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DIGITAL GENERATION OF STOCHASTIC SIGNALS OF ARBITRARY SPECTRAL SHAPE

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Abstract—For computer-simulation experiments in the development of noise monitoring systems three methods of generating ergodic Gaussian random noise with specified spectral properties have been investigated: digital filtering of white noise with optimum symmetric FIR filters, a modified Rice formulation and an approximation of the Kac representation of noise. The proposed modified Rice formulation is a new noise generation method which is most efficient with regard to the computation time. By windowing subsequent Rice sequences a smooth noise record of any desired length can be produced.

1. INTRODUCTION

Reactor-noise analysis is now a well-established field in nuclear science and technology. It has shown important developments and applications during the past few years. One of the aims of application is the detection and diagnosis of anomalies. Since reactor-noise techniques are non-intrusive, which enables noise measurements without any constraints on the reactor operation, there is a clear trend to develop on-line monitoring systems combined with automated computer-supported diagnosis in the real-time analysis mode (Bernard *et al.*, 1985). The use of pattern-recognition techniques and statistical decision algorithms is an interesting area in the application. The required software is at first developed more economically in a higher programming language and then checked on a large-size central computer before it is adapted and implemented in the minicomputer- or microprocessor-supported monitoring system. In such development work of analysis methods, the need for computer-generated noise signals that simulate both normal and anomalous reactor process signals is strongly felt.

There are several reasons for starting method development with artificial noise signals. If one has a signal record available from an actual measurement, it is ordinarily stored on an analogue tape. The digitizing procedure (after appropriate low-pass filtering of the analogue signal) requires an intermediate storage on a digital tape, since central computers are mostly batch operated and not equipped with an A/D converter for real-time data input. The number of available and interesting actual cases containing signals from normal and especially anomalous reactor conditions is ordinarily very small and not sufficient to cover sensitivity studies or to explore fully the features of different analysis methods. On the other hand, many anomalous cases can be simulated by artificial noise in a completely reproducible way. We consider the problem of noise generation from the purely phenomenological view. The generated noise should represent specified spectral characteristics which are estimated using cases of actual measurements. Our problem is quite different in other aspects, like modelling stochastic processes which require deep knowledge or assumptions of the physics involved or solving a mathematical problem by the use of noise which is equivalent to the application of Monte Carlo methods.

In the next section, the problem will be defined in detail. Three noise generation procedures, digital filtering of white noise, a modified Rice formulation and a treatment based on an apparently deterministic formula, have been investigated and are reported in Sections 3–5. In Section 6, the results and experience are summarized. For the investigation, a FORTRAN program package was written which includes several analysis and test procedures on the quality of the generated noise.

2. DEFINITION OF THE PROBLEM

The procedure for generating artificial noise cannot be considered independently from the analysis and evaluation procedures which are provided for the

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development of a noise monitoring and diagnostic system with a specified purpose. The generation algorithm must obviously include all features of the actual noise which are relevant or have influence through the analysis chain up to the final results. It must be efficient, since record lengths of up to 10^6 data points are sometimes required.

We are concerned with univariate Fourier-transform analysis for power spectral density (PSD) pattern-recognition techniques in the frequency domain. The problem is associated with the development of an on-line neutron-noise monitoring system to detect the onset of nucleate boiling at our swimming pool reactor SAPHIR (Behringer *et al.*, 1985) using the first five discriminants of Piety's PSD pattern-recognition algorithm (Piety, 1977). Recent studies show that a significant reduction in the response time due to the appearance of an anomaly can be obtained, if overlap techniques in the PSD estimation are used.

The stochastic signal to be generated should have the following properties.

(1) The stochastic signal should by definition be a random noise signal. It should not contain deterministic (periodic) components. Peaks in the PSD are assumed to be produced by narrow-band random data.

(2) The sequence of digital data points x_n should represent equidistantly sampled values from a signal $x(t)$ being continuous in time. This signal $x(t)$ is assumed to be the output from a linear low-pass filter and must not contain frequency components above the Nyquist cutoff frequency. As in a hardwired fast fourier transform (FFT) analyser, the generation procedure of x_n must start at the digitizer after the input signal has passed the anti-aliasing filter.

(3) The amplitude distribution should be Gaussian with zero mean.

(4) The PSD of the signal should fairly well reproduce a specified PSD or target PSD, which is a positive-valued input function given at the frequency points f_k. It is denoted by $\mathrm{PSD}_k^{(SP)}$. It is zero for $k = 0$ (the d.c. signal is assumed to be zero) and $k = K$, where f_K is the Nyquist cutoff frequency point. From this PSD function we have to distinguish several other PSD functions. In the analysis, the estimation procedure of a PSD is ordinarily performed by signal segmentation. The FFT procedure is applied to a sample of N signal points which we call an N-point transform size assuming that N is equal to a power of 2, and an instantaneous $\mathrm{PSD}_k^{(m)}$ is calculated. From a series of M subsequent instantaneous PSDs the estimated PSD, denoted by $\mathrm{PSD}_k^{(ES)}$, is obtained by using ensemble averaging:

$$\mathrm{PSD}_k^{(ES)} = \frac{1}{M}\sum_{m=1}^{M} \mathrm{PSD}_k^{(m)}; \quad k = 0, 1, \ldots, K = \frac{N}{2}. \tag{1}$$

Thus, if the frequency range and the number of points for the required frequency resolution of the target PSD are given, the transform size for the analysis of the generated noise signal is a fixed parameter.

We define the expected PSD, denoted by $\mathrm{PSD}_k^{(EX)}$, as

$$\mathrm{PSD}_k^{(EX)} = \lim_{M \to \infty} \mathrm{PSD}_k^{(ES)}. \tag{2}$$

The question arises immediately whether the condition

$$\mathrm{PSD}_k^{(EX)} = \mathrm{PSD}_k^{(SP)} \tag{3}$$

can be fulfilled. In most cases the expected PSD is unknown because a theoretical expression cannot be obtained and, in general, only an approach to equation (3) is possible. It should be noted that the estimated or expected PSD depends upon the analysis procedure itself, e.g. if a smoothing window function is used. In our analysis of the artificially generated noise data, we use the boxcar window exclusively which is equivalent to no windowing other than selecting a segment of N data points, or to a uniform weighting of the data points in a segment.

(5) The generated noise signal should be ergodic. This requirement is tacitly assumed for equations (1)–(3). However, a monitoring system for the detection of anomalies is concerned with non-stationary noise signals due to changing process conditions. In principle, one uses analysis methods which have their mathematical foundation in the assumption of the ergodicity hypothesis. The decision as to whether the chosen evaluation method will work under non-stationary signal conditions or not, is best made heuristically. The simulation of non-stationary noise can easily be performed for such cases where the appearance of an anomaly may be assumed to be due to the onset of a new process. Two ergodic noise records must be generated, $x(t)$ with a target PSD representing normal conditions (or whatever is assumed to be normal conditions) and $y(t)$ with a target PSD representing the additional anomalous conditions. The non-stationary signal record $s(t)$ is then obtained, for example, by

$$\begin{aligned} s(t) &= x(t) && 0 \leqslant t < t_0 \\ &= x(t) + A(t)y(t) && t \geqslant t_0, \end{aligned} \tag{4}$$

where $A(t)$ may be a deterministic function representing a slow growth of the record $y(t)$. In the digital treatment, this approach requires only that $A(t)$ is a function sufficiently slowly varying in time in order to avoid excessive production of frequency components above the Nyquist cutoff frequency. In an actual measurement this requirement is automatically fulfilled by the preceeding anti-aliasing filter.

