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# On Functional Observer and State Feedback*

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Abstract

In this paper, we show the relation between state space approach and transfer function approach for functional observer and state feedback design. Two approaches can be transformed into each other, based on this result. More importantly, we find that the state space approach introduces some severe, unnecessary restrictions in solving the problem. The restrictions are, however, reduced to be a trivial condition in transfer function approach. It is believed that the result presented in this paper will be useful in developing both approaches, and motivate some new results for solving the problem.

Key Words:

Functional Observer; State Feedback; Lyapunov Matrix Equation; Minimal Realization.

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## I. Introduction

There are two approaches for functional observer and state feedback design, i.e., state-space approach [1-10] and transfer function approach [11-15]. Although the two approaches share many common points, e.g., they can achieve same design objects and require same order of observer-controller in order to achieve arbitrary state function and state feedback. The algorithms employed are very different. No apparent links between the algorithms of the two approaches have been found yet.

In this paper, we investigate this problem. The relations between two approaches will be found. Based on this result, two approaches can be transformed into each other. More importantly, we find that the state space approach introduces some severe, unnecessary restrictions in solving the problem The restrictions are, however, reduced to be a trivial condition in transfer function approach, which can be satisfied automatically in solving the equation. We believe that the insight provided in this paper will be beneficial to both approaches, and motivate some entirely new algorithms and results for solving the problem.

In sections II and III, we give a brief review for state space approach and transfer function approach, respectively. In section IV, we present a realization scheme, which is useful for the investigation. In section V, we give the results and a discussion. The algorithms to transform two approaches from each other are presented constructively, from which, the main result of this paper is cleared up. Finally, in section VI, we show an example.

## II. State Space Approach

The design of functional observer by state space approach has a long history, see [1-10]. We summarize the design briefly in the following.

Consider a plant G(s) with state space description

$$G(s) = C(sI-A)^{-1}B \tag{1}$$

The functional observer and state feedback can be shown in Figure 1, where all eigenvalues of $\bar{A}$ have negative real part. Let the required feedback state function be Fx, and the state function observed by $(sI-\bar{A})^{-1}(HB\ M^1)$ be $\bar{T}x$, as shown in Figure 1. The following equations can be written out directly.

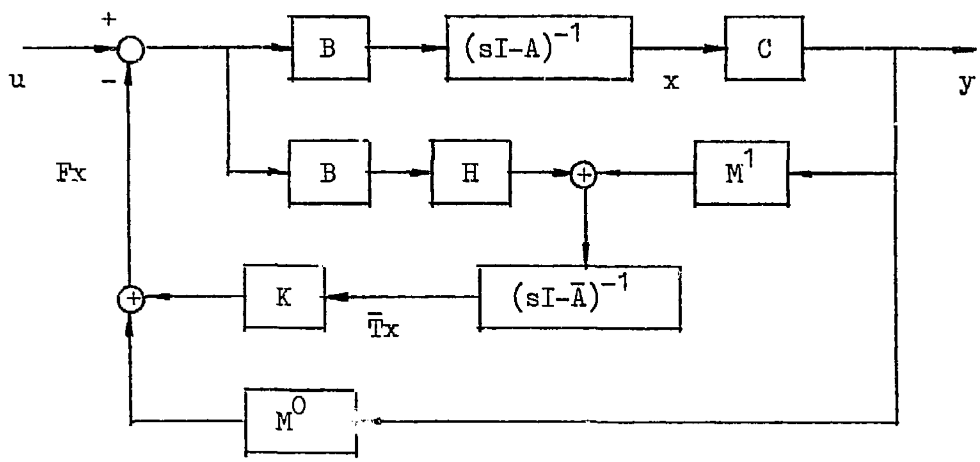


Fig. 1

$$F = K\bar{T} + M^0C \tag{2}$$

$$H(sI-A) + M^1C = (sI-\bar{A})\bar{T} \tag{3}$$

and

$$L = HB \tag{4}$$

where L is the matrix really used for the observer. To solve the equation (3), it needs obviously

$$H = \bar{T} \ . \tag{5}$$

Then, we have

$$\bar{T}A - \bar{A}\bar{T} = M^1 C \tag{3'}$$

and

$$L = \bar{T}B \ . \tag{4'}$$

The solving of equations (2, 3', 4') constructs of the design of functional observer by state space approach. The difficulty in solving these equations is apparent, especially for equation (3'). One of the well known methods in solving the problem needs 4 steps, as shown in the following:

1. Choose $\bar{A}$ such that all eigenvalues of $\bar{A}$ have negative real part and are distinct from those of A.

2. Choose an $M^1$ such that $\{\bar{A}, M^1\}$ is controllable.

3. Solve the $\bar{T}$ from $\bar{T}A - \bar{A}T = M^1 C$.

4. If the matrix $(C' \ \bar{T}')'$ has full column rank, compute (2) and (4'). Otherwise, go back to step 1 or step 2, try something else. □

In essential, the approach is trial and error. Some other algorithms are available. However, as long as the solving of equation (3') is necessary, there is no way to improve the algorithm dramatically.

## III. Transfer Function Approach

If we write the q x p plant (1) as

$$G(s) = N(s)D^{-1}(s)$$

where D(s) an N(s) are right coprime polynominal matrices, D(s) is column reduced with column degrees $\mu_2$, i = 1, 2,..., p. The functional observer and state feedback configuration can be shown in Figure 2, where $D_c(s)$ is Hurwitz.

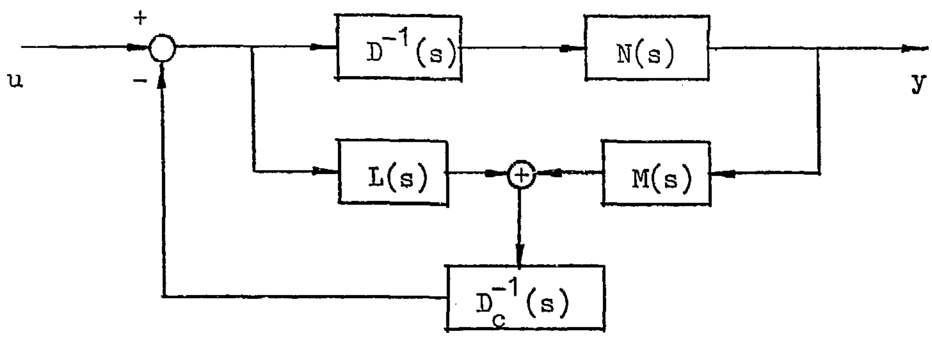


Fig. 2

Let the required feedback function be F(s), we only need to solve the following equation:

$$L(s)D(s) + M(s)N(s) = D_c(s)F(s) \tag{6}$$

The design was first proposed by Wolovich [11]. It is known that if $D_c(s)$ is chosen to be row reduced with all row degrees to be $\nu-1$, where $\nu$ is the observability index of G(s), and all column degrees of F(s) are less than the corresponding ones of D(s), there always exists a compensator $D_c^{-1}(s)[L(s)\ M(s)]$ to meet the equation (6). Furthermore, in general, $D_c^{-1}(s)L(s)$ is always strictly proper, $D_c^{-1}(s)M(s)$ is proper.

To build up a direct relation to state space approach, we write

$$M(s) = D_c(s)M^0 + M^1(s) \tag{7}$$

where $D_c^{-1}(s)M^1(s)$ is strictly proper. Also, we can write

$$F(s) = T(s) + M^0N(s) \tag{8}$$

and finally, we have

$$L(s)D(s) + M^1(s)N(s) = D_c(s)T(s) \tag{9}$$

which is simply from the equation (6)-(8), i.e., from

$$L(s)D(s) + M(s)N(s) = L(s)D(s) + D_c(s)M^0N(s) + M^1(s)N(s) \tag{10}$$

and

$$D_c(s)F(s) = D_c(s)T(s) + D_c(s)M^0N(s) \ . \tag{11}$$

Equating (10) and (11) yeilds (9).

