Theory for Resonant Ion Acceleration by Nonlinear Magnetosonic Fast and Slow Waves in Finite $\beta$ Plasmas

Yukiharu OHSAWA

(Received Oct 18, 1985)

IPPJ-751

Research Report

NAGOYA, JAPAN
Theory for Resonant Ion Acceleration by Nonlinear Magnetosonic Fast and Slow Waves in Finite $\beta$ Plasmas

Yukihiro OHSAWA

Institute of Plasma Physics, Nagoya University, Nagoya 464, Japan

Abstract

A Korteweg de-Vries equation that is applicable to both the nonlinear magnetosonic fast and slow waves is derived from a two-fluid model with finite ion and electron pressures. As in the cold plasma theory, the fast wave has a critical angle $\theta_c$. For propagation angles greater than $\theta_c$, quasi-perpendicular propagation, the fast wave has a positive soliton, whereas for angles smaller than $\theta_c$, it has a negative soliton. Finite $\beta$ effects decrease the value of $\theta_c$. The slow wave has a positive soliton for all angles of propagation. The magnitude of resonant ion acceleration (the $\beta$ acceleration) by the nonlinear fast and slow waves is evaluated. In the fast wave, the electron pressure makes the acceleration stronger for all propagation angles. The decrease in $\theta_c$ due to finite $\beta$ effects results in broadening of the region of extremely strong acceleration. It is also found that strong ion acceleration can occur in the nonlinear slow wave in high $\beta$ plasmas. Possibility of unlimited acceleration of ions by quasi-perpendicular magnetosonic fast waves is discussed.

PACS (i) 52.35.Sb, (ii) 52.35.Tc, (iii) 52.35.Mw, (iv) 94.20.Rr.
I. INTRODUCTION

Recently, extensive studies have been carried out on resonant ion acceleration (the \( \mathbf{v} \cdot \mathbf{B} \) acceleration) by plasma waves propagating in a magnetic field. Sugihara and Midzuno\(^1\) analyzed single particle orbits in a monochromatic electrostatic wave propagating perpendicularly to a magnetic field, and found that some particles are trapped by the large-amplitude wave and are accelerated in the direction parallel to the wave front up to the speed

\[
\mathbf{v} \cdot \mathbf{E} = \mathbf{v} \cdot \mathbf{B} = 0 \quad \mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{v} \mathbf{c} \mathbf{B} \mathbf{c} \mathbf{0} \mathbf{c} > 1.
\]

where \( \mathbf{v} \) is the speed of light, \( \mathbf{E} \) the electric field in the direction of the wave normal, \( \mathbf{B} \) the magnetic field, and \( \mathbf{v} \) the wave phase-velocity. Lembege\(^2\) showed by means of a one dimension, electromagnetic particle simulation that some ions are strongly accelerated by the \( \mathbf{v} \cdot \mathbf{B} \) acceleration in a large-amplitude magnetoionic fast wave. In those simulations, external monochromatic currents, \( j = j_{\text{ext}} \sin \mathbf{kx} - zt \), were imposed to excite and sustain the wave; the wave amplitude is fixed, and the wavenumber \( \mathbf{k} \) and the frequency \( \omega \) are given independently of the amplitude.

The ion acceleration in a self-consistent magnetoionic fast shock wave in a low \( \beta \) plasma was studied in Refs.\(^5\) to \(^7\) by theory and a 2-1-2 dimension, fully electromagnetic particle simulation, where \( \beta \) is the ratio of the plasma pressure to the magnetic pressure. It was shown that in perpendicular laminar shock waves\(^3\),\(^11\) potential jump in the shock region can be much larger than the ion temperature unless the Alfven Mach number \( M_A \) is too close to unity\(^7\), i.e., the potential jump is given by
where $m_i$ is the ion mass, and $v_A$ the Alfvén speed. The characteristic shock width is of the order of the electron inertial length, $c_{De} \approx \frac{m_e}{m_i} \frac{1}{v_A^2}$, which is quite small. Hence the electric field in the shock region is very strong. As a result, trapped ions can be strongly accelerated by the perpendicular shock wave up to the speed
\[\frac{m_i}{m_e} \sim \frac{v_A}{c_{De}} \ll 1,\]
where $m_e$ is the electron mass.

When the propagation angle $\theta$ is below a critical angle $\theta_c$, the characteristic wavelength of the nonlinear magnetosonic wave becomes of the order of the ion inertial length, $c_{in}$, which is about $m_i/m_e$ times larger than the one in the quasi-perpendicular shock. On the other hand, the magnitude of the potential jump is only weakly dependent on the propagation angle. Consequently, the electric field and the $v_A \cdot B$ acceleration are weak in the oblique shock waves with the angles $\tan \theta > 0.1^\circ$.

