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NONLINEAR DYNAMICS ASPECTS OF MODERN STORAGE RINGS*

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ABSTRACT

Particle dynamics in storage rings is considered from the point of view of modern developments in nonlinear dynamics. The paper is intended as a brief introduction for accelerator physicists to the modern world of nonlinear mechanics and for mathematical physicists to the nonlinear world of modern storage rings.

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1. INTRODUCTION

This paper is intended to serve as an introduction to the Proceedings of the 1985 Accelerator School on Sardinia. We try to address the following two questions:

a. Why should accelerator physicists be interested in the recent, sometimes abstract, developments in Nonlinear Dynamics, a field which until recently was mainly studied by mathematicians, theoretical physicists and astronomers ? That such an interest to some extent already exists is apparent from the fact that many accelerator physicists attended this School and several analogous meetings in the past.[1,2]

b. Why should researchers from nonlinear dynamics be interested in modern Storage Rings which are largely designed and built by experimental physicists and engineers ? At the moment few 'nonlinear scientists' work on storage rings (or in the field of accelerator physics). It is a hopeful sign that many (more) attended this School.

We will argue that the nonlinearity of storage rings becomes an *essential* problem as the design parameters of each new machine are pushed further and further. Yet the *familiar* methods of classical mechanics do not allow us to determine single particle orbits over any reasonable length of time. Meanwhile the machines have expanded into multi-hundred-million (and now even multi-billion) dollar landmarks ! Hence we hope that the recent results in nonlinear dynamics[3-9] discussed in this paper (and volume), or their future developments, will alleviate the problems of those physicists who design, build and operate storage rings.

We shall also argue that the single particle dynamics of a storage ring is possibly one of the cleanest and simplest nonlinear dynamical systems available with very few degrees of freedom. It exibits rich and complicated stability problems during 'cosmic' numbers of revolutions. We believe storage rings to be a good experimental testing ground for recent and future theoretical results on the nonlinear mechanics of systems with few degrees of freedom. Hence we hope that specialists in nonlinear dynamics may be attracted to the challenging problems in these machines.

We shall end with a discussion of the more familiar methods of treating nonlinear systems (perhaps too), routinely used in accelerator theory. Without impuning the usefulness of those methods we shall point out some of their limitations and pitfalls and caution the reader not to have excessive faith in their results.

We do not pretend to have written a comprehensive review of the enormous amount of work in each field. The list of references is by no means complete and merely contains those papers which are of direct interest here.

2. STORAGE RINGS

2.1 WHAT IS A STORAGE RING?

A storage ring is a special type of circular accelerator designed and built to accumulate, maintain and collide two beams of charged ultrarelativistic particles rotating against each other in opposite directions.[10] One difference between a storage ring when the beams are colliding and a conventional accelerator is that the passage nearby of the other, counter rotating, charged particles violently disturbs particle orbits.[11] The motion of a particle can be represented as the sum of the motion of an 'equilibrium' point along the ring, at constant revolution frequency, and oscillations of the particle with respect to this equilibrium point. There are transverse (so called *'betatron'*) oscillations in the horizontal plane (of the ring) and the vertical plane as well as longitudinal ones (*'synchrotron'* oscillations).[10,11] One of the basic problems in beam dynamics is to avoid large transverse excurcions, which exceed the transverse size of the vacuum chamber, since particles will be lost then. Many nonlinearities, scattering mechanisms and sources of noise

have an influence on the betatron amplitude and reduce the effective life time of the beam. Nevertheless a life time of several hours ($\approx 10^9 - 10^{10}$ betatron oscillations) is routinely obtained. Sometimes even days are reached.

The main purpose of such a device is to probe the structure and nature of the interactions of particles at the smallest possible distances. Ultra high center-of-mass energy is needed for such studies. Storage rings provide the maximum possible center-of-mass energy for a pair of particles, namely the sum of the energies of each particle.

The effective collision rate ('luminosity') is proportional to the product of the currents in both beams and inversely proportional to the product of the beam widths. At higher particle densities, the beams can be disrupted by different space charge forces. The collective electromagnetic force of one beam acting on each particle of the other, the 'Beam-Beam' force, is an example. As a result the vertical width of the beam increases and the luminosity decreases. At even larger particle densities space charge effects further destroy the stability of particle motion. The life time drops and storage of beams may even become impossible.