It is interesting to note that the equation (9) is similar to the equation (3'). The development of the results of this paper will be mainly related to the comparison of the two equations. In order to do so, we need the realization shown in the next section.

## IV. Realization Scheme

The realization scheme presented in this section was introduced in [15]. Consider the right fraction of the q x p system $G(s) = N(s)D^{-1}(s)$, where $D(s)$ is column reduced with column degrees $\mu_i$, $i = 1, 2, \ldots, P$. Without loss of generality, we assume the column degree coefficient matrix of $D(s)$ to be identity I. Define

$$S_0(s) = \text{diag}\{s^{\mu_i}\};\ i = 1, 2, \ldots, P \tag{12}$$

and

$$S_j(s) = \mathrm{diag}\{s^{\mu_i+j}\} \tag{13}$$

where j is negative integers. Then, we can write

$$D(s) = S_0(s) + D_1S_{-1}(s) + D_2S_{-2}(s) + \ldots + D_\mu S_{-\mu}(s) \tag{14}$$

$$N(s) = N_1S_{-1}(s) + N_2S_{-2}(s) + \ldots + N_\mu S_{-\mu}(s) \tag{15}$$

where $\mu = \max\{\mu_i\}$. Thus, a controllable realization of G(s) is as follows:

$$A = \begin{bmatrix} 0 & I & & & \\ & 0 & I & & \\ & & \ddots & \ddots & \\ & & & & I \\ -D_\mu & -D_{\mu-1} & \cdots & & -D_1 \end{bmatrix} ; \; C = \left(N_\mu \; N_{\mu-1} \; \ldots \; \ldots \; N_1\right) ; \; B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} . \tag{16}$$

Note that the realization (16) is not minimal. However, the minimal realization can be obtained from (16) very easily. Since $\mu$ is the maximum of $\mu_i$'s, in $S_{-\mu}(s)$ and $S_{-\mu+1}(s)$ and so on, there exist some elements with negative s-powers. Of course, these elements are not with any physical meanings, and the corresponding columns of $D_\mu$, $N_\mu$ and $D_{\mu-1}$, $N_{\mu-1}$, and so on, are all zero. Thus, we can delete these columns in A, C, and at meanwhile, delete the same number of rows in A and B. The obtained realization is minimal, see [15]. It is of interest to notice, however, since these columns of $D_\mu$, $N_\mu$ and so on, are all zero, we can keep them in operation and matrix manipulations. In other words, the deletion can be delayed until the last step in order to take the advantage of the neat form of the realization [15].

For the dual factorization $G(s) = D^{-1}(s)N(s)$, we have the observable realization which is in the dual form of (15). Instead of column reducedness of $D(s)$, we require the row reducedness of $D(s)$ there.

## V. Functional Observer and State Feedback

In this section, we investigate the functional observer and state feedback designs by the state space approach shown in Figure 1 and the equations (2), (3'), (4') and the transfer function approach shown in Figure 2 and the equation (6). Some interesting remarks will be reached in concern of the designs.

We start from the state space approach. Consider the system shown in Figure 1. Without loss of any information, by similarity transformations, we can transform the state space description of the plant into the controllable realization form of (16), and transform the observer-controller into the dual form of (16). Let the controllability index and the observability index of $G(s)$ be $\mu$ and $\nu$, respectively. We write the state space description of the plant by (16), and the ones of the observer-controller by the following:

$$\bar{A} = \begin{bmatrix} 0 & & & -D_{c(\nu-1)} \\ I & 0 & & -D_{c(\nu-2)} \\ & I & \ddots & \vdots \\ & & \ddots & \vdots \\ & & I & -D_{c1} \end{bmatrix} ; \; H = \begin{bmatrix} H_{\nu-1} \\ H_{\nu-2} \\ \vdots \\ H_1 \end{bmatrix} ; \; M^1 = \begin{bmatrix} M^1_{\nu-1} \\ M^1_{\nu-2} \\ \vdots \\ M^1_1 \end{bmatrix} \qquad (17)$$

as well as

$$K = (0 \; \cdots\cdots \; 0 \; I) \; . \qquad (18)$$

Meanwhile, on the same basis as the one in (16), we write

$$\bar{T} = \left(\bar{T}_\mu \ \bar{T}_{\mu-1} \ \ldots\ldots \ T_1\right) . \tag{19}$$

Define

$$\bar{S}_0(s) = \mathrm{diag}\{s^{\nu_i}\}; \ i = 1, 2, \ldots, q \tag{20}$$

and

$$\bar{S}_j(s) = \mathrm{diag}\ \{s^{\nu_i + j}\} . \tag{21}$$

We write the equation (3) by the matrices A, C, $\bar{A}$, H, $M^1$ and $\bar{T}$ shown in the equation (16), (17), and (19). Multiplying both sides of the equation by $\left(\bar{S}_{-\nu+1}(s)\ \bar{S}_{-\nu+2}(s)\ \ldots\ \bar{S}_{-1}(s)\right)$, left, and $\left(S_{-\mu}(s)\ S_{-\mu+1}(s)\right.$ $\left.\ldots\ S_{-1}(s)\right)$, right, we have

$$\left(\bar{S}_{-\nu+1}(s)\ \ldots\ \bar{S}_{-1}(s)\right) \begin{bmatrix} H_{\nu-1} \\ N_{\nu-2} \\ \cdot \\ \cdot \\ \cdot \\ H_1 \end{bmatrix} \begin{bmatrix} sI & -I & & \\ & sI & -I & \\ & & \ddots & \ddots \\ & & & -I \\ D_\mu & D_{\mu-1} & & sI+D_1 \end{bmatrix} \begin{bmatrix} S_{-\mu}(s) \\ S_{-\mu+1}(s) \\ \cdot \\ \cdot \\ \cdot \\ S_{-1}(s) \end{bmatrix} +$$

$$+\left(\bar{S}_{-\nu+1}(s)\ \ldots\ \bar{S}_{-1}(s)\right) \begin{bmatrix} M^1_{\nu-1} \\ M^1_{\nu-2} \\ \cdot \\ \cdot \\ \cdot \\ M^1_1 \end{bmatrix} \left(N_\mu\ N_{\mu-1}\ \ldots\ N_1\right) \begin{bmatrix} S_{-\mu}(s) \\ S_{-\mu+1}(s) \\ \cdot \\ \cdot \\ \cdot \\ S_{-1}(s) \end{bmatrix} =$$

$$=\left(\bar{S}_{-\nu+1}(s)\ \dots\ \bar{S}_{-1}(s)\right)\begin{bmatrix} sI & & & D_{c(\nu-1)} \\ -I & sI & & D_{c(\nu-2)} \\ & -I & \ddots & \vdots \\ & & \ddots & \vdots \\ & & -I & sI+D_{c1} \end{bmatrix}\left(\bar{T}_{\mu}\ \bar{T}_{\mu-1}\dots\ \bar{T}_{1}\right)\begin{bmatrix} S_{-\mu}(s) \\ S_{-\mu+1}(s) \\ \vdots \\ S_{-1}(s) \end{bmatrix}. \tag{22}$$

