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A No Hair Theorem for Inhomogeneous Cosmologies

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ABSTRACT

We show that under very general conditions any inhomogeneous cosmological model with a positive cosmological constant, that can be described in a synchronous reference system will tend asymptotically in time towards the de Sitter solution. This is shown to be relevant in the context of inflationary models as it makes inflation very weakly dependent on initial conditions.

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It has been conjectured by Gibbons and Hawking[1] and by Hawking and Moss[2] that cosmologies with a positive cosmological constant approach the de Sitter solution asymptotically in time; this is the so called "no hair" conjecture. Several attempts at proving the conjecture have been made[3], and a general proof has been obtained for homogeneous cosmologies (Bianchi models).[4] The range of validity of no hair theorems is of great importance in several areas of physics such as for example inflationary cosmologies. In these one usually assumes that the universe becomes dominated by a positive vacuum energy, i.e. a cosmological constant $\Lambda > 0$, and for a period of time expands exponentially at the Hubble-rate H= $\sqrt{3\Lambda}$. If the universe undergoes a period of exponential expansion of more than about sixty Hubble times it is possible to explain the cosmological horizon- and flatnes/oldnes puzzles in a natural way[5].

The purpose of this letter is to prove a very general version of the cosmic no hair theorem, and consider some of its immidiate implications. We shall show that a very large class of inhomogeneous and anisotropic cosmologies tend to the de Sitter solution at large times. This implies that inflation becomes a generic feature of a large class of cosmologies provided with a positive cosmological constant. Initial conditions of these cosmologies then becomes almost irrelevant since they all end up in the same asymptotical state, the de Sitter cosmology.

We consider Einstein's equations

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T - g_{\mu\nu} \Lambda \tag{1}$$

Here $g_{\mu\nu}$ is the space-time metric, $T_{\mu\nu}$ the energy-momentum tensor, and $T = T^{\mu}_{\mu}$. We use the sign conventions $(+,-,-,-)$ and the notation of ref.(6). Greek indices run from 0 to 3 and Latin from 1 to 3. The only assumptions we make about the energy-momentum tensor is that it satisfies (i) the dominant energy condition, this means that $T_{\mu\nu} t^{\mu} t^{\nu} \geq 0$ and $T_{\mu\nu} t^{\nu}$ is non spacelike for all timelike t^{ν}, and (ii) the strong energy condition, that

$(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)t^\mu t^\nu \geq 0$ for all timelike t^μ. The dominant energy condition is equivalent to demanding that the energy density is non-negative and the energy flow is causal. All known forms of matter satisfies this condition. (For a perfect fluid it reduces to $\rho \geq |p|$.) The strong energy condition for a perfect fluid reduces to the usual requirement that $\rho + 3p \geq 0$, i.e. a large negative energy density or large negative pressures must be present to violate this condition. We choose to work in a synchronous reference system where $g_{00} = 1$ and $g_{0\nu} = 0$. We shall also introduce a (positive definite) spacial metric tensor $h_{ab} \equiv -g_{ab}$ and define $s_{ab} = \dot{h}_{ab}$. Using this (1) becomes[6]

$$R^0_0 = -\frac{1}{2}\dot{s}^a_a - \frac{1}{4}s^b_a s^a_b = T^0_0 - \frac{1}{2}T - \Lambda$$

$$R^0_a = \frac{1}{2}(s^b_{a;b} - s^b_{b;a}) = T^0_a \tag{2}$$

$$R^b_a = -P^b_a - \frac{1}{2\sqrt{h}}\frac{\partial}{\partial t}(\sqrt{h}s^b_a) = T^b_a - \frac{1}{2}\delta^b_a T - \delta^b_a \Lambda$$

Here P_{ab} is the three dimensional Ricci tensor calculated using h_{ab} and $h = det\ h_{ab}$. ($\sqrt{h}$ can be interpreted as the volume element in three-space.) Let us define the volume expansion by $K \equiv \frac{1}{2}\dot{h}/h = \frac{1}{2}s^a_a$. In what follows we make the assumption that the space is open or flat, i.e. that the scalar spatial curvature $P = P^a_a$ is negative or zero. Eq. (2) then implies

$$-R^0_0 = \dot{K} + \frac{1}{4}s^b_a s^a_b = -T^0_0 + \frac{1}{2}T + \Lambda \tag{3.1}$$

$$-R^a_a = \dot{K} + K^2 + P = -T^a_a + \frac{3}{2}T + 3\Lambda \tag{3.2}$$

To proceed and solve (2) we must first calculate $s^b_a s^a_b$. If we introduce the trace free part of s_{ab}, $2\sigma_{ab} \equiv (s_{ab} - \frac{1}{3}s^c_c h_{ab})$, we find that

$$s^a_b s^b_a = s_{ab}s^{ab} = \frac{1}{3}(s^a_a)^2 + 4\sigma_{ab}\sigma^{ab} = \frac{4}{3}K^2 + 4\sigma_{ab}\sigma^{ab}$$

Substituting this into (3.1) we find

$$\dot{K} = \Lambda - \frac{1}{3}K^2 - \sigma_{ab}\sigma^{ab} - (T^0_0 - \frac{1}{2}T) \tag{4}$$

Eliminating $\dot{K}$ using (3.2) this gives

$$\Lambda - \frac{1}{3}K^2 = -\frac{1}{2}\sigma_{ab}\sigma^{ab} - T_0^0 + \frac{P}{2} \tag{5}$$