Series of ('dipole') magnets are employed 'bending' the particle orbits in order to keep them inside the toroidal vacuum chamber. The (transverse) betatron oscillations would in general be unstable under the inevitable imperfections of each individual dipole. Stability is achieved by employing sets of 'focussing' ('quadrupole') magnets which provide restoring forces in the horizontal and vertical planes. If properly designed, both the horizontal and vertical motions become stable.

Since the amplitude of the betatron oscillations is inversely proportional to the oscillation frequency (for the same oscillation energy), the latter is made large. This is achieved by *alternating* the gradient of the magnetic field ('strong focussing').[12] In that case the coefficients of the equations of motion are periodic in the longitudinal coordinate at the revolution frequency f_0. Betatron oscillations are then described in terms of their Floquet exponents or in accelerator theory by their 'amplitude'- or β-function and their phase function (which can be rewritten as a function solely of the β-function).

In order to avoid the resulting *parametric instabilities* the (linear) betatron frequency f_{lin} must be chosen in such a way that the quantity:

$$Q \equiv f_{lin}/f_0, \tag{2.1}$$

does *not* come too close to a half integer value (cf. Mathieu's and Hill's equations). The parameter Q is called the '*tune*' of the machine. The ratio, to f_0, of the actual frequency of oscillation of an orbit, f_β, is called the tune of that orbit (in Nonlinear Dynamics this is called the 'winding number', cf. section 3.2[13]). It can differ substantially from Q,[14,1] due to the additional (linear and) nonlinear forces discussed in the next section.

The longitudinal motion is made stable by imposing a high frequency (RF) electric field. If the frequency of this field is chosen to be equal to a multiple of the revolution frequency f_0 there exists an electric wave travelling with the particle velocity ('synchronous' wave). Under this condition longitudinal motion is described by the pendulum equation.[15] Correspondingly, unstable fixed points, separatrices (and chaotic regions) appear. Inside the separatrices the synchrotron oscillations can be stable.

The horizontal betatron oscillations are strongly coupled to the longitudinal synchrotron oscillations due to the dependence of the trajectory bending on the total energy of the particle. Since the mass is relativistic the actual betatron frequencies also depend on the total energy. Hence the actual betatron frequencies are modulated by the synchrotron frequency.[16,17] All this creates a wealth of (nonlinear) parametric resonances and possible instabilities.

Small (low energy) storage rings are essentially linear machines: in properly chosen coordinates the particle oscillations are to a good approximation linear in all three degrees of freedom. For transverse motion (betatron oscillations) this is obtained by carefully designing magnets with a magnetic field as linear as possible in the transverse coordinates. The typical magnitude of the nonlinearity of the magnetic field (defined as the ratio of the pole face value of the corresponding component of the field to the value of the bending field) is of the order of 10^{-3}. In the previous section we found parametric instabilities about half-integer values of the tune (2.1). In the nonlinear (one dimensional) case small bands of instability appear also about rational tunes, $Q = p/q$ (p, q integers). To a first approximation the width of a band is proportional to the magnitude of the nonlinearity (and decreases rapidly as $q \to \infty$). Hence, in small storage rings the particle can be kept away from all dangerous betatron resonances.

As mentioned in the previous section the (longitudinal) synchrotron oscillations are nonlinear. The size of the stable region (the 'bucket' within the separatrix of the pendulum equation) can be enlarged by increasing the amplitude of the electric (RF) field. For a small storage ring it is thus possible to make this 'bucket' much larger than the amplitude of the synchrotron oscillations, keeping them essentially linear.

Quite a different situation arises in modern large storage rings. Rapid increases in particle energy (exponential during the last decade) and, correspondingly, in accelerator circumference made the modern storage ring a truly nonlinear machine.[1]

The above mentioned dependence of the betatron frequencies on the particle momentum and the spread in momenta Δp produce a spread ΔQ of actual particle tunes (i.e. a spread of winding numbers[13]). We can to some extent characterise the nonlinearity of the machine by its 'chromaticity' ξ, defined as

$$\xi \equiv \Delta Q/(\Delta p/p) \tag{2.2}$$

To first (Taylor-) approximation the ξ is a machine constant which one would like to keep small. In Table 1 we compare several of the machine parameters including chromaticity for a small (SPEAR), a medium (PEP) and a large (LEP) storage ring.