Considering the property of $\bar{S}_i(S)$'s and $S_i(s)$'s, the equation (22) can be easily written as

$$\left(\bar{H}^2(s)\,\middle|\,\bar{H}^1(s)\right)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \cdots \\ D(s) \end{bmatrix} + M^1(s)N(s) = \left(0\ \dots\ 0\,\middle|\,D_c(s)\right)\begin{bmatrix} \bar{T}^2(s) \\ \cdots\cdots \\ \bar{T}^1(s) \end{bmatrix} \tag{23}$$

and hence

$$\bar{H}^1(s)D(s) + M^1(s)N(s) = D_c(s)\bar{T}^1(s) . \tag{24}$$

Referring to the equation (9), it is clear

$$\bar{H}^1(s) = L(s) \tag{25}$$

$$\bar{T}^1(s) = T(s) . \tag{26}$$

Thus, we have constructed the equation (9) from the equation (3), and two approaches have been linked with each other.

In the following, we show an important observation from (23).

<u>Observation</u>: Consider the equation (23). The submatrices $\bar{H}^2(s)$ and $\bar{T}^2(s)$ have not been used in solving the functional obsever problem. $\square$

Note that $H^2(s)$ is represented by, in fact, the left part of the matrix $(H'_{\nu-1} \dots H'_1)'$ in (22), as well as the left part of H in (3), and $\bar{T}^2(s)$ is represented by, in fact, the upper part of the matrix $(\bar{T}_\mu \dots \bar{T}_1)$ in (22), as well as the upper part of $\bar{T}$ in (3). This means that using the equation (2), (3'), and (4') to solve the observer problem introduces unnecessary restrictions. This is the reason why state space approach to functional observer and state feedback cannot be as simple and natural as the transfer function approach. We state the result as a proposition.

Proposition 1: In solving the functional observer and state feedback problem, the state space approach presented in section II introduces unnecessary restrictions, i.e., the equations (5) and (3'). □

Note that, if we use the equation (2), (3'), and (4') to solve the problem, we do need the equation (5) and (3'), the restrictions. However, the restriction is avoidable in solving the functional observer problem, as can be seen from the transfer function approach (6). In fact, we have the following result.

Proposition 2: The severe restriction $H = \bar{T}$, the equation (5), in solving the observer problem by state space approach is reduced to a trivial restriction in transfer function approach. □

Proof: We write the matrix $(H'_{\nu-1} \dots H'_1)'$ in (22) as

$$H = \begin{bmatrix} H_{\nu-1} \\ H_{\nu-2} \\ \vdots \\ H_1 \end{bmatrix} = \left[\begin{array}{ccc|c} H_{\nu-1,\mu} & \cdots & H_{\nu-1,2} & H_{\nu-1,1} \\ H_{\nu-2,\mu} & \cdots & H_{\nu-2,2} & H_{\nu-2,1} \\ \vdots & & \vdots & \vdots \\ \hline H_{1,\mu} & \cdots & H_{1,2} & H_{1,1} \end{array}\right] \tag{27}$$

and the matrix $(\bar{T}_\mu \; \bar{T}_{\mu-1} \; \cdots \; \bar{T}_1)$ in (22) as

$$\bar{T} = (\bar{T}_\mu \; \bar{T}_{\mu-1} \cdots \bar{T}_1) = \left[\begin{array}{ccc|c} \bar{T}_{\nu-1,\mu} & \cdots & \bar{T}_{\nu-1,2} & \bar{T}_{\nu-1,1} \\ \bar{T}_{\nu-2,\mu} & \cdots & \bar{T}_{\nu-2,2} & \bar{T}_{\nu-2,1} \\ \vdots & & \vdots & \vdots \\ \hline \bar{T}_{1,\mu} & \cdots & \bar{T}_{1,2} & \bar{T}_{1,1} \end{array}\right]. \tag{28}$$

