y_3 = y_2 = \sqrt{\rho}$. The solution (9) is the well-known kink

$$\varphi = \sqrt{-\frac{\alpha}{\beta}}\tanh\left(\sqrt{\frac{\alpha}{2}}(\xi - \xi_0)\right) \tag{11}$$

The antikink is

$$\varphi = -\sqrt{-\frac{\alpha}{\beta}}\tanh\left(\sqrt{\frac{\alpha}{2}}(\xi - \xi_0)\right) \tag{12}$$

The solution of the equation

$$\frac{d^2\varphi}{d\xi^2} + \gamma\frac{d\varphi}{d\xi} + \alpha\varphi + \beta\varphi^3 = h \tag{13}$$

which corresponds to the separatrix joining both saddle points (Figs.2) can be sought in the form

$$\varphi = A \tanh B(\xi - \xi_0) + D \quad , \tag{14}$$

where A,B,D depend only on parameters α, β, γ, h.

Replacing (14) in (13) we obtain the following algebraical system for A,B,D:

$$\begin{aligned} &2AB^2 + \beta A^3 = 0 \\ &\gamma AB - 3\beta DA^2 = 0 \\ &\alpha A + \beta A^3 + 3\beta D^2 A = 0 \\ &h - \alpha D - \beta D^3 - 3\beta D A^2 = 0 \end{aligned} \tag{15}$$

As the phase analysis indicates φ takes values from y_3 (when $\xi = -\infty$) to y_2 (when $\xi = \infty$) then we can write

$$A = \frac{y_2 - y_3}{2} \quad , \quad D = \frac{y_2 + y_3}{2} \tag{16}$$

From (15) we find

$$B = \frac{y_2 - y_3}{2}\sqrt{-\frac{\beta}{2}} \tag{17}$$

But these values of A,B and D are solutions of the system (15) only with the condition

$$\gamma = -3\sqrt{\frac{\beta}{2}}\,(y_2 + y_3) = 3\sqrt{-\frac{\beta}{2}}\,y_1 \tag{18}$$

It is evident that only for a determined value of γ the solution corresponding to a bifurcation trajectory can exist (h fixed).

It is interesting that some particular solutions which correspond to connections of saddle points and the node (Fig.3) can also be sought in the form (14).

These solutions are

$$\varphi = \pm A\tanh B(\xi - \xi_0) + D \quad ,$$

where

$$A = \frac{y_1 - y_2}{2} \quad , \quad D = \frac{y_1 + y_2}{2} \quad , \quad B = \frac{y_2 - y_1}{2}\sqrt{-\frac{\beta}{2}}$$

$$\gamma = \pm 3\sqrt{-2\beta}\,D \tag{19}$$

and

$$A = \frac{y_1 - y_3}{2} \quad , \quad D = \frac{y_1 + y_3}{2} \quad , \quad B = \frac{y_1 - y_3}{2}\sqrt{-\frac{\beta}{2}}$$

$$\gamma = \pm 3\sqrt{-2\beta}\,D \tag{20}$$

There exist other solutions corresponding to connections of saddle points and the node (for other γ, $\gamma^2 \geqslant 4(\alpha + 3\beta y_1^2)$) but they cannot be expressed as (14). They have the form of non-symmetric kink.

4. Following the method used by Puri [7] it is possible to prove that the kink-solitons described by formula (14) are stable. However, the Bell-soliton is unstable. This was shown by the author in [9].

When $h \gtrsim 0$ the Bell-soliton looks like a bound state of a kink and an antikink (if h equals zero the distance between the kinks is infinite). The distance between the walls of the Bell-soliton can be calculated as

$$d = \frac{2\cosh^{-1}\left(\frac{M}{y_1 - y_2} - N\right)}{\sqrt{\beta(p - 3y_2^2)}} \tag{21}$$

where

$$M = \frac{2\sqrt{2p}\,(p - 3y_2^2)}{\sqrt{p(p - y_2^2)}}, \quad N = \frac{2\sqrt{2p}\,y_2}{p - y_2^2}$$

We take d as the distance between the inflexion points of $\varphi(\xi)$.

When $|h|$ grows d decreases (see Fig.4). At the critical value $h_c^2 = -\frac{4}{27}\frac{\alpha^3}{\beta}$ the solution is

$$\varphi = \sqrt{-\frac{\alpha}{3\beta}} .$$

If $R \neq 0$ the Bell-soliton is static ($v = 0$). It exists only if $\gamma = 0$.