Table 1

Ring	Energy Gev	Circum. km	$\beta_{y,max}$ m	ξ_y	Q_y
SPEAR	2	0.2	51	−19	5.17
PEP	15	2.0	500	−100	18.20
LEP	70	27.0	268	−170	78.35

In order to avoid crossing the instability bands we would like to keep ΔQ small. On the other hand one is forced to choose $\Delta p/p$ rather large ($\approx 10^{-2}$) so as to get enough particles in the beam. Since ξ is a constant of the machine these are conflicting requirements.

The chromaticity ξ of the ring can be reduced by adding special sextupole magnets,[18,19] producing a magnetic field which is *quadratic* in the transverse coordinates. Unfortunately this nonlinearity creates strong third order (parametric) resonances (period 3) and a lot of additional problems: the tune of the orbit will now become a nonlinear function of the betatron amplitude (and a 'nonlinear tune spread' arises). Hence more particles reach resonances (i.e. the instability

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orbits in storage rings we would need solutions of the nonlinear equations over cosmic time scales ($10^9 - 10^{10}$ oscillations, as mentioned before). There are no methods available to do this (in general, it *cannot* be done due to the 'Sensitive Dependence on Initial Conditions', cf. section 4.3).[3,4,6,5,7] Earlier attempts to nevertheless use those familiar methods will be discussed in section 4. For storage rings we actually do not need detailed (complete) solutions of the equations. A more modest goal is to determine which solutions remain 'bounded' over a long time (and to calculate the bounds). A region in phase space in which all orbits have the same bounds over the same (particular) long time will be called a region of *Long Time Stability.*

The only available general method in modern nonlinear mechanics to find a region of long time stability (for a nonintegrable system) requires that we compute the location of a '*KAM Torus*' (after Kolmogorov, Arnol'd and Moser).[3-9] KAM tori are $N-$dimensional (N-*d.*) closed surfaces (in $2N$-*d.* phase space) which can have very irregular shapes. If an orbit has one point on a KAM torus the *whole* orbit lies on that (closed) surface (the torus is 'invariant'). Thus all orbits on tori remain bounded forever. Moreover in nonintegrable systems the linearly stable ('elliptic') periodic orbits are in general surrounded by regions in which *most* orbits lie on KAM tori (notwithstanding the presence of chaotic orbits almost everywhere, e.g. in between the tori).[6,3,4,7] Hence there are in general many regions in which most orbits remain bounded forever.

For our test particle (i.e. a system with few degrees of freedom) even more can be said: First consider two degrees of freedom, N=2. Phase space has four dimensions, the surface of constant energy ('s.c.e.') has three, KAM surfaces have two dimensions and orbits have one. It is obvious therefore that orbits starting inside a KAM torus (on the s.c.e.) can never get out. So the entire region contained inside each KAM torus is a region of stability for *all* time, when N=2. Accurate methods to calculate the location of the largest KAM torus (and the parameter interval over which each torus exists) for N=2, will be discussed in section 3. Large regions of all time stability can then be obtained for some 1-d. beam-beam forces.[14] Next consider three degrees of freedom, N=3 (e.g. coupled horizontal and vertical beam-beam interactions plus the uniform motion along the ring): Now phase space has six dimensions, the s.c.e. has five, KAM tori have three but orbits always have one (parametrized only by time). In this case a single torus surface does *not* have enough dimensions to guarantee that all orbits 'inside' it (however one defines 'inside') will always remain 'inside'. Nevertheless, each (closed) torus blocks 'escape' of the orbits in N+1 (out of $2N$-1 dimensions. In view of the large number of tori about each elliptic periodic orbit (and the large number of elliptic orbits, in general) it is clear that escape from these regions (in which *most* orbits lie *on* torus surfaces anyway, as mentioned before) is improbable and in general an extremely slow process. This is called 'Arnol'd Diffusion' ($N \geq 3$).[5,28,4,1] Even compared to the time scales in storage rings it may be slow.[29,5,28]

When the oscillation energy (i.e. the nonlinearity) increases, more and more tori '*break*': Their surfaces become riddled with holes through which orbits escape.[5,7,9,5,4] Thus a violent instability can, and usually does, appear when the '*last*' torus about an elliptic orbit breaks. ***Hence, our most important problem, even in the presence of Arnol'd diffusion, is to calculate the oscillation energy (or nonlinearity) at which the last torus about the equilibrium orbit breaks.*** Research is going on in nonlinear dynamics to extend the techniques to do so from N=2 to N=3.