(6) As far as possible, the noise generation procedure should fulfil the requirement that the frequency points of the estimated PSD are statistically independent, i.e.

$$\text{cov}\,(\text{PSD}_k^{(ES)}, \text{PSD}_{k'}^{(ES)}) \cong 0 \quad \text{for} \quad k \neq k', \qquad (5)$$

and each frequency point follows the χ^2-sampling distribution

$$\frac{\text{PSD}_k^{(ES)}}{\text{PSD}_k^{(EX)}} \cong \frac{\chi^2_{2M}}{2M}. \qquad (6)$$

The condition imposed by equation (5) excludes the use of window functions other than the boxcar function in the analysis (Jenkins and Watts, 1968) and refers to the estimation procedure without overlapped segmentation. The statistical behaviour of the estimated PSD is important for the statistics of the discriminants in PSD pattern-recognition algorithms.

3. DIGITAL FILTERING OF WHITE NOISE

The representation of coloured noise by linearly filtered white noise is a usual procedure in the textbooks (e.g. Bendat, 1958; Jenkins and Watts, 1968). It characterizes a class of noise which fits the requirements specified in the previous section. We feel that a comment is sufficient.

The series of Gaussian random numbers g_0, g_1, g_2, . . . with the expectation values $\langle g_\mu \rangle = 0$ and $\langle g_\mu g_\nu \rangle = \delta_{\mu\nu}$ where $\delta_{\mu\nu}$ is the Kronecker symbol, represents data points of band-limited Gaussian white noise with zero mean and unity variance. If one assumes that the data have been sampled with the time interval Δt, the cutoff frequency F_C is $1/2\Delta t$. Codes exist for the generation of Gaussian pseudorandom numbers. We used a subroutine which is available from the IMSL library.

There are two types of digital-filtering techniques (which of course can also be mixed): recursive and non-recursive (Uhrig, 1970; Stearns, 1975).† They correspond in the noise analysis by autoregressive methods to the AR-modelling and the MA-modelling, respectively (Box and Jenkins, 1976). The transfer function of recursive or feedback filters shows, in general, non-linear phase behaviour. Phase considerations are unimportant here because the PSD represents the correlation of signal frequency components at the same frequency. They become relevant when it is desired to extend the simulation to multivariate analysis. Recursive filtering is an efficient method. But the design of such filters suffers from stability problems which do not appear in non-recursive or finite impulse response (FIR) filters (Lacroix, 1980; Azizi, 1981). Among the FIR filters there is a large class with symmetric filter coefficients. It shows a transfer function with linear phase. The slope of the phase line depends only on the length of the filter or the number L of the filter coefficients, respectively. If L is an odd number, one can make an address shift so that no phase shift exists between the new data record and the original data record to be filtered. In this case, the transfer function from which the filter coefficients are calculated is real-valued and can be set equal to $\sqrt{\text{PSD}^{(SP)}}$. An attempt was made to use this method. It is well-known that non-recursive filtering usually requires a large number L (Bendat and Piersol, 1971). But nevertheless, significant truncation errors were observed between the $\text{PSD}^{(ES)}$ and $\text{PSD}^{(SP)}$ if the latter includes steep transitions. This method of filtering cannot be considered to be useful in generating noise for a highly structured PSD. In particular, it is not practicable for a $\text{PSD}^{(SP)}$ containing sharp peaks.

However, useful and efficient codes exist for designing simple or multiple band-pass filters which allow the generation of noise for flat or smoothly structured PSDs. The $\text{PSD}^{(SP)}$ is no longer an input function but is roughly approximated by the input parameters defining the order and shape of the filter. We used the code developed by McClellan *et al.* (1973) for designing symmetric FIR filters. The code outputs filter coefficients optimized on the Tchebycheff approximation. For such cases, good results with respect to the required noise quality were obtained; in particular, with respect to the requirements of equations (5) and (6). The quality test procedures which were applied are noted in the next section.

4. RICE FORMULATION OF RANDOM NOISE AND ITS MODIFICATION

According to Rice (1944, 1945) an ensemble member of normal random noise can be represented by a Fourier series. Each member of the ensemble is

† There is a tremendous expanse of textbooks and papers about digital-filtering techniques; we must confine our citations to a limited selection.

define in a finite time span T. In the digital version, we write for a member or sequence containing N data points sampled with the time increment $\Delta t = T/N$,

$$x_n = \sum_{k=1}^{N/2-1}\left[a_k\cos\left(\frac{2\pi kn}{N}\right)+b_k\sin\left(\frac{2\pi kn}{N}\right)\right];\quad n = 0, 1, \ldots, N-1. \qquad (7)$$

The amplitude coefficients a_k and b_k, two in each Nyquist co-interval $\Delta f = 1/T$, are assumed to be independent and normally distributed random variables. They have the ensemble averages

$$\left.\begin{aligned}\langle a_k\rangle &= \langle b_k\rangle = 0\\ \langle a_k a_{k'}\rangle &= \langle b_k b_{k'}\rangle = \sigma_k^2\delta_{kk'}\\ \langle a_k b_{k'}\rangle &= 0;\end{aligned}\right\} \qquad (8)$$

σ_k^2 is the variance of the signal components at the centre frequency f_k within Δf, and is related to the expected (two-sided) PSD by

$$\sigma_k^2 = 2\Delta f\,\mathrm{PSD}_k^{(\mathrm{EX})}. \qquad (9)$$

It should be noted that according to a remark by Bendat (1958) the Rice representation is strictly valid only for the limiting case of white noise. For all other cases, the coefficients a_k and b_k are only approximately independent, the degree of independence being determined by the correlation function of the signal. But the use of the Rice representation in general is deemed to be not only convenient but appropriate as well.

If one makes the variance of the contribution of a signal component at a given frequency proportional to the amplitude of the target PSD at that frequency, by selecting the coefficients

$$a_k = Ag_k\sqrt{\mathrm{PSD}_k^{(\mathrm{SP})}} \quad \text{and} \quad b_k = Ag'_k\sqrt{\mathrm{PSD}_k^{(\mathrm{SP})}} \qquad (10)$$

where g_k and g'_k are independent Gaussian random numbers with zero mean and unity variance, a sequence of noise data with any specified colour is obtained. A is a signal amplitude normalizing constant which, for convenience, can be chosen as

$$A = \frac{1}{\sqrt{\sum_k \mathrm{PSD}_k^{(\mathrm{SP})}}} \qquad (11)$$

to achieve unit variance of the signal. Equation (10) defines at most $N/2-1$ occupied frequency components. This has been considered in writing equation (7).

The procedure has been used by Smith and Williams (1972) to develop a stochastic noise generator and to demonstrate its utility for solving stochastic mathematical problems.

The Rice representation is based on ensemble-averaging techniques. Equation (7) represents one sequence of data points whose size must correspond to the transform size used in the analysis. Its main advantage is that such a sequence can quickly be produced by applying the inverse FFT procedure to the random numbers a_k and b_k. If one attempts to utilize such a series in an analysis with an arbitrary starting point other than $n = 0$, there is a crucial point which concerns the problem of continuation, since a Fourier series repeats periodically. Also a record of desired length is not obtainable simply by joining together sequences each subsequently produced with a new set of random numbers because such record does not correspond to the digitized image from a stochastic signal being continuous in time. If one scans such a record in the same manner as it has been produced, it is a trivial conclusion that one must get back exactly what has been put in, i.e. $\mathrm{PSD}^{(\mathrm{EX})}$ must be equal to $\mathrm{PSD}^{(\mathrm{SP})}$. On the other hand, if one starts scanning such sequences simply joined together at a data point x_{n_0} which is not the first in the first sequence, significant leakage occurs and side lobes arise in the $\mathrm{PSD}^{(\mathrm{ES})}$. The $\mathrm{PSD}^{(\mathrm{EX})}$ can be calculated, and reads

$$\mathrm{PSD}_k^{(\mathrm{EX})} = \frac{T}{4}\sigma_k^2\left[1+\left(1-\frac{2n_0}{N}\right)^2\right] + \frac{\Delta t}{N}\left[\sum_{\substack{l=1\\ l\neq k}}^{N/2-1}\sigma_l^2\frac{\sin^2\left(\frac{\pi(k-l)n_0}{N}\right)}{\sin^2\left(\frac{\pi(k-l)}{N}\right)} + \sum_{l=1}^{N/2-1}\sigma_l^2\frac{\sin^2\left(\frac{\pi(k+l)n_0}{N}\right)}{\sin^2\left(\frac{\pi(k+l)}{N}\right)}\right]. \qquad (12)$$

