A numerical procedure is derived for the solutions of the one- and two-dimensional Vlasov-Poisson system equations. This numerical procedure consists of the phase space discretization and the integration of the resulting set of ordinary differential equations. In the phase space discretization, derivatives with respect to the phase space variable are approximated by a weighted sum of the values of the distribution function at properly chosen neighboring points. Then, the resulting set of ordinary differential equations is solved by using an appropriate time integration scheme. The results for linear Landau damping, nonlinear Landau damping and counter-streaming plasmas are investigated and compared with those of the splitting scheme. The proposed method is found to be very accurate and efficient.

1. Introduction

It is well known that the Vlasov equation adequately describes the nonlinear evolution of collisionless plasmas. Since knowledge of the nonlinear behavior of plasmas in two and three dimensions is indispensable in understanding the plasma physics of the controlled thermonuclear fusion, the numerical integration of the Vlasov equation has been studied intensely during recent years [1-13]. However, little progress has been made on the development of fast and accurate integration schemes for the Vlasov equation in two and three dimensions for a magnetized plasma. A splitting scheme by Cheng and Knorr [12], in which the Vlasov equation is integrated in the original phase space by splitting the convective and acceleration terms in such a way that the overall scheme is second-order accurate in $\Delta t$, has been successfully applied and is one of the most promising schemes today. In two and three dimensions, however, the interpolation methods used in the scheme become more and more complicated and time-consuming, especially for magnetized plasmas [13,14]. Development new, accurate and efficient method is still needed before the Vlasov equation can be solved for three-dimensional magnetized plasmas in reasonable computation time.

In this paper, a new numerical method called the modified differential quadrature (M.D.Q.) method is proposed to integrate the Vlasov equation. The new method is an extension of the differential quadrature (D.Q.) method proposed by Bellman et al. [15]. In the present method, derivatives of the distribution function with respect to the phase-space variables are approximated by a weighted sum of the values of the distribution function at properly chosen neighboring points to generate a set of
ordinary differential equations in time, whereas in the original D.Q. method, these are approximated by using values at all mesh points in the computational domain. As a result, computational efficiency is significantly improved with the M.D.Q. method. By changing the weighting coefficients, the spatial derivatives can easily be approximated with as much accuracy as desired. The resulting set of ordinary differential equations is then integrated by using an appropriate time-integration scheme. This solution progress gives a very accurate and flexible method which is simple and straightforward to program. The present method has the feature that it is accurate to arbitrary order in space by changing the weighting coefficients and also in time by choosing a suitable time-integration scheme. Although in this paper, the method is presented only in one dimension in order to illustrate its basic elements clearly, its extension to two and three dimensions is straightforward.

Section 2 describes basic equations for the Vlasov-Poisson system of equations and section 3 describes the M.D.Q. method for the one-dimensional Vlasov equation. In section 4, we present the numerical results obtained for one- and two-dimensional linear Landau damping, nonlinear Landau damping and counter-streaming plasmas. Then, these results are compared with those obtained by the splitting scheme. Section 5 presents the conclusions.

2. The Nonlinear Vlasov Equation

The system of equations under consideration consists of the Vlasov equation for the electron distribution function $f(r,v,t)$

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial r} + E \frac{\partial f}{\partial v} = 0$$

and the Poisson's equation for the electric fields $E(r,t)$

$$\nabla \cdot E = \int_{-\infty}^{\infty} f dv - 1$$

The vectorial notation $r$ and $v$ are corresponding to the position and the velocity in which for the one-dimensional case, $r=x$, $v=v$, and $E=E$, for the two-dimensional case, $r=(x,y)$, $v=(v_x,v_y)$, and $E=(E_x,E_y)$.

These equations are written in dimensionless form. The basic units of time $t$ and velocity $v$ are the reciprocal of the so-called electron plasma frequency $\omega_p$ and the mean thermal velocity $v_t$, respectively. The position vector $r$ is measured in units of the Debye length, $\lambda_D$. A periodic boundary condition is assumed in space. A rectangular mesh will be used to represent the $r$-$v$ phase space with the computational domain

$$R = \{(r,v)|0<r<L, |v|<v_{\max}\}$$

where $L$ is the spatial periodic length and $v_{\max}$ is the cutoff velocity vector, for the one-dimensional case, $L=L$ and $v_{\max}=v_{\max}$, and for the two-dimensional case, $L=(L_x,L_y)$ and $v_{\max}=(v_{\max,x},v_{\max,y})$. 