So far we have ignored dissipation. In an electron ring the rate of energy loss (i.e. the damping due to radiation) per revolution is of the order of 10^{-3}. This means that volumes in phase space contract by a factor 0.999 per revolution (this factor is equal to 1 for a conservative system). Numerical experience in nonlinear dynamics indicates that the main features of phase space (e.g. its resonances) are in that case very well approximated by those of a conservative systems. Some of the higher order resonances (periods of the order of 10^3) may be wiped out by the dissipation and each (formerly) elliptic orbit, acquires in general a very small 'basin of attraction' about it.[4] We therefore believe that the major instabilities of electron rings can be

obtained from Hamiltonian mechanics.[25]

Radiation by a proton in the field of the same strength is 10^6 times smaller due to its larger mass and may be neglected. Hence, a proton ring represents a good example of a conservative dynamical system.

3. Recent Nonlinear Dynamics Results Applicable to Storage Rings

In this section we outline some recent methods which allow us to calculate the regions of long time stability, discussed in section 2.3, and the parameter range over which they exist. We saw that this requires finding the 'last' KAM torus about the equilibrium orbit and the critical oscillation energy at which that torus 'breaks'. We briefly describe two methods to do so for systems with two degrees of freedom (N=2). Research is being done to extend these methods to three degrees of freedom.

3.1 Surface of Section Maps

A system of Hamiltonian differential equations with *two degrees of freedom* can always be reduced to a mapping of the plane to itself: We take a $2-d.$ plane in the $3-d.$ surface of constant energy. Consecutive intersection points of the orbit with this (Poincaré-) 'Surface of Section' constitute a mapping of the plane to itself.[3,4,5,7] For storage rings of all kinds these mappings are used explicitly in analytical form (as difference equations).[1,2,14,30] The origin is chosen at the point where the equilibrium orbit crosses the plane. Hence it is a 'fixed point' (period 1) of the mapping. The KAM tori we are looking for intersect the plane in 'KAM circles' (actually nonlinearly deformed ellipses) about the origin. The original proof established the existence of such 'circles' at very low oscillation energy E (nonlinearity $\approx 10^{-10}$). Numerically one seems to observe that such a circle becomes larger and larger as we increase E (even beyond the maximum value allowed by present day proofs) and breaks at some critical energy E_{br} which is different for each torus. For all (relevant) circles we can now extremely accurately compute these individual breaking points E_{br}, as we shall see below, and plot the corresponding circle at its largest size. We have to repeat this calculation for several other circles as well if we want to find the largest circle at some given energy E.

3.2 Calculating the Breaking Point of a KAM Circle

To some orbits in the plane (sequences of intersection points) one can assign a *'Winding Number'* ω, i.e. the average angle of rotation (about the origin) of an orbit, divided by 2π.[13] The tune (parameter) Q (2.1) of a storage ring merely provides the the winding number of the linear(ized) betatron oscillations (e.g. without even adding the linear force contribution from the beam-beam interaction).[14] Periodic orbits have rational winding numbers, $\omega = p/q$ (p, q integers). Orbits with irrational ω are called *'quasiperiodic'*. Each orbit on a KAM circle is quasiperiodic (and must even have a 'sufficiently' irrational ω).[3,4,6–9] The least difficult to compute are the periodic orbits. We therefore approximate a quasiperiodic orbit on a KAM torus, of irrational winding number ω_{KAM}, via a sequence of periodic orbits with winding numbers ω_k, where

$$\omega_k \equiv p_k/q_k \quad (p_k, q_k, k \text{ integers}) \tag{3.1}$$

with

$$\lim_{k\to\infty} \omega_k = \omega_{KAM} \tag{3.2}$$

For each value of ω_k (with $q_k > 4$) there are at least two periodic orbits (at some oscillation energy E), one is 'elliptic' (linearly stable) the other 'hyperbolic' (linearly unstable in one direction).