Equation (12) is symmetric around $n_0 = N/2$ if one replaces n_0 by $N-n_0$. The worst case showing the largest difference between $\mathrm{PSD}^{(\mathrm{EX})}$ and $\mathrm{PSD}^{(\mathrm{SP})}$ appears when scanning the segments with a 50% time shift, i.e. with $n_0 = N/2$. Each expected spectral point suffers from a 50% leakage in the main lobe, while the other 50% of power is distributed in the side lobes. This is best seen when the target PSD is a single-frequency spectrum and a small transform size is used in the generation and analysis of the noise record. Figure 1 shows such a case of a retrieved PSD. The points connected by dashed lines are the $\mathrm{PSD}^{(\mathrm{SP})}$ or the $\mathrm{PSD}^{(\mathrm{EX})}$ for $n_0 = 0$; the points connected by solid lines represent the $\mathrm{PSD}^{(\mathrm{ES})}$ for $n_0 = N/2$. The values obtained for the $\mathrm{PSD}^{(\mathrm{ES})}$ approach very closely (within

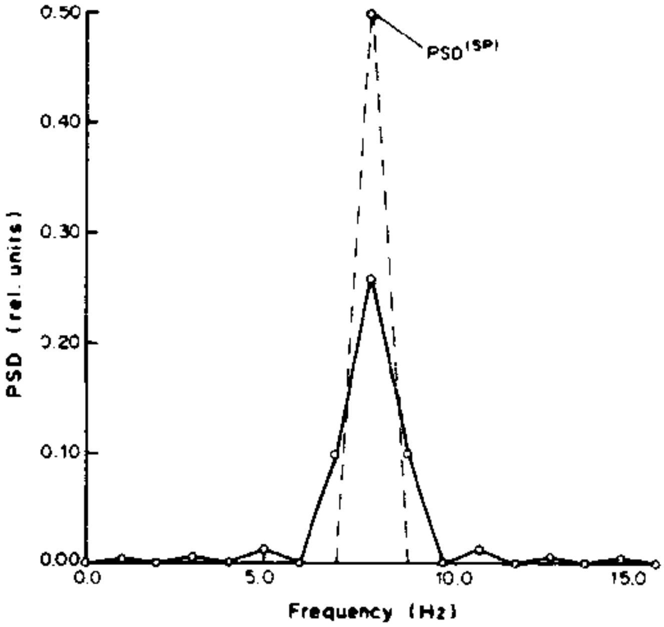

Fig. 1. Retrieved PSD: target PSD = single-frequency spectrum (8 Hz); transform size = 32 points; generation procedure = no window; analysis procedure = 1200 averages, 50% time shift.

the statistical limits) the theoretical values of the $PSD^{(EX)}$ when one calculates them from equation (12). For convenience, in this and the following plots of numerical examples, values of subsequent spectral points are connected by straight lines and the frequency range has been normalized, assuming a Nyquist cut-off frequency at 16 Hz. Another case is shown in Fig. 2 where a larger transform size is used. The noise

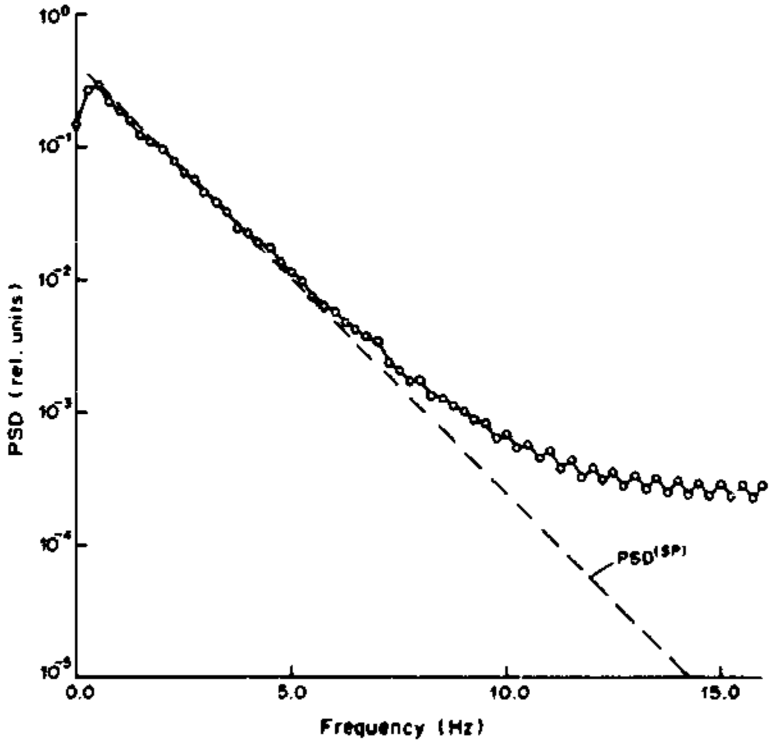

Fig. 2. Retrieved PSD: target PSD = exponentially decaying spectrum 0.25–15.75 Hz, 5 decades; transform size = 128 points; generation procedure = no window; analysis procedure = 600 averages, 50% time shift.

record is generated with an exponentially shaped target PSD specified for 50 dB decay from 0.25 to 15.75 Hz. The tail of the estimated PSD retrieved by scanning the segments again with a 50% time shift is markedly influenced by the side lobes. In general, the side lobes which appear can be considered as a background confining the spectral amplitude range in the reproduction of the target PSD, in particular feigning the presence of a d.c. component in the signal, and introducing appreciable correlations between neighbouring spectral points.

A noise generation procedure where the retrieval of the noise signatures strongly depends upon the starting point in the analysis is not a useful method, in particular, it is not suitable for performing analyses with overlapped segmentation. Consider whether two subsequent Rice sequences, m and $m+1$, could be smoothly joined together by overlapping them at the last and first points. If one imposes the conditions

$$x_0^{(m+1)} = x_{N-1}^{(m)} \quad \text{and} \quad \dot{x}_0^{(m+1)} = \dot{x}_{N-1}^{(m)} \tag{13}$$

two variables in the sets $a_k^{(m+1)}$ and $b_k^{(m+1)}$ to be used for generating the Rice sequence $m+1$ are bounded. They could, for example, be randomly selected in the frequency range where the $\text{PSD}^{(SP)}$ has large amplitudes, and values must then be attributed to fulfil the required continuation conditions. However, this procedure of continuing Rice sequences creates a stability problem. The noise thus generated mostly diverges after a few sequences. Generally, if the random numbers $g_k^{(m+1)}$ and $g_k'^{(m+1)}$ are combined into a vector, any orthonomial transformation leads to a new vector without changing the required statistical properties. One can show that, if such a transformation fulfilling the conditions of equations (13) is possible, then ordinarily many orthonomial transformations exist, and one can choose one of them. However, the existence of such a transformation is not guaranteed.