Clearly $\sigma_{ab}\sigma^{ab}$ is non negative and zero only when $\sigma_{ab} = 0$. The strong and dominant energy conditions imply that $T_0^0 - \frac{1}{2}T$ and T_{00} are positive, so from (4) and (5) we find

$$\dot{K} \leq \Lambda - \frac{1}{3}K^2 = -\sigma_{ab}\sigma^{ab} - T_0^0 + \frac{P}{2} \leq 0 \tag{6}$$

This shows that $K^2 \geq 3\Lambda$. Also, after integration of the first inequality of (6) we find that $K \leq \sqrt{3\Lambda}/\tanh(\sqrt{\frac{\Lambda}{3}}(t + t_0(x_c))$, where t_0 only depends on space. (Here we have chosen the positive square root so $K_0 = K(t_0)$ is positive corresponding to an expanding universe.) This implies that asymptotically

$$0 \leq K - \sqrt{3\Lambda} \leq 4\sqrt{3\Lambda}e^{-2\sqrt{\frac{\Lambda}{3}}(t+t_0)}$$

From (6) it then follows that $\sigma_{ab}\sigma^{ab}$, T_{00} and $-P$ all are suppressed by the same exponential factor. This shows that the expansion rate K of the volume tends to the de Sitter rate of $\sqrt{3\Lambda}$ and that $\sigma_{ab} = 0$ asymptotically, i.e.

$$\dot{h}_{ab} - 2\sqrt{\frac{\Lambda}{3}}h_{ab} = 0$$

This implies that asymptotically, $h_{ab}(t, x_c) = e^{2\sqrt{\frac{\Lambda}{3}}t}\tilde{h}_{ab}(x_c)$, where $\tilde{h}$ only depends on space.

From the dominant energy condition it follows that for all timelike t^ν

$$(T_{0\nu}t^\nu)^2 \geq T_{a\nu}t^\nu h^{ab}T_{b\mu}t^\mu \geq 0 \tag{7}$$

Choosing $t^\nu = \delta^{0\nu}$ this shows that

$$T_{00}^2 \geq T_{a0}h^{ab}T_{b0} \geq 0$$

Since T_{00} vanishes faster than h^{ab} this forces T_{0a} to vanish asymptotically. Substituting this back into (7) we similarly find that T_{ab} vanishes asymptotically. Therefore asymptotically Einstein's equations for P_{ab} becomes

$$P_{ab} = 0 \tag{8}$$

The only solution to this is flat Euclidean three-space, i.e. $\tilde{h}_{ab}$ becomes equal to a (constant) Euclidean metric. This shows that in suitable coordinates the full space-time metric $g_{\mu\nu}$ asymptotically becomes equal to the de Sitter metric

$$g_{\mu\nu} dx^\mu dx^\nu \rightarrow dt^2 - e^{2\sqrt{\frac{\Lambda}{3}}t}(dx^a)^2$$

In particular this applies to the case of homogeneous cosmologies, the Bianchi models. Except for Bianchi IX all these are flat or open, so in the presence of a positive cosmological constant these will all approach de Sitter space, in agreement with previous results[4].

We also wish to point out that our argument holds in any number of dimensions (except that (8) does not imply that space is Euclidean when the space dimensions exceeds three).

As we have mentioned in the beginning , the fact that such a large class of cosmologies under the influence of a positive cosmological constant tend to the de Sitter space-time has great importance for the inflationary cosmology scenarios[7]. In these gravity is coupled to a massless scalar field ϕ with a "flat" potential $V(\phi)$. The cosmological constant is fine tuned so that the energy density vanishes at the minimum of V corresponding to the absence of a cosmological constant today. The equation of motion for ϕ is

$$\ddot{\phi} + K\dot{\phi} = -V'(\phi) \tag{9}$$

where ϕ is taken to be smooth so that we can neglegt gradient terms in (9), and K is given by (5). We notice that if $P \leq 0$, then K is greater than the Hubble constant $\sqrt{3\Lambda}$ of de

Sitter phase. Therefore the "friction force" felt by the field, $K\dot{\phi}$ is greater than in de Sitter phase making the field roll slower over the potential; in the same token a greater K means faster expansion. We shall assume that initially ϕ is stabilized (by for example initial conditions or thermal corrections) on the "flat" part of V. Then we find that at least as much inflation is produced in the general case with the universe being inhomogeneous prior to inflation than in the usual case with Friedmann-Robertson-Walker (FRW) cosmology prior to inflation (similar arguments were used in ref.(8) for the anisotropic case). The resulting universe is highly homogeneous and isotropic on scales much larger than the horizon and after ϕ has returned to its minimum it evolves like the usual FRW model, but without the extra assumption of initial homogeneity or isotropy. We should comment that there in no need to assume that $P \leq 0$ in all of space, in order to have succesful inflation. We just need $P \leq 0$ in some region large enough that surface effects can be neglected, then this region will eventually evolve into de Sitter space and may then become our observable universe. This strongly supports the belief that inflation is a very universal feature largely independent of the initial conditions of the universe.

We also pointed out above that our result is true in higher dimensions: under the conditions given above an $(n+1)$-dimensional cosmology under the influence of a positive cosmological constant will eventually expand at a rate of $\sqrt{\Lambda/n}$ in each of the n spatial directions, and space becomes "Ricci-flat". This rules out inflation in Kaluza-Klein type theories in which P (the sum of the curvature of the internal space and three space) is negative, since eventually the internal dimensions will expand and become observable. (Although it may be possible to have $P \leq 0$ if one is willing to violate the energy conditions.)

We have shown that the "no hair conjecture" is true for a wide class of spatially open and flat cosmological models. The only requirements is the existence of a synchronous reference frame, a positive cosmological constant and an energy momentum tensor that satisfies both the strong and dominant energy conditions. Furthermore we have argued that in the case where gravity is coupled to a homogeneous, massless scalar field inflation is unavoidable.

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