It was pointed out in Ref.14 that the simulation results well explain the ion heating in interplanetary shocks observed by ISEE spacecraft. The simulation results and observations by ISEE spacecraft have the following common features. First, plasmas are strongly heated in quasi-perpendicular shock waves even with subcritical Mach numbers. Second, significant ion acceleration occurs within the shock ramp by resonant interactions. Third, the ions are heated primarily in the direction perpendicular to a magnetic field. Fourth, the electrons predominantly heated by adiabatic compression. The fourth point implies
that micro-instabilities are unimportant in the heating process.

In this paper we will develop a nonlinear theory for the magnetosonic fast and slow waves on the basis of a two-fluid model with finite ion and electron pressures. Since we use the two-fluid model, kinetic effects such as the Landau damping and finite Larmor radius are neglected \[^{16-18} \]
. From the two-fluid model we derive a Korteweg-de Vries equation that is applicable both to the magnetosonic fast and slow waves. It will be shown that the fast wave has a positive soliton in the propagation angles \( \theta > \theta_c \geq 2 \), whereas in the region \( 0 < \theta < \theta_c \) it has a negative soliton. The critical angle \( \theta_c \) is shifted to lower values by finite \( \beta \) effects, hence the region of the positive soliton is wider in a high \( \beta \) plasma than in a low \( \beta \) plasma. The slow wave has a positive soliton for all propagation angles \( \theta \): the density perturbation is positive, with propagation speed greater than that of the linear slow wave. The sign of the dominant magnetic field perturbation is opposite to that of the density perturbation as in the linear slow waves and slow shocks \[^{16,19-22} \]
.

Using the nonlinear theory, we evaluate the magnitude of the \( \lambda \cdot B \) acceleration in the nonlinear fast and slow waves. The fast wave has stronger acceleration in a finite \( \beta \) plasma than in a zero \( \beta \) plasma. The magnitude of the acceleration increases for all angles \( \theta \), mainly due to finite electron temperature. Also, the region of extremely strong acceleration, \( \theta < \theta_c \leq 2 \), becomes larger as the \( \beta \) value is increased. The nonlinear slow wave can strongly accelerate ions in a high \( \beta \) plasma. We also discuss the possibility of unlimited acceleration \[^{13} \]
 of ions by the magnetosonic fast wave. Preliminary results of our work were reported in Ref. 24.

The plan of this paper is as follows. In the next section, we study linear dispersion relations for the magnetosonic fast and slow waves with
long wave-length. The third section is devoted to a derivation of the Korteweg-de Vries equation for the nonlinear fast and slow waves. In Sec.IV we evaluate the magnitude of the \( \nu_p \cdot B \) acceleration of ions in the nonlinear fast and slow waves. In Sec.V we summarize our results.

II. BASIC EQUATIONS AND LINEAR DISPERSION RELATIONS

We assume that an external magnetic field is in the \( r,z \) plane, \( \vec{B} = \cos \theta \hat{r} + \sin \theta \hat{z} \), and that the waves propagate in the \( r \) direction in a collisionless plasma with finite ion and electron pressures. We consider low-frequency waves, and hence we neglect displacement currents in Maxwell's equations. In addition, we assume charge neutrality, \( n = n_i = n_e \). Here, subscripts \( i \) and \( e \) refer to the ions and electrons, respectively. The charge neutrality condition leads to the relationship, \( v_{z i} = v_{z e} = v_z \), with \( v_z \) being the velocity in the \( r \) direction. Then, basic equations for the ions and electrons can be written as.

\[
\frac{\partial n}{\partial t} + \frac{\partial}{\partial r} \left( n v_z \right) = 0. \tag{4}
\]

\[
\frac{\partial}{\partial t} v_z + v_z \frac{\partial v_z}{\partial x} = a_j R_j \vec{E} - \vec{v}_j \cdot \vec{B} - \frac{R_j}{\mu R_i} \frac{1}{R_z} \nabla \rho_j, \tag{5}
\]

where \( j = i \) or \( e \), and \( a_j = 1 \) for \( j = i \), and \( a_j = 1 \) for \( j = e \).

\[
\frac{\partial}{\partial t} v_z + v_z \frac{\partial v_z}{\partial x} = -\gamma_j \rho_j \frac{\partial v_z}{\partial r}. \tag{6}
\]

where \( \gamma_j \) is the specific heat ratio. We have normalized the length, time.
and velocity to characteristic length \( l_0 \), time \( t_0 \), and speed \( v_0 \), respectively, with \( u_0 = l_0 / t_0 \). The density and the magnetic field are normalized to the quantities \( n_0 \) and \( B_0 \) far upstream of the waves, respectively. The electric field and the pressures are normalized to \( u_0 B_0 \ c \) and \( n_0 m_i - m_e u_0^2 \), respectively. The quantity \( R_i \) is defined by
\[
R_i = \omega_i t_0 \quad \text{with} \quad \omega_i = \omega_c,\ \text{the cyclotron frequency,} \quad \omega_c c = B_0, m_i c.
\]

In addition to the above equations, we need Maxwell's equations to determine field quantities.