In solving (24), we can have the solution

$$\left[\begin{array}{c} H_{\nu-1,1} \\ \vdots \\ H_{2,1} \\ \hline H_{1,1} \end{array}\right] \tag{29}$$

and

$$\left(\bar{T}_{1,\mu} \cdots \bar{T}_{1,2} \,\middle|\, \bar{T}_{1,1}\right) \tag{30}$$

which represent $H'(s)$ and $\bar{T}'(s)$ in (24), respectively. Referring to (27) and (28), we can find that $H'(s)$ and $\bar{T}'(s)$ are represented by the right side block in H and the bottom block in $\bar{T}$, respectively, which says that the only restriction to meet is

$$H_{1,1} = \bar{T}_{1,1} \; . \tag{31}$$

The meaning of (31) is that row degree coefficient matrix of L(s) in (9) must be equal to the column degree coefficient matrix of $\bar{T}(s)$ in (9), since we assume that the column degree coefficient matrix of D(s) and row degree coefficient matrix of $D_c(s)$ are identities. This restriction, however, is really trivial, and more importantly, it can be met automatically in solving (9) on (6). □

In the following, we give a discussion for the result by a number of remarks.

Remark 1: Once the matrices (29) and (30) are obtained by solving the equation (24) or (9), we can assign the rest part of H and $\bar{T}$ in (27) and (28) very easily in order to meet the requirement (5). Thus, the equations (2), (3') and (4'), of state space approach, can be solved through the transfer function approach.

Considering (27) and (28), in order to ensure $H = \bar{T}$, we simply let

$$\left(H_{1,\mu} \cdots H_{1,2}\right) = \left(\bar{T}_{1,\mu} \cdots \bar{T}_{1,2}\right) \tag{32}$$

and

$$\begin{bmatrix} \bar{T}_{\nu-1,1} \\ \cdot \\ \cdot \\ \cdot \\ \bar{T}_{2,1} \end{bmatrix} = \begin{bmatrix} H_{\nu-1,1} \\ \cdot \\ \cdot \\ \cdot \\ H_{2,1} \end{bmatrix}. \tag{33}$$

The right sides of (32) and (33) are known from (29) and (30).

The rest part of H and $\bar{T}$ can be directly solved by taking the advantage of the special structure of corresponding submatrices in (22). In particular, we have

$$
-\begin{bmatrix} H_{\nu-1,\mu} & \cdots & H_{\nu-1,2} \\ H_{\nu-2,\mu} & \cdots & H_{\nu-2,2} \\ \vdots & & \vdots \\ H_{2,\mu} & \cdots & H_{2,2} \end{bmatrix} + \begin{bmatrix} H_{\nu-1,1} \\ H_{\nu-2,1} \\ \vdots \\ H_{2,1} \end{bmatrix} (D_{\mu-1} \cdots D_1) + \begin{bmatrix} M^1_{\nu-1} \\ M^1_{\nu-2} \\ \vdots \\ M^1_2 \end{bmatrix} (N_{\mu-1} \cdots N_1)
$$

$$
= \begin{bmatrix} D_{c(\nu-1)} \\ D_{c(\nu-2)} \\ \vdots \\ D_{c2} \end{bmatrix} (\bar{T}_{1,\mu-1} \cdots \bar{T}_{1,2}\ \bar{T}_{1,1}) - \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \bar{T}_{\nu-1,\mu-1} & \bar{T}_{\nu-1,\mu-2} & \cdots & \bar{T}_{\nu-1,1} \\ \bar{T}_{\nu-2,\mu-1} & \bar{T}_{\nu-2,\mu-2} & \cdots & \bar{T}_{\nu-2,1} \\ \vdots & \vdots & & \vdots \\ \bar{T}_{3,\mu-1} & \bar{T}_{3,\mu-2} & \cdots & \bar{T}_{3,1} \end{bmatrix} \quad (34)
$$

Note that all matrices in (34) are known, except the first one on the left side, i.e., a part of H in (22), and the last one on the right side, i.e., a part of $\bar{T}$ in (22). We also note that we should have $H = \bar{T}$. Thus, the equation (34) can be solved row by row. After solving the first row of (34), we obtain

$$
(H_{\nu-1,\mu}\ H_{\nu-1,\mu-1} \cdots H_{\nu-1,2}) = (\bar{T}_{\nu-1,\mu},\ \bar{T}_{\nu-1,\mu-1} \cdots \bar{T}_{\nu-1,2})
$$