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3. The Numerical Method

We consider the one-dimensional Vlasov equation in order to explain the present numerical method,

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + E \frac{\partial f}{\partial v} = 0.$$  (4)

If the distribution function $f$ satisfying Eq. (4) is sufficiently smooth, we can write the approximate relations

$$\frac{\partial f_{ij}(t)}{\partial x} = \sum_{k=1}^{N} a_{ik} f_{kj}(t), \quad i = 1, 2, \ldots, N \quad (5a)$$

$$\frac{\partial f_{ij}(t)}{\partial v} = \sum_{l=1}^{2M} a_{jl} f_{il}(t), \quad j = 1, 2, \ldots, 2M \quad (5b)$$

where we adopt the notation $f_{ij}(t)=f(x_i, v_j, t)$ and $N$ and $2M$ indicate the number of mesh points used along the directions $x$ and $v$, respectively.

In this paper, we have modified the approximate relations, Eqs. (5a) and (5b), to use the values of $f$ at the nearest $N_p$ and $M_p$ points centered $x_i$ and $v_j$, respectively, instead of using those at all mesh points in the computational domain, as is the case in the original D.Q. method [15]. By doing this, the number of arithmetic operations to be performed for every points is significantly reduced; moreover, in the case of a uniform mesh, the weighting coefficients $a_{ik}$ and $a_{jl}$ become independent of indeces $i$ and $j$. Therefore, the approximate relations, Eqs. (5a) and (5b), can be rewritten as

$$\frac{\partial f_{ij}(t)}{\partial x} \approx \sum_{k=-n}^{n} a_{ik} f_{i+k,j}(t) \equiv D_x(f_{ij}) \quad (6a)$$

$$\frac{\partial f_{ij}(t)}{\partial v} \approx \sum_{l=-m}^{m} a_{jl} f_{i,l+j}(t) \equiv D_v(f_{ij}) \quad (6b)$$

where $a_{ik} = a_{nk}, n=(N_p + 1)/2$ in Eq. (5a) and $a_{jl} = a_{ml}, m=(M_p + 1)/2$ in Eq. (5b).

There are many ways of determining the weighting coefficients, $a_{ik}$ and $a_{jl}$. In the original D.Q. method, these coefficients are determined explicitly, choosing $x_i$ to be the root of shifted Legendre polynomial of degree of $N$, $P^*_N(x)$. In this paper, we have determined $a_{ik}$ numerically, similarly to Lagrangian interpolation and choose the test function in the following form,

$$\phi_k(x) = \Pi(x)/[(x-x_{i+k})\Pi'(x_{i+k})] \quad (7)$$

where $\Pi(x)$ is the polynomial of degree $N_p$,

$$\Pi(x) = (x-x_{i-n}) \cdots (x-x_i) \cdots (x-x_{i+n}) \quad (8)$$

It follows that $\phi_k(x)$ is a polynomial of degree $N_p-1$ such that $\phi_k(x_{i+1}) = \delta_{kl}$ and
\( \Pi(x_{i+1}) = 0 \), where \( \delta_{ki} \) denotes the classical Kronecker delta. If the values of \( f(x,v,t) \) are known at \( N \) points, \( x=x_{i-n},\ldots,x_i,\ldots,x_{i+n} \), a polynomial of degree \( N-1 \), \( \tilde{f}(x,v,t) \), which coincides with \( f(x,v,t) \) at these collocation points, can be written as

\[
\tilde{f}(x,v,t) = \sum_{k=-n}^{n} \phi_k(x)f(x_{i+k},v_j,t).
\] (9)

By differentiating Eq. (9) with respect to \( x \), we have the relation,

\[
\tilde{f}'(x,v,t) = \sum_{k=-n}^{n} \phi_k'(x)f(x_{i+k},v_j,t).
\] (10)

Using the fact that such a relation as Eq. (6a) is to be exact for \( f(x,v,t) = \Phi_k(x) \), we see that

\[
a_{ik} = \Pi'(x_i)/[(x_{i+k} - x_{i+k})^2 \Pi'(x_{i+k})], \quad k \neq 0.
\] (11)