\[
\frac{\partial B_{z_0}}{\partial t} = -\frac{\partial E_{z_0}}{\partial x} \quad \tag{7}
\]

\[
\frac{\partial B_{z_0}}{\partial t} = -\frac{\partial E_{y_0}}{\partial y} \quad \tag{8}
\]

\[
\frac{\partial B_{z_0}}{\partial x} = -Ke (v_{ni} - v_{ne}) \quad \tag{9}
\]

\[
\frac{\partial B_{z_0}}{\partial y} = Ke (v_{zi} - v_{ze}) \quad \tag{10}
\]

Here, the coefficient \( K \) is given by
\[
K = \frac{t_0}{c} \frac{2}{\omega_{ni} \omega_{ni}} \left( \frac{1}{\omega_{ni} t_0} \right). \quad \tag{11}
\]

If we linearize Eqs. (4) to (10) assuming that perturbed quantities vary with \( \exp i k_1 z \), we obtain, after tedious but straightforward calculations, linear dispersion relations for the long-wavelength modes.
$R_0 R_t^2 R_e - R_t^3, 3 \cos^3 \theta \left( \frac{v_p^2 - v_0^2 \cos^2 \theta}{v_p} \right) + \frac{4}{3} \left( \frac{v_0^2 - c_s^2}{v_p^2} - \frac{2 \cos^2 \theta}{v_p} \right) - \frac{2 R_e^2 - R_t^2 c_s^2}{v_p} \right) \right.$

$- \frac{k^2 v_p^2}{R_e} \cos^2 \theta \left( \frac{v_p^2 - v_0^2 \cos^2 \theta}{v_p} \right) - R_t^2 - R_e R_t - \frac{2 R_e^2 - R_t^2 c_s^2}{v_p} \right) \right.$

$- \frac{2 R_e^2 - R_t^2 c_s^2}{v_p} \right) \right.$

$- \frac{2 R_e^2 - R_t^2 c_s^2}{v_p} \right) \right.$

Here, $v_t$, $v_t^*,$ and $c_s^2$ are defined as.

$\dot{v}_t = \frac{\gamma}{\gamma^2} - k^2.$

$\dot{v}_t^* = R_e R_t, \quad R_e - R_t.$

$c_s^2 = \gamma \rho_0 \gamma - \gamma \rho_0.$

with $\rho_0$ the equilibrium pressure. In Eq. (12), the terms up to $k^2$ are retained.

In the long-wavelength limit, we can neglect the second term in Eq. (12) that is proportional to $k^2$, and we obtain the dispersion relations.

$\dot{v}_0 = v_0^2 \cos^2 \theta.$

and

$\dot{v}_0 = v_0^2 - c_s^2 \dot{v}_0 - v_0^2 c_s^2 \cos^2 \theta = 0.$

It follows from Eq. (17) that

$\dot{v}_0^2 = \frac{1}{2} \left( v_0^2 - c_s^2 \right) \dot{v}_0 = v_0^2 + c_s^2 - 4 v_0^2 c_s^2 \cos^2 \theta.$
Equations (16) and (18) show three magnetohydrodynamic waves: the shear Alfvén wave, Eq. (16), the fast wave, upper sign in Eq. (18), and the slow wave, lower sign in Eq. (18). Since we are concerned with the fast and slow waves, we will make use of Eqs. (17) and (18).

The $\beta_{\nu}$ defined by Eq. (17) or (18) is independent of the wavenumber $k$. Let $\delta p$ be a small change in the phase velocity from $v_{\nu}$ due to a finite wavenumber.

$$\delta p_{\nu} k$$

Then linearizing Eq. (12) with respect to $\delta p_{\nu} k$, and using the dispersion relation in the long-wavelength limit, Eq. (17), we obtain an equation for $\delta p_{\nu} k$.

$$\delta p_{\nu} k = \frac{k^2 \rho_{\nu}^2}{4R_0 R_1} \frac{\tilde{\omega} - \omega}{\tilde{\omega} - \omega} \frac{\tilde{\omega} - \omega}{\tilde{\omega} - \omega} - 2 \left( R_{\infty} R_1 \frac{\tilde{\omega} - \omega}{\tilde{\omega} - \omega} \right).$$

For the angles $\cos^2 \theta = 1$, the dispersion relation for the slow wave given by Eqs. (18) and (20) reproduces Eq. (2.1) in Ref. 16.

In the cold plasma limit, $c_s^2 = 0$, the slow wave disappears, and Eq. (17) gives the dispersion relation only for the fast wave.

$$v_{\nu}^2 = v_{f}^2,$$

and $\delta p_{\nu} k$ is reduced to

$$\delta p_{\nu} k = -k^2 \frac{v_{\nu}^2}{2R_0 R_1} \left( 1 - \frac{R_{\infty} R_1}{R_0 R_1} c_s^2 \right).$$

This equation coincides with the one obtained by Kakutani et al. 12, 13 In the cold plasma limit, the magnetosonic wave has positive dispersion.
\( \partial \nu_p \partial k > 0 \), for propagation angles, \( 0 < \theta < \theta_c \), where the critical angle \( \theta_c \) in the cold plasma is given by

\[
\theta_c = \arctan \left( \frac{R_e - R_i}{R_e R_i} \right)^{1/2} = \arctan \left( \frac{m_i}{m_e} \right)^{1/2} - \frac{n_e}{m_i} \right)^{1/2},
\]

whereas for the angles, \( \theta_c < \theta < \pi/2 \), it has negative dispersion.