When $h^2 \ll -\frac{4}{27}\frac{\alpha^3}{\beta}$ the roots of the cubic equation (4) can be approximated as

$$y_1 \approx \frac{h}{\alpha}, \quad y_2 \approx \sqrt{-\frac{\alpha}{\beta}} - \frac{h}{2\alpha}, \quad y_3 \approx -\sqrt{-\frac{\alpha}{\beta}} - \frac{h}{2\alpha}$$

and the condition (18) for the existence of the kink corresponding to connections of saddle points can be written in the form

$$\frac{RV}{\sqrt{1 - \frac{v^2}{c^2}}} = \pm 5\sqrt{-\frac{\beta}{2}}\,\frac{h}{\alpha} \tag{22}$$

Formula (22) enables us to calculate the velocity of the soliton when h is small.

A similar expression was derived by Fogel et al. [3] by the use of perturbation theory about the pure sine-Gordon soliton profile. As pointed out by them, it has the form of Stokes' law for a particle driven by a constant force in a viscous medium. For very high values of v and h it is necessary to use our exact condition (18). The maximum velocity for this type of kink is determined by

$$\frac{RV}{\sqrt{1 - \frac{v^2}{c^2}}} = \sqrt{\frac{3}{2}\alpha} \tag{23}$$

These results may be applied to Bloch walls in impure materials and charged-phase solitons in one-dimensional charge-density wave condensates.

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FIGURE CAPTIONS

Figs.1 - (a) Phase plane portrait for $\gamma = 0$, $h > 0$; (b) shape of the Bell-soliton.

Figs.2 - (a) Phase plane portrait for $\gamma = 3\sqrt{-\frac{\beta}{2}}\, y_1$; (b) shape of the kink-soliton.

Fig.3 - Phase plane portrait for $\gamma^2 > 4(\alpha + 3\beta\, y_1{}^2)$.

Fig.4 - When $|h|$ grows d decreases and the structural deformation of the Bell-soliton becomes more evident.