We contrived a method to realize appropriate noise records by windowing Rice sequences. The method preserves signal stationarity and makes the $\text{PSD}^{(FX)}$ fairly well independent from the starting ooint in the analysis, thereby reducing leakage and side-lobe production. If $x_n^{(m)}$ are Rice sequences and two sequences are added together with weights, the resulting sequence $y_n^{(m)}$ is also a stochastic time-series. The new sequence $y_n^{(m)}$ is obtained by

$$y_n^{(m)} = w_{n+N}x_n^{(m)} + w_n x_n^{(m+1)}. \tag{14}$$

The weight function w_n is a two-sequence window defined in the interval $0 \leqslant n \leqslant 2N$ and has the properties

$$w_0 = w_{2N} = 0, \quad w_n > 0 \quad \text{for} \quad 0 < n < 2N \quad \text{and} \quad w_n^2 + w_{n+N}^2 = 1. \tag{15}$$

The last condition is necessary to preserve the variance of the resulting noise signal. This weight function covers two subsequently identical Rice sequences which are a representative sample of a continuous periodic signal function when equation (7) is used with the same amplitude coefficients for a time longer than T.

Two weight functions have been investigated which fulfil the requirements specified by equations (15):

(a) a sin/cos weight function, hereafter called a two-sequence cosine window, defined by

$$w_n = \sin\left(\frac{\pi n}{2N}\right); \quad 0 \leqslant n \leqslant 2N \tag{16}$$

(b) a square-root weight function, hereafter called a two-sequence square-root window, defined by

$$w_n = \sqrt{\frac{n}{N}} \quad \text{and} \quad w_{n+N} = \sqrt{1-\frac{n}{N}}; \quad 0 \leqslant n \leqslant N. \tag{17}$$

The use of 'windows' refers to the noise generation process and should not be confused with windowing in the analysis procedure where a sequence of noise data are weighted by a one-sequence window function before the FFT procedure is applied. Our generation procedure bears some similarity to windowed data analysis with 50% overlapped segmentation (Enochson, 1977), but the use of this technique for the generation of a stochastic signal record, as well as the concept of the square-root window, is believed to be new.

The proposed generation procedure may be called a process of looking at time-series produced by the Rice formulation through incomplete or 'cracked' windows of finite length. To understand the concept of this 'cracked' window, let us first consider the case of a uniform window in the analysis procedure. For a time interval of length T that runs exactly over one

Rice sequence, the instantaneous spectrum is given by

$$X_k = \Delta t \sum_{n=0}^{N-1} x_n \exp\left(-\frac{2\pi ikn}{N}\right) = \frac{T}{2}(a_k - ib_k). \quad (18)$$

The instantaneous PSD is estimated by

$$\mathrm{PSD}_k = \frac{X_k^* X_k}{T} = \frac{T}{4}(a_k^2 + b_k^2). \quad (19)$$

Equation (19) means that through the uniform analysis window the original 'image' is perfectly observed. In this case, the window function is given in the frequency domain by†

$$V(f) = \Delta t \sum_{n=0}^{N-1} \exp(-2\pi ifn\Delta t) = \Delta t \exp(-\pi if(N-1)\Delta t) \times \frac{\sin(\pi fN\Delta t)}{\sin(\pi f\Delta t)}. \quad (20)$$

For the discrete values of $f = k/N\Delta t$, the window has a value only for the main lobe, or $k = 0$, it vanishes for other values of k and the function can be treated as if no window exists.

When the Rice sequences are scanned with a time shift, we look through the time interval T always at two incomplete sequences. The time interval T is divided into two parts. On the l.h.s. there are $N - n_0$ data points of the sequence m, and the r.h.s. covers n_0 data points belonging to the sequence $m+1$. Using equation (18), if the instantaneous spectra are denoted by $X_k^{(m)}$ and $X_k^{(m+1)}$, respectively, these spectra are uncorrelated and have the expectation values

$$\langle X_{k'}^{(m)} X_k^{(m+1)} \rangle = 0, \quad (21)$$

even at the same frequency points $k' = k$. As a consequence, the window function comprises two parts which are given in the frequency domain by

$$V_k^{(L)} = \Delta t \sum_{n=0}^{N-n_0-1} \exp\left(-2\pi i\frac{kn}{N}\right) = T\delta_{0k} - \Delta t \exp\left[\frac{\pi ik(n_0+1)}{N}\right] \frac{\sin\left(\frac{\pi kn_0}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)} \quad (22)$$

and

$$V_k^{(R)} = \Delta t \sum_{n=N-n_0}^{N-1} \exp\left(-2\pi i\frac{kn}{N}\right) = T\delta_{0k} - V_k^{(L)}. \quad (23)$$

Equations (22) and (23) represent incomplete windows or 'cracked' windows. The spectrum is given as the sum of convolutions of the incomplete window functions and the spectra $X_k^{(m)}$ and $X_k^{(m+1)}$. Therefore, side lobes are produced. The expected PSD results from

$$\mathrm{PSD}_k^{(EX)} = \frac{\Delta f}{2} \sum_{l=1}^{N/2-1} \sigma_l^2 [|V_{k-l}^{(L)}|^2 + |V_{k+l}^{(L)}|^2 + |V_{k-l}^{(R)}|^2 + |V_{k+l}^{(R)}|^2]. \quad (24)$$

Equation (24) reproduces equation (12) exactly, which was originally obtained directly by using ensemble averaging. For a single-frequency target PSD it gives the composite squares of these window functions, as shown in Fig. 1 by the solid line where scanning was performed with a 50% time shift.

The preceeding considerations should give deeper insight into explaining equation (12) in another way, by the concept of a 'cracked' analysis window. It is not worthwhile to attempt analytical transfer of the proposed noise generation procedure by equations (14)–(17) as a means of overcoming a problem in the Rice formulation for equivalent cracked analysis windows. The generation procedure uses a two-sequence window while analysis is always made with a one-sequence window. From a practical point of view, the analysis should not complicate analysis windows but should be performed with simple windowing. If for any purpose additional smoothing is required, one should select an appropriate window other than the boxcar function. A number of such smoothing analysis windows are known (Harris, 1978).

Similarly to Fig. 1, Figs 3 and 4 show the PSDs retrieved without and with a 50% time shift from scanning noise records generated with a single-frequency target PSD by equation (14), using the two-sequence cosine window of equation (16) or the two-sequence square-root window of equation (17). From these figures it is seen that the retrieved PSDs are insensitively affected by sliding scanning. The Rice sequences are joined smoothly together to adequately reproduce the sample from a continuous stochastic signal. It seems that the two-sequence square-root window behaves slightly better than the two-sequence cosine window.

As a further example, Fig. 5 shows the PSDs retrieved from noise records generated by the new method with the same exponentially decaying target PSD and the same transform size as used in the case referring to Fig. 2. We show here only cases where the two-sequence square-root window is used in the generation process. Comparable results obtained with

† We have purposely used a different letter to distinguish clearly between the generation and analysis windows.