In a warm plasma with finite \( T_i \), the terms \( \nu_i^2 - c_i^2 \), \( \nu_i^2 - c_i^2 \), \( 2 \), and \( \nu_i^2 - c_i^2 \), in Eq. (20) are expressed as,

\[
\nu_i^2 - c_i^2 = \frac{1}{2} (\nu_i^2 - c_i^2) - (\nu_i^2 - c_i^2)^2 - 4\nu_i^2 c_i^2 \sin^2 \theta \]

\[
\nu_i^2 - c_i^2 = \frac{1}{2} (\nu_i^2 - c_i^2) - (\nu_i^2 - c_i^2)^2 - 4\nu_i^2 c_i^2 \sin^2 \theta \]

and

\[
\nu_i^2 - c_i^2 \cos^2 \theta = \frac{1}{2} \left( \nu_i^2 (1 - 2\cos^2 \theta - c_i^2) - \nu_i^2 (1 - 2\cos^2 \theta - c_i^2) \right.
\]

\[
- 4\nu_i^2 c_i^2 \sin^2 \theta^{1/2} \right).
\]

The above equations, Eqs. (24) to (26), show that the three terms in Eq. (20) are all positive for the fast wave, upper sign, while for the slow wave, lower sign, they are all negative. Thus, it is readily seen that the slow wave always has the negative dispersion, \( \delta i_p \partial k \cdot 0 \).

For the fast wave, \( \delta i_p \partial k \) becomes zero when

\[
1 - \frac{R_e - R_i}{R_e R_i} \left( \nu_i^2 - c_i^2 \cos^2 \theta \right) = 0.
\]

If we assume that \( \cos^2 \theta \) in Eq. (27) is small, we have an approximate form
for $\theta_c$.

$$\theta_c = \arctan \left( \frac{\sqrt{\frac{v^2 (R_e - 1)^2}{R_e - 1}}}{\sqrt{\frac{v^2 - c_s^2}{R_e - 1}} R_e R_e} \right)^{1/2}.$$

The assumption of small $\cos^2 \theta$ is valid as long as $m_e, m_i, v_i^2 > c_s^2$. For an extremely high $\beta$ case, we must use Eq. (27) instead of Eq. (28) to calculate the critical angle. (Our fluid theory will cease to be valid for such high $\beta$ plasmas.) The fast wave has the positive dispersion for $0 < \theta_c$, and has the negative dispersion for $\theta_c < \theta_c < \pi / 2$. Comparing Eqs. (23) and (28), we see that the value of $\theta_c$ is reduced by finite $\beta$ effects; the region of the negative dispersion for the fast wave is broadened.

III. REDUCTION TO KORTEWEG-DE VRIES EQUATION

In this section we will show that by using the reductive perturbation method the basic system of equations, Eqs. (4) through (10), for the magnetosonic waves (the fast and slow waves) can be reduced to a Korteweg-de Vries equation.

We introduce following stretched variables.

$$\tau = \epsilon^{3/2} t,$$

$$x = \epsilon^{1/2} (x - v_{ph} t).$$

where $\epsilon$ is a smallness parameter, $\epsilon < 1$. The $v_{ph}$ in Eq. (30) can be the phase velocity of the fast wave or the slow wave. We then expand the plasma variables as.
\[ n = 1 - \varepsilon n_1 + \varepsilon^2 n_2 - \ldots \]  
\[ \psi = \varepsilon \psi_1 + \varepsilon^2 \psi_2 - \ldots \]  
\[ \nu_{ij} = \varepsilon \nu_{ij1} + \varepsilon^2 \nu_{ij2} - \ldots \quad \text{for} \quad j \neq i \]  
\[ p_i = p_{i0} - \varepsilon p_{i1} + \varepsilon^2 p_{i2} - \ldots \]  
\[ E_{ij} = \varepsilon E_{ij1} + \varepsilon^2 E_{ij2} - \ldots \]  
\[ B_{ij} = \sin\theta - \varepsilon B_{ij1} + \varepsilon^2 B_{ij2} - \ldots \]

and

\[ \nu_{ij} = \varepsilon^3 \nu_{ij1} + \varepsilon^5 \nu_{ij2} - \ldots \]  
\[ E_i = \varepsilon^3 E_{i1} + \varepsilon^5 E_{i2} - \ldots \]  
\[ E_z = \varepsilon^3 E_{z1} + \varepsilon^5 E_{z2} - \ldots \]  
\[ B_y = \varepsilon^3 B_{y1} + \varepsilon^5 B_{y2} - \ldots \]