For the case \( k = 0 \), use of l'Hopital's rule gives

\[
a_{i0} = \Pi''(x_i)/[2 \Pi'(x_i)].
\] (12)

Another weighting coefficients \( a_{j1} \) are determined similarly. Then the coefficients \( a_k \) and \( a_1 \) are determined, using the relations \( a_k = a_{nk} \) and \( a_1 = a_{ml} \) in which \( n=(N+1)/2 \) and \( m=(M+1)/2 \). The coefficients \( a_k \) and \( a_1 \) are computed once and for all at the beginning of the calculation and stored.

Substitution of Eqs. (6a) and (6b) into Eq. (4) yields the set of \( N \times 2M \) ordinary differential equations (ODEs) in time,

\[
\frac{df_{ij}(t)}{dt} = F_{ij}(x_{i},v_{j},D_{x}(f_{ij}),D_{v}(f_{ij}),E(f_{ij}),t).
\] (13)

The numerical solution of such a system, Eq. (13), is a simple task using a standard scheme for ODEs. In this paper, we tested three schemes, namely, leapfrog, corrected leapfrog and Stetter's method [16]. All of these schemes are second-order accurate in time. As applied to Eq. (13), these schemes take the following forms;

1) leapfrog method

\[
f_{ij}^{n+1} = f_{ij}^{n-1} + 2\Delta t F_{ij}^{n}.
\] (14)

2) corrected leapfrog method

\[
f_{ij}^{(0)} = f_{ij}^{n-1} + 2\Delta t F_{ij}^{n},
\]

\[
f_{ij}^{n+1} = f_{ij}^{n} + \Delta t (F_{ij}^{n} + F_{ij}^{(0)})/2.
\] (15)
3) Stetter's method

\[
\begin{align*}
 f_{ij}^{(0)} &= f_{ij}^{n} + \Delta t f_{ij}^{n} \\
 f_{ij}^{(1)} &= f_{ij}^{n} + \Delta t (\frac{f_{ij}^{n} + f_{ij}^{(0)}}{2}) \\
 f_{ij}^{(2)} &= f_{ij}^{n} + \Delta t (\frac{f_{ij}^{n} + f_{ij}^{(1)}}{2}) \\
 f_{ij}^{n+1} &= \alpha f_{ij}^{(1)} + (1 - \alpha)f_{ij}^{(2)}, \quad 0 < \alpha < 1
\end{align*}
\]  

(16)

where the superscripts \( n \) and \( n+1 \) refer to time \( t \) and \( t+\Delta t \), respectively, and the parameter \( \alpha \) is determined by the stability for the scheme.

The electric field \( E \) is computed using the standard direct Fourier transform method. In this paper, using fast Fourier transformation (FFT), the computational time is reduced.

4. Numerical Results

In this paper, we will present the results of integrating the one- and two-dimensional Vlasov equation and demonstrate the accuracy and efficiency of the present method in comparison with the splitting scheme [12].

4.1 Results for one-dimensional case

As described in section 2, we will use a rectangular mesh to represent the \( x-v \) phase space with the computational domain,

\[
R = \{(x,v) | 0 < x < L, \quad |v| \leq V_{\text{max}}\}.
\]

Throughout the following examples, the cutoff velocity, \( V_{\text{max}} \), is taken to be 5.0. The symbols \( N \) and \( 2M \) designate the number of mesh points used along the \( x \) and \( v \) directions, respectively, while \( N_p \) and \( M_p \) denote the degree of M.D.Q. method in each direction, corresponding to \( N_p-1 \) and \( M_p-1 \) order of accuracy, respectively.