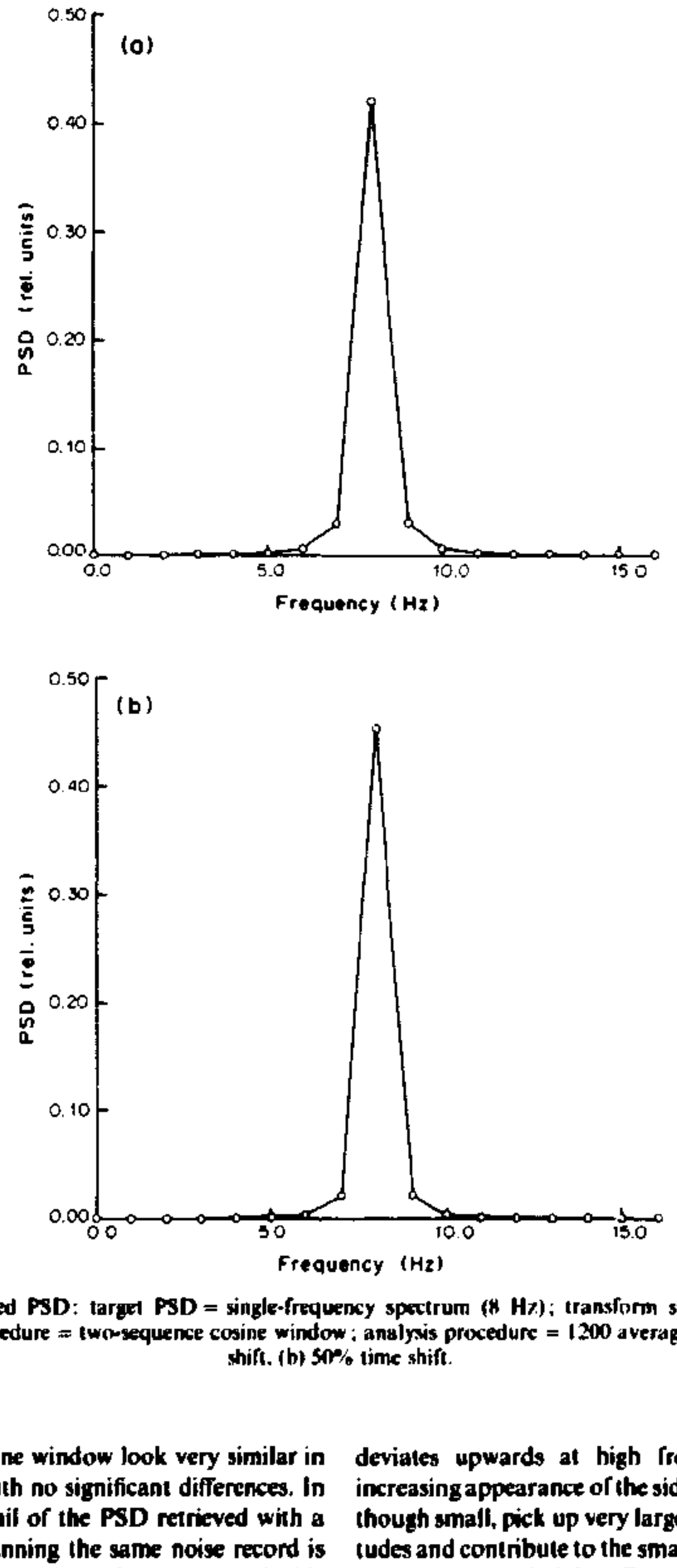

Fig. 3. Retrieved PSD: target PSD = single-frequency spectrum (8 Hz); transform size = 32 points; generation procedure = two-sequence cosine window; analysis procedure = 1200 averages. (a) zero time shift, (b) 50% time shift.

the two-sequence cosine window look very similar in the retrieved PSDs with no significant differences. In this figure only the tail of the PSD retrieved with a 50% time shift in scanning the same noise record is represented because the points of the curves coincide in the low-frequency region. The reproduction of the target PSD is good over the first 2.5 decades, but deviates upwards at high frequencies due to the increasing appearance of the side lobes. The side lobes, though small, pick up very large low-frequency amplitudes and contribute to the small high-frequency components. The side-lobe effect is now always present and depends only slightly on the applied time shift in scanning the noise record. If other time-shift values

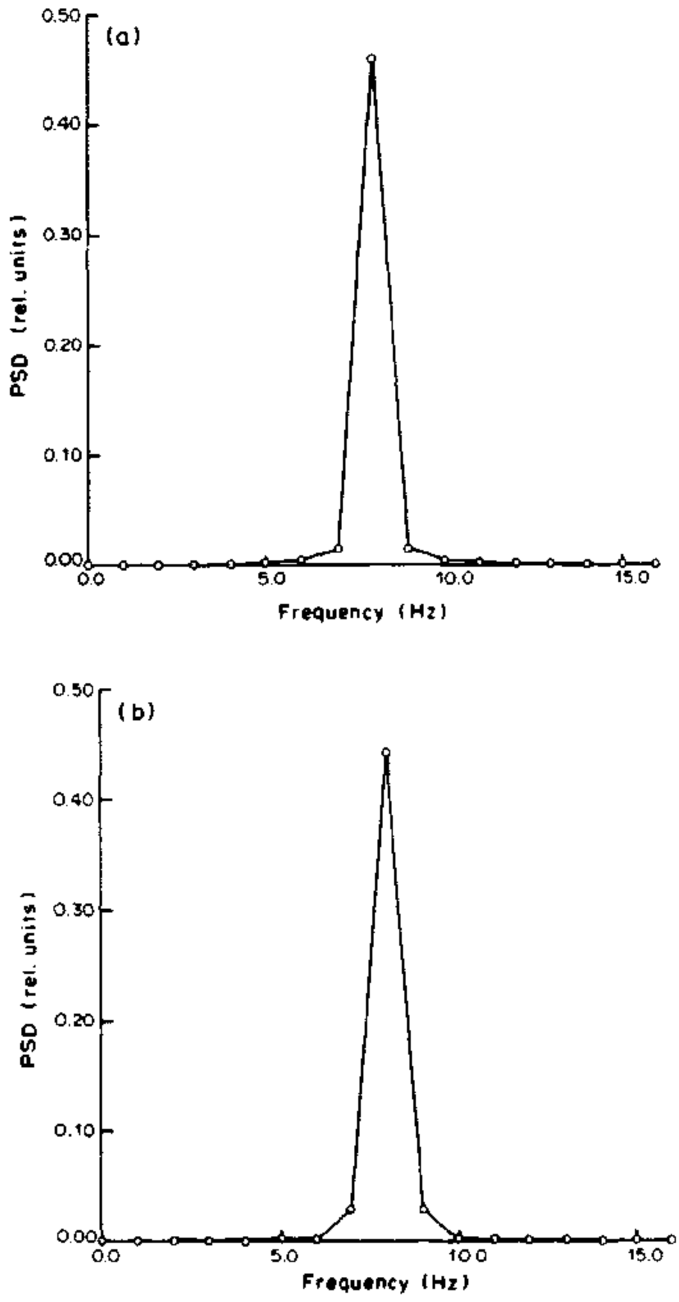

Fig. 4. Retrieved PSD: target PSD = single-frequency spectrum (8 Hz); transform size = 32 points; generation procedure = two-sequence square-root window; analysis procedure = 1200 averages. (a) zero time shift. (b) 50% time shift.

are chosen, it spreads over only the region between the two curves shown in this figure.

Side lobes always arise in the spectral analysis of actual noise signals. Their appearance can be reduced by applying a proper smoothing window. An effective method for reduction is by increasing the transform size. Figure 6 shows the same case as represented in Fig. 5, but the noise has been generated and analysed with a transform size 8 times larger. We confine the representation to only the PSD retrieved from zero time-shift scanning. The range of the reproduced target PSD is clearly increased and now extends over about 3.5 decades, proportional to the transform size. The selection of an adequate transform size is not only