Now, applying the above stretching and expansion to the basic equations, Eqs.(4) to (10), we have, in the order of \( \varepsilon \), the following equations.

\[ R_i E_{ij1} + R_i (v_{z1} \cos \theta - v_{z1} \sin \theta) = 0. \]  
\[ -R_o E_{ij1} - R_o (v_{z1} \cos \theta - v_{z1} \sin \theta) = 0. \]  
\[ K(v_{z1} - v_{z1}) = 0. \]
By virtue of Eq. (35), Eq. (34) becomes identical to Eq. (33), and \( v_z = v_{z1} = v_{z2} \) is expressed in terms of \( E_{q1} \) and \( v_t \) as

\[
v_{z1} = -E_{q1}\sec\theta - v_t\tan\theta.
\]

In \( \varepsilon^{3,2} \), we have

\[
l_{t1} = l_{t0}n_1.
\]

\[
-l_{p0}\frac{\partial l_{t1}}{\partial \xi} - R_1 E_{t1} - \sin\theta v_{t1} = -\frac{R_0 - R_1}{R_0} \frac{\partial p_{11}}{\partial \xi}.
\]

\[
l_{p1} = \gamma_1 l_{p1} n_1.
\]

\[
-l_{p0}\frac{\partial v_{t1}}{\partial \xi} = -R_0 E_{t1} - \sin\theta v_{t1} = -\frac{R_0 - R_1}{R_0} \frac{\partial p_{11}}{\partial \xi},
\]

\[
-l_{p0}\frac{\partial v_{z1}}{\partial \xi} = -R_0 E_{z1} - \cos\theta v_{z1} = -\frac{R_0 - R_1}{R_0} \frac{\partial p_{11}}{\partial \xi}.
\]

\[
-p_{11} = \gamma_1 p_{10} l_{t1} n_1 = \gamma_1 p_{10} n_1.
\]

\[
-p_{t1} = -R_0 E_{t1} - \sin\theta v_{t1} = -\frac{R_0 - R_1}{R_0} \frac{\partial p_{11}}{\partial \xi},
\]

\[
p_{z1} = \gamma_1 p_{z0} n_1.
\]

\[
E_{q1} = v_{p0} B_{z1}.
\]

\[
\frac{\partial B_{z1}}{\partial \xi} = -K \cdot v_{z1} - v_{z1}.
\]
From the equations in \( \Omega^2 \), we get

\[
B_r = E_\Omega \mathcal{L}_{\Omega}.
\]

\[
\mathcal{L}_{\Omega} = \frac{\mathcal{E}_i}{\mathcal{E}} = R. R. E. \sin R. R. - \frac{L_{\Omega}^2}{R_i} \frac{L_{\Omega}^2}{R_i}.
\]

Combining Eqs. \( \text{36} \) through \( \text{49} \), we can express the lowest order perturbations in terms of \( \xi \) and the equilibrium plasma variables as

\[
B_{\xi} = \frac{E_{\Omega p}}{E_{\Omega}} \frac{E_{\xi p}}{E_{\Omega}} \frac{1 - \cos^2 \theta}{v_{s}} R_i.
\]

\[
\frac{\partial \xi}{\partial t} = \frac{1}{2} \left( \frac{\partial r}{\partial \xi} \frac{1 - \cos^2 \theta}{v_{s}} R_i \right) + \frac{\partial m}{\partial \xi}.
\]

\[
\frac{\partial \xi}{\partial t} = \frac{1}{2} \left( \frac{\partial r}{\partial \xi} \frac{1 - \cos^2 \theta}{v_{s}} R_i \right) + \frac{\partial m}{\partial \xi}.
\]
where use has been made of the equation for \( \nu_{el} \), Eq. (17). Integration of
Eq. (50e) yields the electric potential

\[
\phi = \frac{\epsilon_0}{\epsilon_R} \left( \frac{1}{R_e} - \frac{1}{R_i} \right) \phi_{el} - \frac{1}{R_i} \phi_{ec} + \frac{1}{R_i} \int n_i \, d\xi. \tag{50f}
\]

We now proceed to the next order. It follows from the
momentum equations for the ions and electrons, Eq. (5), that

\[
\frac{1}{R} \frac{d}{d\xi} \left( \frac{1}{R} \right) v_{el} = \frac{1}{R_e} \phi_{el} - \frac{1}{R_i} \phi_{ec} = \sin \theta (v_{el} - v_{e2}) \tag{51}
\]

\[
3v_{el} + v_{e2} = \frac{R_e - R_i}{R_e R_i} \frac{1}{R} \frac{d}{d\xi} (p_{i2} - p_{e2} + n_1 \frac{\partial}{\partial \xi} p_{i1} - \frac{\partial}{\partial \xi} p_{e1}) \tag{52}
\]

\[
\cos \theta (v_{el} - v_{e2}) \tag{53}
\]