In the first example we show results for the free-streaming case with \( E(x,t) = 0 \) in Eq. (1). The initial condition is

\[
f(x,v,0) = f_0(v)(1 + A \cos mx)
\]

(18)

with \( A = m = 0.5 \) and \( f_0(v) = (2\pi)^{-\frac{1}{2}}\exp(-v^2/2) \). The analytic solution corresponding to this initial condition is given by

\[
f(x,v,t) = f_0(v)[1 + A \cos m(x - vt)]
\]

(19)

Here, we define the density as...
\[ \rho(x,t) = \int_{-\infty}^{\infty} f(x,v,t) dv \]  \hspace{1cm} (20)

Eq. (19) combined with Eq. (20) leads to the following expression for the density

\[ \rho(x,t) = 1 + \exp(-m^2 t^2/2)A \cos mx \]  \hspace{1cm} (21)

Numerical calculation of Eq. (20), on the other hand, gives the following expression for the density

\[ \rho(x,t) = \Delta v \sum_{j=1}^{2M} f_0(v_j)[1 + A \cos m(x - v_j t)] \]  \hspace{1cm} (22)

The numerical results are shown in Fig. 1 where the computed densities at \( x = 0 \) (curve A) and \( x = L/8 \) (curve B) are plotted against time. As can be predicted from Eq. (21), \( \rho + 1 \) as \( t \rightarrow \infty \), and the curves in Fig. 1 tend asymptotically to 1. The right-hand side of Eq. (22) is the sum of periodic functions of time, which results in a quasi-periodic behavior for \( \rho(x,t) \) called recurrence. In the present calculation with \( A = m = 0.5, V_{\max} = 5.0, N = 8, \) and \( M = 20, \) the predicted recurrence time is \( T_R = 2\pi/(m\Delta t) = 49.00. \) The numerical results presented in Fig. 1 agree very well with this value.

The second example shown in Fig. 2 tests linear Landau damping for the same initial condition as Eq. (18), but in this case, the electric field \( E \) is retained in Eq. (1). We used \( A = 0.01, m = 0.5, N = 8, M = 20, \) and \( \Delta t = 1/8. \) The abscissa is the time nondimensionalized by \( \omega_p^{-1} \) and the ordinate shows the first Fourier mode \( E(0.5) \) of the electric field \( E. \) The solid curve in Fig. 2 has been obtained by the present method with \( N_p = M_p = 7 \) and Stetter's scheme for time integration.
while the dashed curve has been obtained by the splitting scheme. In this case the electric field decays exponentially and agreement of the numerical results with Landau's theory is quite good up to \( t = 40 \omega_p^{-1} \), except for a slight deviation in the dashed curve shortly after \( t = 32 \omega_p^{-1} \). The recurrence effect occurs at \( t = 46.63 \omega_p^{-1} \) in the present method and at \( t = 46.50 \omega_p^{-1} \) in the splitting scheme, and the times are comparable to the theoretical value \( T_r = 49.00 \) obtained for the free-streaming case.

The third example tests nonlinear Landau damping. The effect of a strong nonlinear perturbation, \( A = 0.5 \), is shown in Fig. 3 for \( m = 0.5 \), in which the evolution of the first three Fourier modes of the electric field \( E \) is shown. This problem has been solved by many authors [6,10, and 12] and, therefore,
is appropriate for evaluating the present method. We used $N = 16$, $M = 64$, and $\Delta t = 1/8$. The solid curve and the dashed curve in the figure show the results of the present method and the splitting scheme, respectively. In this case we used the present method with $N_p = 15$ and $M_p = 7$. Initially, the first mode damp much more than predicted by the linear theory while the second and third modes damp much less than Landau’s theory. After $t = 15 \omega_p^{-1}$ all modes grow exponentially until $t = 40 \omega_p^{-1}$, where saturation occurs. The results of the first and second Fourier modes obtained by the present method are both qualitatively and quantitatively equal to the results obtained by the splitting scheme, except for a slight differences after $t = 50 \omega_p^{-1}$. For the third mode, the results show considerable differences between $t = 10 \omega_p^{-1}$ and $t = 30 \omega_p^{-1}$. We can see, however, that the overall behavior of the three Fourier modes by the present method is still the same as that of the splitting scheme.