One criterion for finding $E_{br}(\omega_{KAM})$, the '*Residue Criterion*', is due to Greene and MacKay.[8] While the underlying 'renormalization theory'[8] may not (yet) be rigorously established, their criterion does provide extremely accurate values of $E_{br}(\omega_{KAM})$ and is easy to use numerically: After obtaining a periodic orbit with $\omega_k = p_k/q_k$ we get q_k different linear mappings (i.e. 2×2 matrices) by linearizing our (storage ring) mapping about each point of the periodic orbit. Consider the 2×2 matrix which is the product of those q_k matrices. Its trace, Tr_k determines the linear stability of the periodic orbit.[31] Greene and MacKay prefer to look at the corresponding '*Residue*' R_k:

$$R_k \equiv (2 - Tr_k)/4 \tag{3.3}$$

The orbit is elliptic if $0 < R_k < 1$ (or hyperbolic if $R_k < 0$ or $R_k > 1$).[31] We then plot these residues as a function of the (oscillation) energy E in Fig. 1. One could just as well plot it as a function of the tune Q (2.1) or the beam current and get similar graphs. With Greene and MacKay we note that *all residues appear to intersect at one R value* ($R_{br} \approx 0.2500888...$as $k \to \infty$). The corresponding energy value in Fig. 1 is the critical energy, E_{br}, at wich the KAM circle breaks, for $\omega_{KAM} = (3 - \sqrt{5})/2$ in this[32] mapping. Working with just a few orbits of reasonable period ($< 10^4$), and extrapolating[8] the results, one can thus obtain the value of E_{br} to very high accuracy. The residue criterion could also be used with perturbation theory to produce explicit (approximate) analytical expressions.[8]

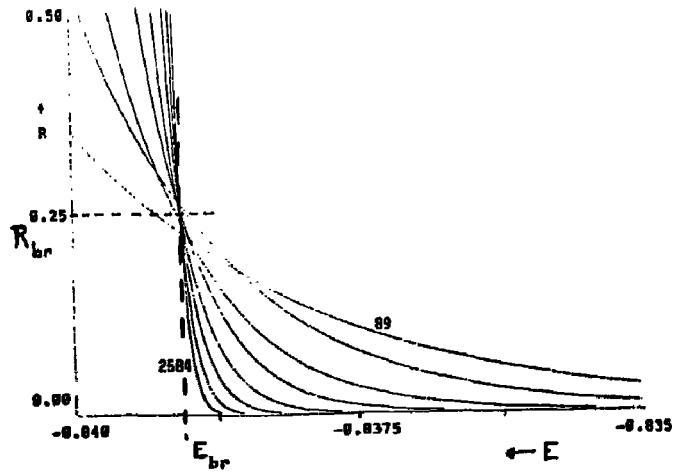

Figure 1: Residues R_k (3.3) for 8 elliptic periodic orbits as a function of the (linear oscillation) energy E. The winding numbers ω_k (3.1) approximate $\omega_{KAM} = (3 - \sqrt{5})/2$, with $\omega_k = F_{k-2}/F_k$, where the F_k are Fibonacci numbers: 34, 55, 89, 144...2584.[8] This figure was obtained by J.van Zeijts.[32] The curves are ordered according to period, from 89 to 2584.

Another criterion is due to Aubry and Mather.[9,8] They consider the nonlinear mapping equations as variational (Euler-Lagrange) equations of the '*Action*' W (per unit time). Evaluating this action for the kth periodic orbit of our sequence, we obtain values W_k^{ell} and W_k^{hyp}, for the

elliptic and hyperbolic orbits, respectively. The difference between these two actions,

$$\Delta W_k \equiv W_k^{ell} - W_k^{hyp}, \tag{3.4}$$

can then be plotted as a function of the (oscillation) energy, for each value of k used. Mather and (independently) Aubry proved that,

$$\lim_{k\to\infty} \Delta W_k = 0, \tag{3.5}$$

if, and only if, the (unbroken) KAM circle exists ($\lim|\Delta W_k| = \infty$, when it is broken).[9] Unfortunately the convergence of the ΔW_k (especially near E_{br}) appears to be very slow, numerically. Perhaps it might numerically be easier to detect the divergence and thus show that a circle is definitely broken beyond some (higher) E value. Another simple criterion to test for the absence of KAM circles was given by Percival.[7]

While the action criterion seems to be more cumbersome to apply numerically than the residue criterion it might perhaps be simpler to extend to more degrees of freedom than two.

4. Earlier Nonlinear Methods Applied to Storage Rings

In this section we briefly discuss more familiar nonlinear methods including Perturbation Theory, Chirikov's 'Resonance Overlap Criterion' and numerical 'Tracking'.