We have from the equation for the pressure, Eq. (6),

\[
\gamma_i \frac{\partial}{\partial \xi} (p_{i1} + p_{e1}) = \frac{\partial}{\partial \xi} \left( t_{i1} \frac{\partial}{\partial \xi} p_{i1} - p_{e1} \right) - c_s^2 \frac{\partial^2 v_{el}}{\partial \xi^2} \tag{54}
\]
Equations (53) and (9) give

$$\frac{\partial B_{z1}}{\partial \tau} = \frac{\partial E_{z2}}{\partial \xi} - K_{\nu\rho} \cdot u_{v1} - u_{v2} \cdot \frac{\partial \nu_{v1} - \nu_{v2}}{\partial \xi} - K_{\nu\rho} n_{1} \cdot \nu_{v1} - \nu_{v2} \cdot . \quad (54)$$

We wish to eliminate the second-order quantities with subscript 2 from the above equations and to derive an equation that contains only the equilibrium and first-order quantities. Combining Eqs. (47), (49), (52), and (54), we can eliminate the quantities \(u_{z1} - v_{z2} R_{1}, v_{z1} - v_{z2} R_{2}, \nu_{v1} - \nu_{v2}, \) and \(v_{z1} - v_{v2} \), and we have the following equation

$$- \frac{x}{y} \frac{\partial E_{z2}}{\partial \xi} - K_{\nu\rho} \frac{\partial B_{z1}}{\partial \tau} = \frac{\partial B_{z1}}{\partial \xi}$$

Also, from Eqs. (51), (53), and (54), it follows that

$$\frac{\partial E_{z2}}{\partial \xi} = - \frac{\nu_{v1} \cdot \text{cosec} \theta}{v_{\gamma}} \cdot \frac{\partial}{\partial \tau} - \nu_{v1} \frac{\partial}{\partial \xi} \cdot \nu_{v1} + \frac{\nu_{v1} \cdot \text{cosec} \theta}{v_{\gamma}} \frac{\partial \nu_{v1}}{\partial \xi} \cdot \frac{\partial B_{z1}}{\partial \tau}$$

$$- K_{\nu\rho} \cdot \text{cosec} \theta \cdot u_{v1} - \nu_{v1} \cdot B_{z1} - \frac{\text{cosec} \theta}{v_{\gamma}} \left( \frac{\partial}{\partial \tau} + \nu_{v1} \frac{\partial}{\partial \xi} \cdot \nu_{\gamma} \cdot p_{v1} - p_{v1} \right)$$

$$- K_{\nu\rho} n_{1} \cdot \nu_{v1} - \nu_{v1} \cdot \frac{\text{cosec} \theta}{v_{\gamma}} \frac{\partial \nu_{v1}}{\partial \xi}$$

$$- \frac{\text{cosec} \theta}{v_{\gamma}} \left( \nu_{v1} - \nu_{v2} \right) \frac{\partial \nu_{v1}}{\partial \xi} - \frac{\nu_{v1} \cdot \text{cosec} \theta}{v_{\gamma}} \cdot \nu_{v1} \cdot \frac{\partial p_{v1}}{\partial \xi} . \quad (56)$$
If we substitute Eq. (56) into Eq. (55), \( t_{2} \) as well as \( E_{\mu 2} \) is eliminated by virtue of the equation for \( t_{p0} \), Eq. (17). Using the relationship among the first-order quantities, Eqs. (37), (40), (43), and (50), we obtain the Korteweg-de Vries equation:

\[
\frac{\partial n}{\partial z} - \alpha n \frac{\partial n}{\partial \xi} - \mu \frac{\partial^3 n}{\partial \xi^3} = 0. \tag{57}
\]

where,

\[
\alpha = \frac{3t_{p0}^2 - c_{s}^2 \cos^2 \theta - \left( t_{p0}^2 - t_{c}^2 \cos^2 \theta \right) \left( c_{l}^2 - \gamma p_{0} - \gamma p_{\phi} \right)}{4t_{p0}^2 - \left( t_{c}^2 - t_{s}^2 \right) \gamma p_{0}}. \tag{58}
\]

\[
\mu = \frac{t_{p0}^2 - t_{c}^2 - c_{s}^2}{4R_{s}R_{c} - t_{p0}^2 - t_{c}^2} \left( 1 - \frac{R_{c} - R_{s} \gamma p_{0}^2 \cos^2 \theta}{R_{c}R_{s} - t_{c}^2 \cos^2 \theta} \right). \tag{59}
\]

Equation (57) represents the Korteweg-de Vries equation for both the magnetosonic fast and slow waves in finite \( \beta \) plasmas: note that we have used Eq. (17) but not used Eq. (18) to derive Eq. (57). The \( t_{p0} \) in Eqs. (58) and (59) can be the phase-velocity of the fast wave (upper sign in Eq. (18)) or the slow wave (lower sign in Eq. (18)). In the cold plasma limit, the slow wave disappears, and Eqs. (57) to (59) reproduce the nonlinear equation for the magnetosonic fast wave in zero \( \beta \) plasmas derived by Kakutani et al.\textsuperscript{12,13}