Then, a part of the matrix on the right side of the equation, i.e., $(\bar{T}_{\nu-1,\mu-1} \cdots \bar{T}_{\nu-1,2})$, along with (33), provide the necessary conditions in solving the second row, and so forth. This iterative algorithm provides a simple method to the design of state space approach. □

<u>Remark 2</u>: The reason why the state space approach introduces unnecessary restrictions is that it does not take into account of the fact that the number of rows in B is larger, or much larger, than the rank of B; and so for the matrix

K, see Figure 1. In fact, this advantage could and should be used in solving the problem, as can be seen in transfer function approach, see (23), (24). □

Remark 3: Due to the unnecessary restriction, the state space approach solve the equation (3) in an unusual way, see Section II. It seems that $\bar{T}$ should be assigned and $M^1$ should be solved, however, instead, in state space approach, we have to choose an $M^1$ and solve $\bar{T}$. This problem has been straightened in transfer function approach (6), where the matrices $D_c(s)$ and $\bar{T}(s)$ can be chosen, then, we solve $L(s)$ and $M^1(s)$. □

Remark 4: We note that in transfer function approach, we need not to solve the equations (7-9). The design can be accomplished in one step procedure, i.e., to solve the equation (6) directly. Since $D_c^{-1}(s)\, M(s)$ in general is proper, the constant part of the observer-controller included automatically in the procedure. □

Remark 5: It is therefore not surprising that for both approaches the orders of the observer-controllers for arbitrarily assigning the pole of the observer and the state function are all equal to $p(\nu-1)$ [10,12]. Recently, some algorithms for designing lower order observers are proposed by the state space approach [10]. Similar result is also available by transfer function approach, where, the algorithm is much simpler, it is achieved simply by a search of the resultant of $D(s)$ and $N(s)$. □

## VI. Example

The following example is shown to illustrate the result of section V. We solve the functional observer and state feedback problem by transfer function approach (6). Then, by using the realization of section IV. We translate the result in the state space form of (2), (3') and (4').

Consider the 2 x 2 plant

$$G(s) = N(s)D^{-1}(s) = \begin{bmatrix} 0 & s-1 \\ 1 & s+1 \end{bmatrix} \begin{bmatrix} s^2+1 & s \\ s & s^3+1 \end{bmatrix}^{-1}$$

where D(s) and N(s) are right coprime, D(s) is column reduced with column degree coefficient matrix I, and $\mu_1=2$, $\mu_2=3$. It can be found that the observability index of G(s), $\nu$, is 3. We choose

$$D_c(s) = \begin{bmatrix} s^2+s+1 & \\ & s^2+s+1 \end{bmatrix}$$

and

$$F(s) = \begin{bmatrix} 2s & -s \\ -s & 3s^2+3s \end{bmatrix}.$$

Solve the equation (6), we have

$$L(s) = \begin{bmatrix} 2s+3 & 0 \\ -s-7 & 3s+3 \end{bmatrix}$$

$$M(s) = \begin{bmatrix} -2s-3 & -s^2-3 \\ 7s+10 & 3s^2-3s+7 \end{bmatrix}.$$

From (7), it is clear,

$$M^0 = \begin{bmatrix} 0 & -1 \\ 0 & 3 \end{bmatrix}$$

and

$$M^1(s) = \begin{bmatrix} -2s-3 & s-2 \\ 7s+10 & -6s+4 \end{bmatrix}.$$

From (8), we have

$$T(s) = \begin{bmatrix} 2s+1 & 1 \\ -s-3 & 3s^2-3 \end{bmatrix}.$$

Following the equation (14) and (15), we write

$$D(s) = \begin{bmatrix} s^2 & \\ & s^3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{D_1}\begin{bmatrix} s & \\ & s^2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{D_2}\begin{bmatrix} 1 & \\ & s \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{D_3}\begin{bmatrix} s^{-1} & \\ & 1 \end{bmatrix}$$