With the mesh points used in this case, total time of execution (CPU time) was 55 sec using a Fujitsu FACOM M-200 computer. In order to demonstrate the accuracy and efficiency of the present method, we compare the results obtained by the present method with those of the splitting scheme with twice as many mesh points in each direction. In the former method we used $N = 16$ and $M = 64$, while in the latter, $N = 32$ and $M = 128$. The initial conditions as well as the other parameters are the same as in Fig. 3. The time evolution of the first Fourier mode of the electric field, $E(0.5)$, is plotted in Fig. 4, in which the solid curve shows the result of the present method and the dashed curve shows that of the splitting scheme. Although the difference between the two curves at times later than $t = 50 \omega_p^{-1}$ is slightly larger than that noted in Fig. 3, the agreement can be judged as excellent. Actually the difference in Fig. 4 is much less than that existing between the splitting scheme itself with $N = 16$, $M = 64$, and $N = 32$, $M = 128$. We can see from Fig. 4 that with the present method, results with nearly the same accuracy can be obtained using half number of mesh points along each direction. Although the present method requires slightly more computation time per mesh point per time step, we can conclude that the present method is more efficient than the splitting scheme even for the one-dimensional cases.

Next, we demonstrate the accuracy and efficiency of the present method in the two-dimensional problems.

4.2 Results for two-dimensional case

A rectangular mesh will be used to represent the $r$-$V$ phase space with the computational domain

![Fig. 4 Effect of different number of mesh points](image-url)
\[ R = \{(r,v) | 0 \leq x \leq L_x, 0 \leq y \leq L_y, |v_x| \leq V_{x_{\max}}, |v_y| \leq V_{y_{\max}}\} \] (23)

where \( L_x = (2\pi/k_x) \) and \( L_y = (2\pi/k_y) \) are the spatial periodic length in the \( x \) and \( y \) directions, respectively, \( V_{x_{\max}} \) and \( V_{y_{\max}} \) are the cutoff velocities for \( v_x \) and \( v_y \), while \( N_x, N_y, 2M_x, \) and \( 2M_y \) designate the number of mesh points used along the directions \( x, y, v_x, \) and \( v_y \), respectively.

The first example shows the linear Landau damping case. In this case the initial value of the distribution function \( f \) is given by

\[ f(r,v,0) = \frac{1}{2\pi} \exp\left[-(v_x^2 + v_y^2)/2\right] (1 + A \cos k_x x \cos k_y y). \] (24)

We choose \( k_x = k_y = 0.5 \) and \( A = 0.05 \).

The calculations carried up to \( t = 20 \omega_p^{-1} \), using a time step \( \Delta t = 1/8 \), while the other parameters were set, the number of mesh points \( N_x = N_y = 8 \) and \( M_x = M_y = 16 \), cutoff velocities \( V_{x_{\max}} = V_{y_{\max}} = 4.0 \). These parameters, except \( k_x, k_y, \) and \( A \), keep the constant values throughout the following examples.

The results in this case are shown in Fig. 5, where the absolute values of Fourier mode \( E_{xk}(0.5,0.5) \) are plotted on a logarithmic scale against time.

The solid curve in Fig. 5 has been obtained by the present method with \( N_p = M_p = 7 \), and leapfrog scheme for time integration, while the dashed curve has obtained by the splitting scheme. The two fundamental modes \( E_{xk}(0.5,0.5) \) and \( E_{yk}(0.5,0.5) \) remained exactly equal in magnitude. The result by the present method shows the peaks of the curve are damped linearly, while that by the splitting scheme is damped linearly until \( t \approx 17 \omega_p^{-1} \).

The second example shown in Fig. 6 tests strongly nonlinear Landau damping for the same initial condition as Eq. (24). We used \( A = 0.5, k_x = k_y = 0.25, \) and \( \Delta t = 1/8 \). Fig. 6 gives the plot, on a logarithmic scale of \( E_{xk}(0.25,0.25) \) against time.

The solid curve shows the result by the present method with \( N_p = M_p = 7 \), and leapfrog scheme for time integration, while the dashed curve shows the result by the splitting scheme. Two results are equal qualitatively and quantitatively until \( t = 19 \omega_p^{-1} \). After that, there are the differences between the two results. But these differences are very slight. For the higher diagonal mode \( E_{xk}(0.5,0.5) \), shown in Fig. 7, the amplitudes of these modes are found to remain one or two orders of magnitude smaller than that of the fundamental mode \( E_{xk}(0.25,0.25) \), and these two
The third example tests the time evolution of an instability decayed large amplitude oscillation in counter-streaming plasmas. The initial distribution function \( f \) is taken as

\[
f(r, v, 0) = \left(\frac{v^2}{2\pi}ight)\exp\left[-\frac{(v_x^2 + v_y^2)}{2}\right](1 + A \cos k_x \cos k_y).
\]  