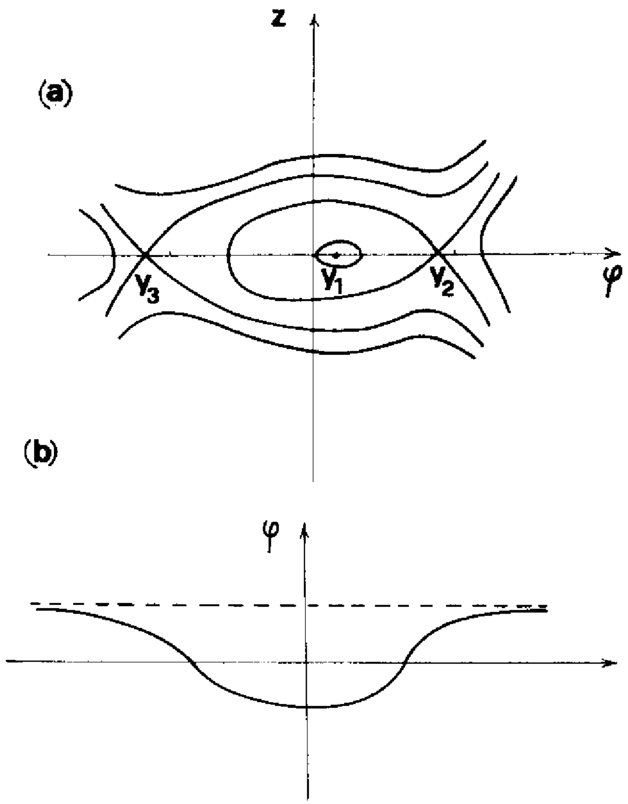

Fig.1

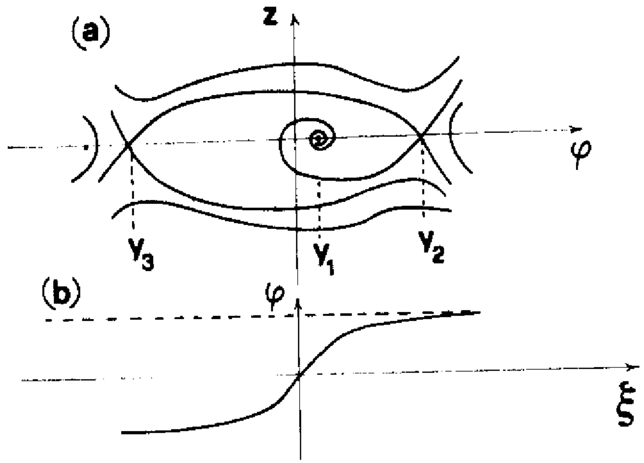

Fig.2

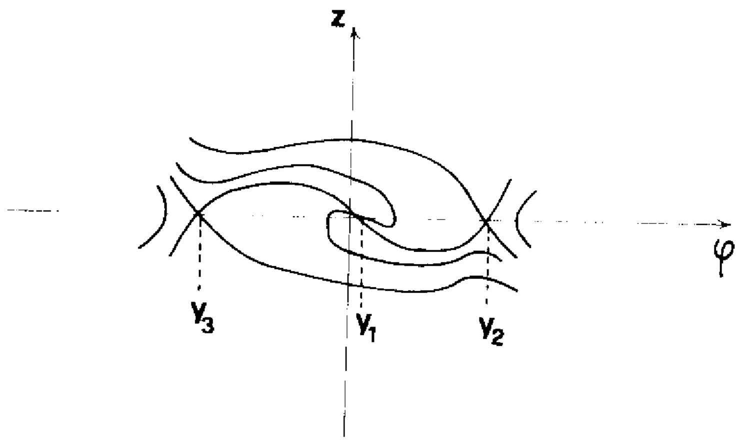

Fig.3

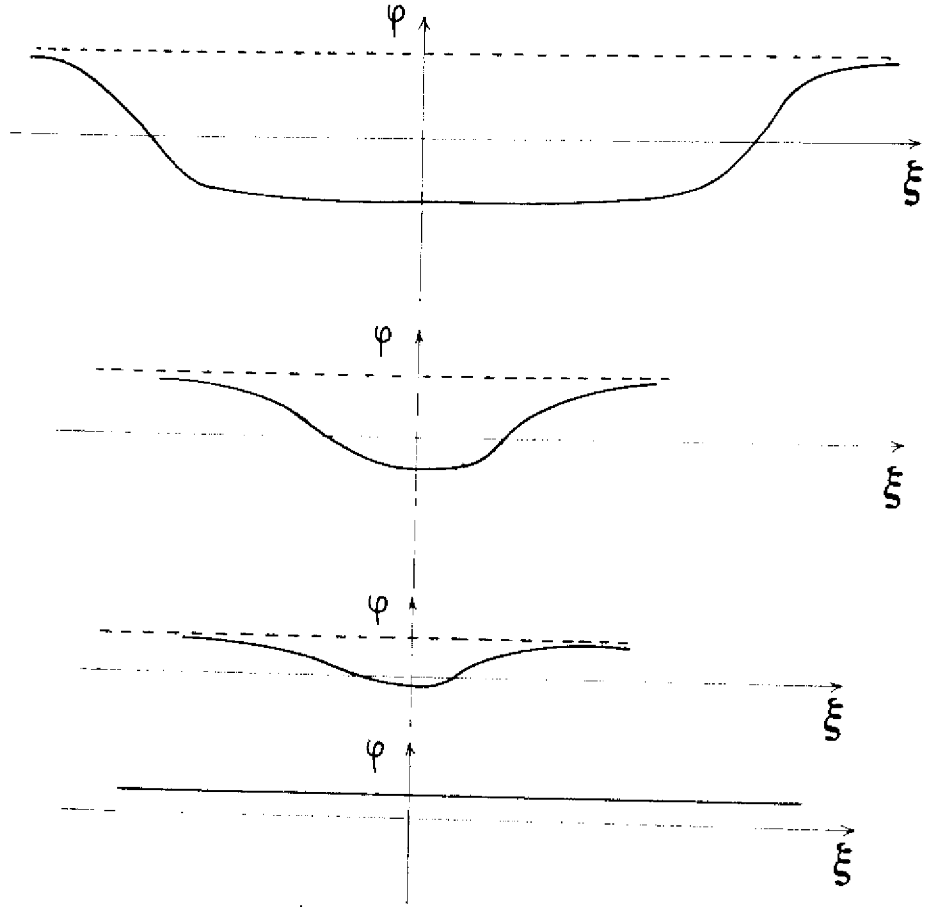

Fig.4