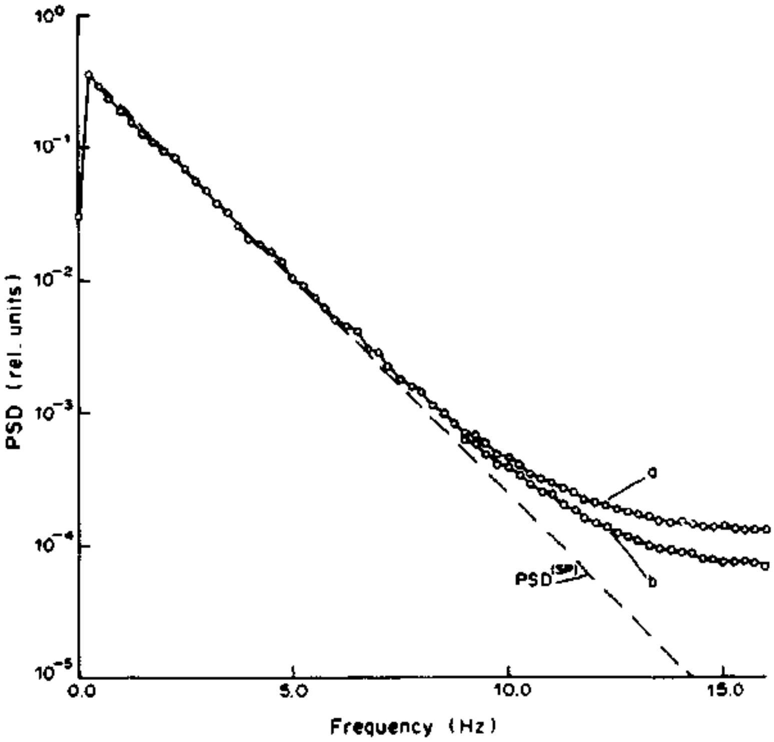

Fig. 5. Retrieved PSD: target PSD = exponentially decaying spectrum 0.25–15.75 Hz, 5 decades; transform size = 128 points; generation procedure = two-sequence square-root window; analysis procedure = 600 averages. (a) zero time shift, (b) 50% time shift.

a matter of the required spectral resolution but also concerns the desired dynamic range which, in the analysis, is limited due to the appearance of side lobes.

With regard to the other statistical properties of the generated noise, the following items were examined:

(a) normality,

(b) stationarity

and

(c) statistics of the $PSD^{(ES)}$.

Tests on these items check not only on the generation method itself but also the quality of the Gaussian random number generator used and the correctness of the codes. Items (a) and (b) were tested directly on the noise data by using standard procedures, as described by Bendat and Piersol (1971). As an example, numerical results obtained from the noise record generated for the PSD analysis cases shown in Fig. 5 are given in Table 1.

Two procedures were applied to test normality. The simplest procedure is to look at the values of the skewness and kurtosis. They were calculated from estimated moments up to the fourth. From these values alone we could infer that the generated noise behaves well normally. The other more refined procedure of testing normality is the χ^2-goodness-of-fit test which checks the χ^2-distribution as a measure of the discrepancy between an estimated probability density function and the theoretical density function. Assuming the hypothesis of a normal distribution, the amplitude histogram was obtained by selecting class intervals which provide for equal expected amplitude frequencies. The position of these intervals on the amplitude scale was adapted to the estimated apparent d.c. signal and the estimated signal variance. The reduced χ^2 determined (i.e. χ^2 divided by the number of degrees of freedom) was acceptable in an interval with an upper-sided boundary based on a 5% level of significance. In our examinations, no case was observed which had to be rejected. There may possibly be a very small trend in the random number generator to underproduce frequencies of large-amplitude numbers. Nevertheless, we can state that our proposed generation procedure and the Gaussian random number generator used give excellent normal noise.

As regards the stationarity test, the generated noise record must be segmented into time intervals. For convenience, segmentation was performed with mul-

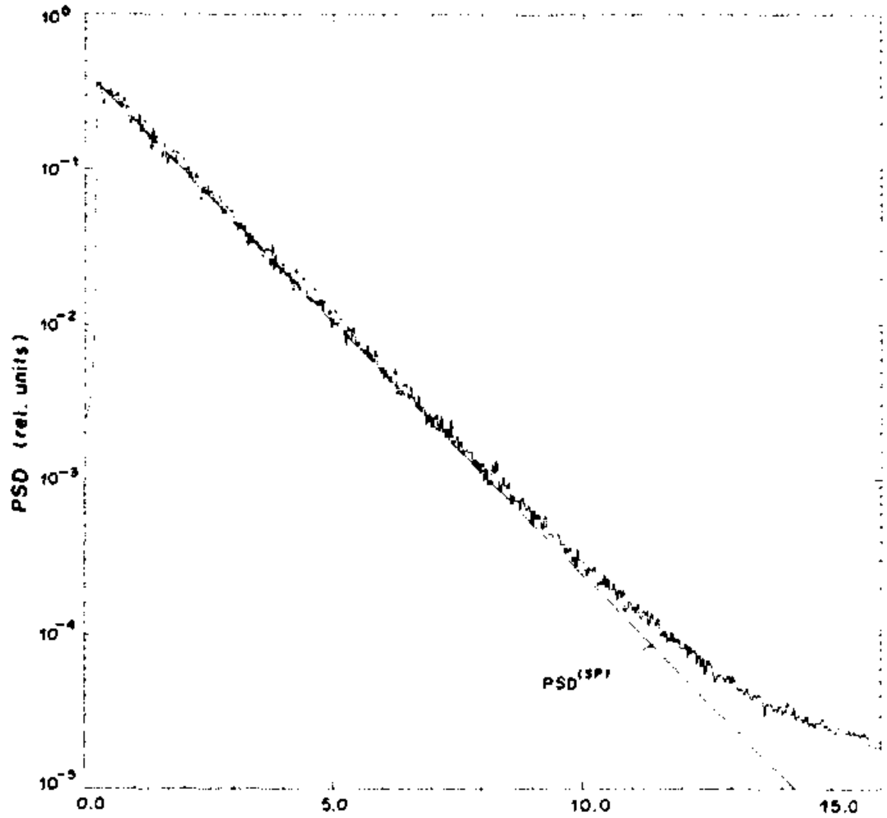

Fig. 6. Retrieved PSD: target PSD = exponentially decaying spectrum 0.25–15.75 Hz, 5 decades; transform size = 1024 points; generation procedure = two-sequence square-root window; analysis procedure = 140 averages, zero time shift.

tiples of the transform size. The distribution of the second moment estimated in each segment around the median value was examined by the run test, assuming that the sequence of second moment values are independent random observations. The hypothesis of stationarity is accepted if the number of runs is close to the expected value. Since the associated probability distribution function is a discrete function, the two-sided acceptance region cannot be given for any required level of significance. It was determined for an approx. 5% level of significance using the (conditional) probability function of the number of runs, as given by Fisz (1963). Stationarity was always found to be excellent. This is as expected as the signal generation process using the Rice formulation of noise is based on preserving the expected variance of each data point.

The last item of interest, i.e. the statistical behaviour of the spectral points of the $PSD^{(FS)}$, presents a more difficult problem. It was tested in an indirect way by an available code which applies to the PSD pattern-recognition algorithm of Piety (1977). There are eight statistical decision discriminants. Improved limit criteria on the statistics of these discriminants have recently been elaborated by Behringer and Spickerman (1984) and by Behringer *et al.* (1985). These criteria assume stationary Gaussian noise and are based on the strict validity of equations (5) and (6). Thus, if stationarity and normality are fulfilled, significant deviations between the predicted and observed statistical behaviour of the discriminants should give an indication of the non-validity of equations (5) and (6). Our investigation showed that for spectral points in large-amplitude regions, equations (5) and (6) may be considered to hold in the sense of a satisfactory approach. They are not strictly valid and are not expected to behave as such. If the aim of the noise generation is the investigation of the statistics of discriminants in pattern-recognition methods, the obtained results should be interpreted cautiously. As an example, the noise record generated with the exponentially decaying target PSD with 62 occupied spectral points (the retrieved PSDs shown in Fig. 5) was analysed by 380 short-time PSDs, each estimated

Table 1. Normality/stationarity tests: numerical results obtained from the noise record generated for the cases shown in Fig. 5.