As we have seen in Eqs. (24) to (26), the terms \( t_{p0}^2 - c_{s}^2 \), \( t_{p0}^2 - t_{c}^2 \cdot 2 \), and \( t_{p0}^2 - t_{c}^2 \cos^2 \theta \), are all positive for the fast wave, while they are all negative for the slow wave. Similarly, it is readily shown that the term \( t_{p0}^2 - c_{s}^2 \cos^2 \theta \) is also positive for the fast wave and negative for the slow wave. Therefore, the coefficient \( \alpha \) is always positive for both the the fast and slow waves. The coefficient \( \mu \) is
positive for the slow wave. For the fast wave, \( \mu \) is negative for the angles \( 0 < \theta < \theta_c \), and positive for \( \theta_c < \theta \leq \pi/2 \), where the critical angle \( \theta_c \) is defined by Eq. (27), or approximately by Eq. (28).

IV. EVALUATION OF THE \( \mathbf{i}_p \cdot \mathbf{B} \) ACCELERATION

In Refs. 5-7, the magnitude of the \( \mathbf{i}_p \cdot \mathbf{B} \) acceleration of ions was estimated by using a nonlinear theory for the magnetosonic soliton in a zero \( \beta \) plasma. The values thus estimated were in good agreement with those measured in the simulation for the magnetosonic shock waves in a low \( \beta \) plasma. In this section we will also evaluate the magnitude of the \( \mathbf{i}_p \cdot \mathbf{B} \) acceleration of ions in the nonlinear fast and slow waves in finite \( \beta \) plasmas using the soliton theory developed in the previous section. For the sake of definiteness we consider a case of one soliton solution of the Korteweg-de Vries equation, Eq. (57).

Assuming that \( n_1 \) approaches 0 as \( |\xi| \rightarrow \infty \), we have the one soliton solution of Eq. (57) as

\[
\psi(t) = \frac{a_0}{2} \sech^2 s,
\]

where \( a_0 \) is the amplitude, and \( a = 1 \) when \( \mu > 0 \), and \( a = 1 \) when \( \mu < 0 \). The argument \( s \) is

\[
s = \frac{\delta}{12\beta \mu} \left( \frac{1}{2} \alpha \xi - \frac{\delta^2}{3} \right).
\]

\[
= \frac{\xi_0}{12\beta \mu} \left( \frac{1}{2} \alpha \xi - \frac{\xi_0^2}{3} \right).
\]

- 17 -
with \( \varepsilon \) being the "true" amplitude \( \varepsilon_0 = \varepsilon \phi \).

The slow wave has a positive soliton, because \( \alpha \) and \( \mu \) are both positive. The density perturbation is positive, with a propagation speed greater than that of the linear slow wave. The magnetic field perturbation \( B_z \) is, however, negative because of Eq.\((50b)\): in the slow wave the sign of the magnetic field perturbation \( B_z \) is opposite to that of the density perturbation. On the other hand, the fast wave has a positive soliton for the angles \( \theta_c < \theta < \theta_c \) and a negative soliton for \( 0 < \theta < \theta_c \). In the fast wave, the perturbations of the magnetic field \( B_z \) and the density have a same sign.

The magnitude of the \( v_p \cdot B \) acceleration is roughly given by the sum of the \( E \cdot B \) drift velocity and the wave phase-velocity \( v_p \). Since we can calculate the electric field strength \( E \) from Eqs.\((50e)\) and \((60)\), we can evaluate the magnitude of the \( v_p \cdot B \) acceleration in the nonlinear magnetosonic fast and slow waves in finite \( \beta \) plasmas. The electric field in the direction of the wave normal is

\[
E_{\parallel} = \frac{2\delta^3}{\alpha \omega_i \mu} \left\{ \frac{R_0 - R_1}{R_e R_1} \frac{v_p^2 - C_s^2 \gamma_p^2}{(v_p^2 - v_A^2 \cos^2 \theta)} \right\} \left( \frac{\gamma_e \gamma_p^2}{R_e} \right) \left( \frac{\gamma_i \gamma_p^2}{R_i} \right)
\]

\[
\times \text{sech}^2 \psi \left( \text{tanh} \, \psi \right).
\]

The term \( \text{sech}^2 \psi \left( \text{tanh} \, \psi \right) \) has its maximum value when \( \text{tanh}^2 \psi = 1.3 \) ( \( \text{sech}^2 \psi = 2.3 \) ), therefore the maximum value of \( E_{\parallel} \) is

\[
E_{\parallel} = \frac{2\delta^3}{9 \alpha \omega_i \mu} \left\{ \frac{R_0 - R_1}{R_e R_1} \frac{v_p^2 - C_s^2 \gamma_p^2}{(v_p^2 - v_A^2 \cos^2 \theta)} \right\} \left( \frac{\gamma_e \gamma_p^2}{R_e} \right) \left( \frac{\gamma_i \gamma_p^2}{R_i} \right).
\]