$$N(s) = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{N_1}\begin{bmatrix} s & \\ & s^2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{N_2}\begin{bmatrix} 1 & \\ & s \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}}_{N_3}\begin{bmatrix} s^{-1} & \\ & 1 \end{bmatrix}$$

and

$$D_c(s) = \begin{bmatrix} s^2 & \\ & s^2 \end{bmatrix} + \begin{bmatrix} s & \\ & s \end{bmatrix}\underbrace{\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}}_{D_{c1}} + \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}}_{D_{c2}}$$

$$L(s) = \begin{bmatrix} s & \\ & s \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}}_{L_1} + \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 3 & 0 \\ -7 & 3 \end{bmatrix}}_{L_2}$$

$$M^1(s) = \begin{bmatrix} s & \\ & s \end{bmatrix} \underbrace{\begin{bmatrix} -2 & 1 \\ 7 & -6 \end{bmatrix}}_{M_1^1} + \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -3 & -2 \\ 10 & 4 \end{bmatrix}}_{M_2^1} .$$

Then, we can write all matrices in Figure 1.

$$A = \begin{array}{c} \left[\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ \hline & & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \end{array}\right] \\ \;\; -D_3 \qquad\; -D_2 \qquad\; -D_1 \end{array} ; \quad C = \begin{array}{c} \left[\begin{array}{cc|cc|cc} 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \\ N_3 \qquad N_2 \qquad N_1 \end{array} ; \quad B = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right]$$

and

$$\bar{A} = \begin{array}{cc} \left[\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right] & \begin{array}{c} D_{c2} \\ \\ \\ D_{c1} \end{array} \end{array} ; \quad H = \begin{array}{cc} \left[\begin{array}{c|cc} & 3 & 0 \\ {}_1H & -7 & 3 \\ \hline & 2 & 0 \\ {}_2H & -1 & 3 \end{array}\right] & \begin{array}{c} H_2 \\ \\ \\ H_1 \end{array} \end{array} ; \quad M^1 = \begin{array}{cc} \left[\begin{array}{cc} -3 & -2 \\ 10 & 4 \\ \hline -2 & 1 \\ 7 & -6 \end{array}\right] & \begin{array}{c} M_2^1 \\ \\ \\ M_1^1 \end{array} \end{array}$$

$$T = \begin{array}{c} \left[\begin{array}{cccc|cc} & {}_1\bar{T} & & & {}_2\bar{T} & \\ \hline 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & -3 & -3 & 0 & -1 & 3 \end{array}\right] \\ \bar{T}_3 \qquad \bar{T}_2 \qquad \bar{T}_1 \end{array} ; \quad K = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

By using (32), (33), we have

$$ {}_2H = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & -3 & -3 & 0 \end{bmatrix} $$

$$ {}_2\bar{T} = \begin{bmatrix} 3 & 0 \\ -7 & 3 \end{bmatrix} . $$

By using (34), we have

$$ {}_1H = {}_1\bar{T} = H_{2,2}\,(D_2\ D_1) + M_2^1\,(N_2\ N_1) - (\bar{T}_2\ \bar{T}_1) $$

$$ = \begin{bmatrix} 3 & 0 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ 10 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 2 & 0 \\ -3 & 0 & -1 & 3 \end{bmatrix} $$

$$ = \begin{bmatrix} 0 & -2 & -2 & 0 \\ 0 & 7 & 4 & -3 \end{bmatrix} . $$

Thus, we have

$$ H = \bar{T} = \begin{bmatrix} 0 & -2 & -2 & 0 & 3 & 0 \\ 0 & 7 & 4 & -3 & -7 & 3 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & -3 & -3 & 0 & -1 & 3 \end{bmatrix} \qquad \square $$

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