In this case, we used \( A = 0.25, k_x = k_y = 0.25 \), and \( \Delta t = 1/8 \). Fig. 8 shows the plot, on a logarithmic scale of the fundamental mode \( E_{0.25,0.25} \) against time. The solid curve and the dashed curve in Fig. 8 show the results by the present method and the splitting scheme, respectively. In the present method, the degree of M.D.Q. are \( N_p = M_p = 7 \) and leapfrog scheme is adopted for the time integration. The results by both methods are qualitatively and quantitatively equal and have two peaks since the initial distribution function, Eq. (25), has the term \( (v^2/2\pi)\exp[-(v_x^2 + v_y^2)/2] \), which is double peaked Maxwellian. The periods of these oscillations including each peak are both about 9.0 \( \omega_p^{-1} \).

Fig. 9 shows the evolution of the one-dimensional mode \( E_{0.5,0} \). This mode indicates a one-dimensional instability for counter-streaming plasmas. Another one-dimensional mode \( E_{0,0.5} \) is
shown in Fig. 10. In the Landau damping case, two one-dimensional modes, \( E_{xk}(0.5,0) \) and \( E_{yk}(0,0.5) \), are equal but in this case these two modes are totally different as can be easily verified.

Next, we compare the results with different time integration schemes, leapfrog and Stetter's method. In Fig. 11, solid curve denotes the result with leapfrog method and the dashed curve denotes that with Stetter's method. Testing problem is chosen strongly nonlinear Landau damping case. The parameters are the same ones as the second example. In the fundamental mode \( E_{xk}(0.25,0.25) \), the both results are exactly equal. Furthermore, in the higher diagonal mode \( E_{xk}(0.5,0.5) \), until \( t = 15 \omega_p^{-1} \) the both results are exactly equal. After that, the differences between two
Table 1 Comparison of the CPU time per time step

<table>
<thead>
<tr>
<th></th>
<th>M.D.Q. (Leapfrog)</th>
<th>Time Splitting</th>
<th>M.D.Q. (Stetter)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson Eq.</td>
<td>62</td>
<td>124</td>
<td>186</td>
</tr>
<tr>
<td>Vlasov Eq.</td>
<td>1626</td>
<td>3666</td>
<td>5206</td>
</tr>
<tr>
<td>Total</td>
<td>1688</td>
<td>3790</td>
<td>5392</td>
</tr>
</tbody>
</table>

COMPUTER: FACOM M - 200

results are too slight. Then, using leapfrog method for time integration, we have a good resolution.

From above three examples, it is clear that the method proposed in this paper gives at least the same accuracy as that of the splitting scheme. And the solutions by the present method can be demonstrated the physical natures of each problem, well.

Finally, we show the comparison of the CPU time per time step between three methods, the present method with leapfrog and Stetter's methods for time integration and the splitting scheme. Table 1 gives the comparative CPU time per time step on a Fujitsu FACOM M-200 computer to solve the Poisson's equation, the Vlasov equation, and the total. It is clear that the present method with leapfrog time integration scheme requires less than 1/2 computer time of the splitting scheme.

5. Conclusions

In the present paper we have developed a numerical procedure, based on the modified differential quadrature (M.D.Q.) method, for the solutions of one- and two-dimensional Vlasov-Poisson system of equations. The derivatives with respect to phase-space variables in the Vlasov equation are approximated as a weighted sum of the distribution function at properly chosen neighboring points. The weighting coefficients are determined similarly Lagrangian interpolation. As a result, the approximation to phase-space derivatives can be of arbitrary order of accuracy by changing the weighting coefficients. The time integration of the distribution function is performed as in the case of ordinary differential equations.

We have shown some example that demonstrate the accuracy of the present numerical method. The results for one- and two-dimensional linear Landau damping, strongly nonlinear Landau damping and counter-streaming plasmas are good agreement with those using the splitting scheme. The important feature is that the computation time
is quite low in comparison with the splitting scheme. The computation time is more and more important problem when multi-dimensional case is solved. Then, compared with the splitting scheme, the present method has the same accuracy and a good efficiency.

Although we have shown the accuracy and efficiency of the present method in one- and two-dimensional problems, the extension of this method to three-dimensional Vlasov equation is simple and straightforward.

References