Total number of analysed data points = 76800
Record segmentation
  Number of segments = 150
  Number of data points within a segment = 512
Amplitude segmentation = 40 intervals providing for equal probability and covering the theoretical amplitude range from $-\infty$ to $+\infty$

	Observed	Theoretical	Accepted region	Accepted level of significance (%)
First moment (over the analysed record length)	−3.980E−04	0		
Variance (over the analysed record length)	1.023	1		
Skewness (over the analysed record length)	−3.854E−02	0		
Kurtosis (over the analysed record length)	−1.586E−02	0		
Reduced χ^2 in the normality test	1.329	1	< 1.411	5
If only the 38 inner amplitude intervals are considered ($\|x\| \leq 1.96$)	1.175	1	< 1.417	5
Median of segment second moments	1.006	1		
Number of runs of segment second moments around the median	74	76	64–88	4.013

by $M = 8$ averages. (This noise record was effectively very long; only a part of it was used in the previous analyses.) The frequency range was divided into three intervals, 1.0–4.75 Hz, 5.0–8.75Hz and 9.0–12.75 Hz, each containing 16 spectral points. The 'diagnosis' was performed on the spectral points in each frequency interval. In the learning period, the first 80 PSDs were utilized to establish the reference PSD and to obtain the sample mean and standard r.m.s. value of the discriminants. The second 300 PSDs were used to count the discriminant values which fall inside the limits calculated for given confidence coefficients. According to Piety (1977), the limits of the first five discriminants have been slightly corrected with the previously determined sample mean and standard r.m.s. values of these discriminants. The results show that Piety's discriminants behave more or less satisfactorily for the two lower frequency ranges, but behave completely erratically for the highest frequency range. The latter point may be due to the fact that in the highest frequency range very large low-frequency amplitudes contribute to the small high-frequency components through the side-lobe effect discussed earlier. The side lobes introduce a significant corrrelation between the spectral points. The numerical results of the diagnosis in the two lower frequency ranges are given in Table 2. One can observe that there are systematic differences between the predicted and sample values of the mean and standard r.m.s., particularly for the discriminants D-4 and D-5. These discriminants are conjectured to be sensitive to correlations between spectral points. With regard to the counting statistics, there are also count values which fall outside the acceptance regions which have been based on an approx. 5% level of significance. But in our opinion, they are not so far outside that one must completely reject the hypothesis of an approximate validity of equations (5) and (6) for spectral points in large-amplitude regions.

The method of noise generation is efficient. The computation time of our code which is not optimized with regard to speed, is only slightly dependent on the chosen transform size. For the generation of the Rice sequences the Gaussian random number generator available from the IMSL library and the FFT routine developed by Brigham (1974) are used, combined with data shuffling to profit effectively from an $N/2$-point FFT transform size only. The computation time for generating a record of 10^5 data points is about 8 min CPU-time on a CDC 6500.

Table 2. Test on the statistics of the PSD[15]: theoretical and sample results for the statistical behaviour of Piety's discriminants applied to the PSD frequency intervals 1.0–4.75 and 5.0–8.7 analysis was performed by 380 short-time PSDs, each estimated by $M = 8$ averages, on the same noise record generated for the cases shown in Fig. 5)

(a) Learning period: mean and r.m.s. of the discriminants based on the first 80 short-time PSDs

	Analysed frequency interval 1.0–4.75 Hz						Analysed frequency interval 5.0–8.75 Hz				
	Mean			r.m.s.			Mean			r.m.s.	
Discr.	Theoretical	Sample	Δ (%)	Theoretical	Sample	Δ (%)	Theoretical	Sample	Δ (%)	Theoretical	Sample
D-1	−3.266E−03	−3.243E−03	−0.7	5.393E−02	5.338E−02	−1.0	−4.494E−03	−4.505E−03	0.2	6.333E−02	6.266E−02
D-2	−3.314E−01	−3.612E−01	9.0	1.084E−01	1.066E−01	0.2	−3.314E−01	−3.430E−01	3.5	1.084E−01	1.046E−01
D-3	2.317E−01	2.530E−01	9.2	6.962E−02	7.300E−02	4.9	2.317E−01	2.413E−01	4.1	6.962E−02	6.574E−02
D-4	−2.737E−02	−3.329E−02	21.6	3.985E−02	4.564E−02	14.5	−2.737E−02	−3.247E−02	18.6	3.985E−02	5.987E−02
D-5	2.593E−02	3.130E−02	20.7	1.010E−02	1.161E−02	15.0	2.593E−02	3.068E−02	18.3	1.010E−02	1.355E−02
D-6	9.571E+00	9.750E+00	1.9	1.238E+00	1.392E+00	12.4	9.571E+00	1.003E+01	4.8	1.238E+00	1.475E+00
D-7	8.500E+00	8.400E+00	−1.2	1.936E+00	2.022E+00	4.4	8.500E+00	8.275E+00	−2.6	1.936E+00	2.019E+00
D-8	4.338E+00	4.462E+00	2.9	1.518E+00	1.533E+00	1.0	4.338E+00	4.588E+00	5.8	1.518E+00	1.794E+00

(b) Working period: percentage of discriminant counts in bound discriminant acceptance intervals (the discriminant values were obtained from a further 300 short-time PSDs)

Discr.	Rejection region	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
D-1	Two-sided	66.7	64.3	67.7	61.3—71.3	90.0	87.3	91.3	86.7—92.7
D-2	Lower-sided	66.7	66.3	58.7	61.3—71.3	90.0	87.0	84.3	86.7—92.7
D-3	Upper-sided	66.7	67.7	63.3	61.3—71.3	90.0	90.3	86.7	86.7—92.7
D-4	Two-sided	66.7	70.0	68.7	61.3—71.3	90.0	91.0	94.3	86.7—92.7
D-5	Upper-sided	66.7	63.7	65.3	61.3—71.3	90.0	89.3	94.3	86.7—92.7
D-6	Upper-sided	54.5	52.3	41.7	49.0—59.7	79.0	75.7	70.7	74.3—83.0
D-7	Two-sided	69.8	78.7	80.7	64.7—74.3	88.2	77.3	82.0	84.7—91.3
D-8	Upper-sided	62.8	64.3	56.0	57.3—67.7	91.5	89.7	86.3	88.3—94.0

Columns: (1) confidence coefficient (%); (2) observed percentage of counts, analysed frequency interval 1.0–4.75 Hz; (3) observed percentage of counts, analysed frequency interval 5.0–8.75 Hz; (4) acceptance region—percentage of counts, based on an approx. 5% level of significance.

5. KAC FORMULATION OF RANDOM NOISE AND ITS APPROXIMATION

Kac, based on his earlier paper (Kac and Steinhaus, 1938), gave an account of another interesting way of generating random noise artificially. A summary article written by Kac is contained in the recent book *The Making of Statisticians*, edited by Gani (1982). Independently from Kac, the same formula was used by Goto and Toki (1969), for simulating earthquake motion, and by Shinozuka (1971) to develop a general simulation theory of multivariate stationary random processes with a specified cross-spectral density matrix. Despite the fact that we did not succeed in the practical application of this formula, we should like to report briefly on our investigations because this formula is of some interest. According to Kac, the formula reads

$$x_n(t) = \sqrt{\frac{2}{n}}\sum_{\nu=1}^{n}\cos(2\pi f_\nu t). \qquad (25)$$

There are two conditions which are required to fully represent the random signal $x(t)$ as a member of the ensemble:

$$x(t) = \lim_{n\to\infty} x_n(t); \qquad (26)$$

(b) the frequencies f_ν must be linearly independent over the field of rationals, i.e.

$$f_\nu / f_{\nu'} = \text{irrational for any pair } \nu \neq \nu'. \qquad (27)$$

Under the conditions given by equations (26) and (27) the noise signal has unity variance, is stationary and Gaussian according to the central limit theorem. If ν_k denotes the number of (non-negative valued) frequencies in a small frequency interval Δf around the centre frequency f_k, then for any random set of these frequencies the density function $\nu_k/n\Delta f$ is related to the target PSD, normalized to unity variance, by

$$\lim_{\nu_k, n\to\infty}\frac{\nu_k}{n\Delta f} = 2\mathrm{PSD}^{(SP)}(f_k). \qquad (28)$$

In the Rice formulation of a random signal the sine and cosine terms are fixed at determined frequencies and the amplitudes are stochastically 'modulated'.