4.1 Perturbation Theory

The standard method for 'solving' nonlinear problems, in the familiar textbooks on classical mechanics, is to find a canonical transformation of our variables to (true-) *'Action Angle Variables'*, J,Φ. These are defined as new variables such that the Hamiltonian will not depend on Φ, after the transformation. If such a transformation exists the N elements of J become constants of the motion ('integrals') and the angles Φ(t) are found trivially as well. The transformation itself is constructed via power series (e.g. for the 'generating function' of the transformation) whose coefficients must be determined iteratively, i.e. perturbatively. A *nonintegrable* system does *not* have N smooth (i.e., Taylor-expandable) constants of the motion, by definition.[3,6,5] Hence such *canonical perturbation series must diverge* (i.e. have a zero radius of convergence in general) for nonintegrable systems. Moreover, *most* Hamiltonian systems are nonintegrable.[6,4]

Other familiar methods like Birkhoff Series,[6] Lie Transformations,[7] Hamilton-Jacobi Theory, Bogolyubov-Krylov's Averaging Method,and Kolmogorov's Superconvergence Method[35] may be more efficient and powerful but must all be implemented iteratively. For the same systems they are divergent again, for the same reasons. The methods of Bogolyubov-Krylov and of Kolmogorov are extensively used in beam dynamics.[36] Purely numerical solutions of the equations of motion (or the Hamilton-Jacobi equation) may be possible in principle but suffer other problems in practice, as mentioned in section 4.3.

In practice we truncate all series at some finite order. The 'solution' thus obtained is the *exact* solution for some *integrable* Hamiltonian which may or may not be "close" to our actual, *non*integrable Hamiltonian. Very little is known (a priori) about the accuracy, if any, of such *'approximations to something that does not exist'*.[6,3,4,5,7]

4.2 The Resonance Overlap Criterion (Chirikov)

In this approach one truncates *in effect* the perturbation series at the lowest possible order that yields two neighboring elliptic periodic orbits ('Resonances'). This low order truncation is the exact solution of an integrable Hamiltonian which is easily obtained. The width of the 'islands' ('buckets') about its two elliptic orbits are approximated analytically.[28,8,37] It is Chirikov's hypothesis that instabilities and chaos arise in the actual nonintegrable system when the sum of the island widths (in the integrable approximation) exceeds the actual distance between the periodic orbits of the nonintegrable system.[26,8,37,36] Mathematically it merely indicates the point where low order perturbation theory is guaranteed to fail. The intuitive idea behind this 'Resonance Overlap Criterion' is that orbits (of the actual system) might then be able to go from the first island chain to the second, i.e. that any KAM tori separating the islands (in the actual system) might then be broken. A direct application to the so called 'Standard Map' (and its last KAM circle), which depends on a parameter K, produces a critical value $K_{br} \approx 2.5$.[28,8,37] Chirikov has later extended the calculation to include higher order perturbation terms (and higher order resonances) yielding $K_{br} \approx 1.35$,[28] whereas the residue criterion (of section 3.2) gives $K_{br} = 0.971635406...$.[8]

Note however that the overlap criterion does produce explicit analytical expressions, whereas the residue criterion has so far merely been used as a numerical procedure, but cf. Ref. 8.

The overlap criterion has been applied to the 1-d. beam-beam effect.[38] It produces an estimate for the 1-d. beam-beam limit (cf. section 2.2), e.g. for the maximum beam currents, which is much too high compared with numerical results for the same equations. The overlap criterion has been extended and applied to a 2-d. beam-beam effect (vertical betatron and longitudinal synchrotron oscillations). The resulting estimate agrees reasonably well with real experimental data for the beam-beam limit (obtained with bunched beams).[39] In this case the overlap of resonances already occurs at much smaller values of the perturbation (or coupling constant).

4.3 Tracking with Surface of Section Maps

In this section we discuss the numerical computation of orbits in surface of section mappings ('tracking'), cf. section 3.1.

Most of the nonlinear forces are highly localized at points along the ring (e.g. the multipole magnets). One assumes that a particle, moving at the speed of light, experiences these nonlinear forces as (periodic) δ-functions in time. It is at those (periodic) moments that we construct a separate nonlinear surface of section map for each such nonlinear 'elements' of the ring. Explicit nonlinear difference equations are obtained for these maps.[1,2,14] The motion due to the (non-localized) linear forces can of course be solved explicitly and re-expressed as linear mapping (from one nonlinear element to the next). Thus the total *'return' map* for the ring is given by a chain of all these separate maps (for each 'element' of the 'lattice'). Numerical 'tracking' of orbits in such maps is faster and more accurate than a numerical solution of the basic differential equations. Tracking is so time consuming nevertheless that it is usually done *separately* for each nonlinear effect. Unfortunately this gives little information about the abundance of new resonances and instabilities that may arise when combining different nonlinear effects. This needs to be understood (and the new resonances avoided) before one could really interpret and use a *total* return map. Long time tracking of orbits in any nonintegrable system may trigger several numerical and financial instabilities: One can rapidly (exponentially fast) lose precision (and track of the actual orbit). This happens in (chaotic) regions of *'Sensitive Dependence on Initial Conditions'*.[4,5] An explicit but *typical* example of such behavior is provided by the simple 1-d. mapping, $x_{n+1} = \cos(10 x_n 2\pi)$, where it is clear that we loose one decimal precision everytime we iterate the map, e.g. all information from x_0 is already lost after 19 iterations (when computing