The former method may be called an 'AM' method. The Kac formulation may be referred to as a 'frequency modulated' or 'FM' method of generating random noise. Here, the frequencies are the independent random variables while the amplitude of each cosine term is kept constant. Each random set of frequencies following equation (28) produces a signal $x(t)$ which is a member of the ensemble. The difference to the Rice formulation is that $x(t)$ continues over the entire time axis. When dealing with temporal averages, the f_ν are not considered anymore as random variables but are treated as sample values of these random variables (Shinozuka, 1971). Kac's representation of noise is based on elements of number theory. The noise looks as if it could be derived from a simple deterministic formula.

Neither of the requirements specified by equations (26) and (27) can be fulfilled in practice. At best, a pseudo-random noise record could be generated. Equation (27) implies independence of the cosine terms in equation (25). One can at first put the question of how many independent cosine terms are required to obtain sufficient normality of the noise under the strict mathematical validity of equation (27). A simple inequality results from the condition that the possible maximal amplitude of the signal $x_n(t)$ must be much larger than the signal r.m.s., i.e. $\sqrt{2n} \gg 1$. Referring to Appendix A of the paper by Behringer *et al.* (1985), the probability density function $p_n(s)$ to observe the amplitude s on the signal $x_n(t)$ is given by

$$p_n(s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, e^{i\omega s} J_0^n\left(\omega\sqrt{\frac{2}{n}}\right). \qquad (29)$$

where J_0 is the zero-order Bessel function of the first kind.

Analytical solutions of equation (29) are obtainable for $n = 1$ and $n = 2$. p_1 is well-known (e.g. Bendat and Piersol, 1971). It is a strongly peaked density function at the endpoints of the amplitude range, i.e. when $|s| \to \sqrt{2}$, and suggests that a large number of superposed cosine terms may be necessary to obtain approximate normality of the signal. p_2 Can be expressed by a complete elliptic integral function of the first kind and shows, on the other hand, a strong peaking at $|s| \to 0$. An attempt was made to numerically integrate equation (29) by the inverse FFT procedure. Such an integration method is not suitable for small values of n because of the damped oscillatory behaviour of the function J_0. But by proper selection of the truncation points and choice of a very large transform size (8 K) it was possible to obtain acceptable results for values $n > 10$. The results show that about 50 terms are required to obtain a density function which approaches to within a few percent of the normal density function over the 3σ amplitude range.

When approximating the Kac formula by a finite set of frequencies, a much more serious problem concerning the required number of terms arises from the dynamic range considerations following from equation (28). If the noise is coloured by a target PSD for values from $PSD^{(SP)}_{(min)}$ to $PSD^{(SP)}_{(max)}$, the frequency interval Δf at the frequency of $PSD^{(SP)}_{(min)}$ must be occupied by at least one term. The frequency interval Δf at the frequency of $PSD^{(SP)}_{(max)}$ has to contain at least $PSD^{(SP)}_{(max)}/PSD^{(SP)}_{(min)}$ terms. This involves an almost unreasonably large number of cosine calls on the computer. Let us consider a rather extreme case of a structured $PSD^{(SP)}$ where the noise is to be generated for an analysis by a 1024-point transform size, each of the 511 occupied frequency intervals containing 100 cosine terms on average. A computation time of about 7.2 days on a CDC 6500 is estimated in order to obtain a record of 10^5 data points.

It is therefore imperative to think of approximate methods other than that of restricting the noise generation procedure to a finite set of frequencies only. An attempt was made to group the ν_k frequencies in each frequency interval Δf around the centre frequency f_k with equidistant spacing δf_k, thus allowing an analytical partial summation of the cosine terms. The approximation of equation (25) reads as follows:

$$\left.\begin{aligned} x_n(t) &= \sqrt{\frac{2}{n}} \sum_{k=1}^{N/2-1} A_k(f_k, \nu_k, t), \\ &\text{with} \\ n &= \sum_{k=1}^{N/2-1} \nu_k \\ &\text{and} \\ A_k &= \frac{\sin(\pi \delta f_k \nu_k t)}{\sin(\pi \delta f_k t)} \cos[2\pi(f_k + \delta f_k \varepsilon_k)t + \varphi_k]. \end{aligned}\right\} \quad (30)$$

In equation (30), initial random phase numbers of equal likelihood, defined in the interval $|\varphi_k| \leqslant \pi$, have been introduced. ε_k Denotes other initial random numbers of equal likelihood to be introduced over the interval $|\varepsilon_k| \leqslant 0.25$, which provide for a slight random shift of one frequency package against another. In this procedure, only three calls for trigonometric functions rather than ν_k calls are needed for each specified spectral point. But even the generation of this noise is more expensive than by the modified Rice formulation for comparable record lengths.

In the above approximation a phenomenon occurs which may be called a beating effect. The ratio of the sine terms in A_k of equation (30) takes periodically the value ν_k, alternating in sign, when t reaches multiples of $1/\delta f_k$. At these time points a frequency package causes the creation of a large amplitude excursion like a beat. At $t = 0$, the beats from all the frequency packages superpose. This starting time region of the noise must be excluded in the application. Furthermore, the beating effect indicates that the noise generation procedure via equation (30) is not suitable for simulating random signals with flat or nearly flat PSDs. Satisfactory results in the retrieved PSDs and with regard to the signal stationarity can only be obtained for target PSDs which provide for a more or less uniform distribution of the beats, and all beats must be contained in the time limit to be used for the analysis. For example, when the case given in Fig. 5 is treated by our simplified Kac formulation, with ν_k-values ranging from 1 to 10^5, and the noise is analysed in exactly the same way, the generated noise record showed non-stationary behaviour. The retrieved PSD failed to reproduce the target PSD due to missing beats. If the amplitude range of the target PSD is reduced to an exponential decay over 1 decade using ν_k-values from 12 to 120, an acceptable stationarity for the generated noise record was observed. The target PSD was well-reproduced by the retrieved PSD but, in all the cases investigated, the noise generated by equation (30) was never normal, not even approximately over the 3σ amplitude range.

6. CONCLUSIONS

For computer-simulation experiments in the development of noise monitoring systems, three methods of generating ergodic Gaussian random noise with specified spectral properties have been investigated.

Digital filtering of white noise is a known method. We confined our considerations to symmetric FIR filters and used an existing program for designing optimum filters. Coloured noise is obtainable within the limitations of the code. Simulation of noise with strongly structured PSDs is not possible. However, the method is efficient and fulfils the required conditions for noise quality.

A new generation method using the Rice formulation of random noise is proposed. By windowing subsequent Rice sequences a smooth record of any desired length can be produced. The method is very efficient since the inverse FFT procedure can be used for the generation of the Rice sequences. The Rice representation of noise allows the reproduction of any specified spectral characteristics. As a future extension of applications one may consider simulations of correlated peaks in the target PSD by mixing together the Gaussian random numbers from different frequency regions within the same Rice sequence. For the detection of correlated peaks, the 'criss'-spectrum techniques of Väth (1979) or the envelope cross-correlation techniques recently suggested by Saxe (1985) are possible analysis methods. Also, an extension to the simulation of multivariate processes should be possible by adapting the theoretical background given by Shinozuka (1971) to our proposed generation method.

The noise representation of Kac is an interesting scientific application but its correct use for the noise generation is hindered by unacceptably long computation times. Perhaps the next generation of fast computers, like the Cray machine, may be suitable. The approximate method investigated by grouping the frequencies equidistantly within packages suffers from the beating effect.

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