with 18 decimals). Such 'chaotic' regions are present virtually everywhere in the phase space of a nonintegrable system. Even in the regions that look 'regular' (in a surface of section plot), i.e. with an abundance of KAM tori, there usually are infinities of (perhaps microscopically) small chaotic regions. The so called *'Reversal Test'* turns out to be one of the sharpest numerical tests for the accuracy of an orbit: For example, in a 2-d. map we would stop after n iterations, having obtained values for x_{n-1} and x_n, and then restart with initial conditions x_n, x_{n-1} (i.e. we reversed their order). The mapping must be 'time reversible' since we obtained it from a Hamiltonian system, i.e. each Hamiltonian orbit should back-track exactly when we reverse its velocities. So, if our orbit returned to $\approx x_1, \approx x_0$ with only the first four decimals identical, it would be unreasonable to assume much better accuracy for x_{n-1}, x_n. Usually the reason for not performing this test is given as: 'It already cost us so much to get out to x_n that it would be a waste to spend the same amount just to get back to x_0'. In that case, it seems to us, there is all the more reason to test the wisdom of such an investment. Usually the test provides ample justification for substantial savings. Another often stated reason for not performing the test is that fluctuations (similar to those initiated by the round off errors of the computer ?) also exist in storage rings. This may be true but does not clarify the inherent properties of the mapping. It does however give a better idea of the motion under the influence of some kind of noise. One should perhaps repeat the tracking with different kinds of noise in order to see how the results depend on those differences. Whatever the counterarguments, it always is of interest to know the accuracy of any numerical 'experiment'.

The next problem is one of *'Sampling'*: for how many (and for which) initial conditions should one do the tracking in order to get insight into, let alone confidence in, the stability of the total system? Answers to this question cannot be very convincing since bounded and unbounded orbits may be interwoven in any region of a nonintegrable system, no matter how small the region. Yet it is important to note that we can locate the larger regions of *short* time instability and/or stability via a modest number of samples. It is often assumed that one need not sample many initial conditions when tracking orbits in strongly nonlinear situations, since the orbit in the course of time will come "nearly everywhere" in the chaotic region. Even if this were true, would it really come everywhere *equally* often ('ergodicity')? Only if the motion in some regions were ergodic could we replace the space- (sample)-averaging by a time average. Ergodicity is precluded however by the many small KAM tori that are embedded even in the chaotic regions. Global ergodicity is even rarer, i.e. a general Hamiltonian system is neither ergodic nor integrable.

In electron storage rings one usually does not track any longer than (a few times) the radiation damping time ($\approx 10^4$ oscillations). For many initial conditions however this (long) damping time only provides the duration of the (small) *transient* before an orbit settles down on an ('attracting')[4] orbit which is well approximated by that of a Hamiltonian system, cf. section 2.3. Hence, in order to get resonance structure of phase space it would be wise to occasionally do long time tracking for electron machines as well, neglecting both damping and quantum fluctuation.

The collective effects we mentioned in section 2.2 are studied by simultaneously tracking some M 'macroparticles', i.e. 'particles' with the usual mass but one Mth the charge of the total bunch of particles. The relativistic mass of each particle is so large that we can for some time ignore the effect of particle interactions within each bunch. Yet the *total* electromagnetic (Lorentz-) force due to one bunch on each particle of the oncoming bunch is considerable and is recomputed for every macroparticle each time the bunchs collide. The most realistic situation one has been able to track is the interaction between two bunches of 50 macroparticles oscillating in all three directions but without being able to include other nonlinearities of the ring, as noted earlier.

Unfortunately, the more effects one wishes to combine the fewer alternatives there are to tracking, at this time.

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