We now express the physical quantities by unnormalized variables to
estimate specific values of the $v_p \cdot B$ acceleration. In unnormalized variables, $v_\parallel^2$ and $c_s^2$ defined by Eqs. (14) and (15) coincide with ordinary definitions of the Alfven and sound speeds,

$$v_\parallel^2 = B^2 \cdot 4\pi n_i (m_i - m_0),$$

$$c_s^2 = c_l^2 - c_p^2,$$

where

$$c_j = \frac{\gamma \cdot \rho_j}{\rho_i \cdot \frac{m_i}{m_j} - c_l^2}, \quad j = i \text{ or } e.$$

The magnitude of the $E_i \cdot B$ drift velocity is written with the unnormalized variables as

$$\frac{\mu}{\rho_i} = \frac{2 \cdot 3 \cdot \left| \gamma - 1 \right| \cdot \frac{3}{2}}{\rho_i \cdot \rho_j} \left( 1 - \frac{m_i}{m_j} \cdot \frac{v_p^2 - c_s^2 \cdot c_p^2}{v_p^2 - \frac{3}{2} \cdot c_s^2 \cdot \cos^2 \theta} - c_s^2 \cdot \frac{m_j}{m_i} \cdot c_l^2 \right),$$

where

$$\bar{a} = \frac{3 \left( v_p^2 - c_s^2 \cdot \cos^2 \theta \right) - v_p^2 - \frac{3}{2} \cdot c_s^2 \cdot \cos^2 \theta \cdot (c_l^2 - \gamma_i \cdot c_s^2 - \gamma_p \cdot c_l^2)}{4v_p^2 \cdot (v_p^2 - (v_i + c_l^2) \cdot 2)},$$

$$\bar{\mu} = \frac{v_p^2 \cdot (v_p^2 - c_s^2)}{4 \cdot \rho_i \cdot \rho_j \cdot v_p^2 - (v_i + c_l^2) \cdot 2} \left( 1 - \frac{\gamma_i \cdot m_i \cdot m_j \cdot v_p^2 \cdot \frac{3}{2} \cdot \cos^2 \theta}{\rho_i \cdot \rho_j \cdot v_p^2 - \frac{3}{2} \cdot \cos^2 \theta} \right),$$

and $N$ is the Mach number, which is the ratio of a soliton speed to the (fast or slow) magnetosonic speed $v_{ps}$ and is related to the amplitude $u_0$ as $N = 1 - \alpha u_0 / 3$. Here, the characteristic speed, $u_0 = l_0 / t_0$, to which the velocities were normalized, is taken to be the phase velocity $v_{ps}$. The potential is given by
In the limit of $\theta = 2$, the potential is expressed as

$$e^* = \frac{3m_i \gamma^2}{\alpha} \left\{ \frac{1}{\alpha} \left( \frac{v_i^2 - c_i^2}{\gamma_i^2 c_i^2 + \gamma_e^2 c_e^2} \right) \left[ 1 - \frac{m_e}{m_i} \left( \frac{v_i^2 - c_i^2}{m_i c_i^2} \right) \right] \right\} \text{sech}^2 x. \quad 70$$

Further, in the limit of $\rho, \beta \to 0$, and $\theta = 2$, the potential jump is reduced to a simple form: $2m_i v_i^2 \gamma^2 \gamma_i$.

Let us discuss the acceleration in the fast and slow waves separately.

A. Fast Wave

The acceleration in a low $\beta$ plasma was studied in detail by theory and simulation in Refs. 5 to 7. Our theory in the low $\beta$ limit agrees with those results. As can be seen from Eq. (61), $|\mu|^1$ gives a measure of the soliton width. In the cold plasma limit, the soliton width is of the order of the electron inertial length, $c/\omega_{pe}$, for the angles $\theta < 0 : e < 2$. While it is of the order of the ion inertial length, $c/\omega_{pi}$, for $0 < \theta < \theta_c$. Since the magnitude of the potential jump is independent of $\mu$, the electric field strength $E_i$ is about $m_e m_i^{1/2}$ times larger for the angles $\theta < 0 : e < 2$ than for $0 < \theta < \theta_c$. Therefore, the $\nu_p \cdot B$ acceleration is stronger in the region $\theta_i < 0 : e < 2$; in this region the magnitude of the $\nu_p \cdot B$ acceleration is given by Eq. (3).

In a high $\beta$ case, the $\nu_p \cdot B$ acceleration is strong also in the quasi-perpendicular propagation. However, we should note that the value of the critical angle $\theta_c$ decreases as the total $\beta$ value is increased (see
Eq. (28)). This leads to the broadening of the region of extremely strong acceleration. Another important point is that the electron pressure raises the electric potential in the soliton, resulting in a stronger electric field $E_x$ (see Eqs. (67) and (70)). This makes the magnitude of the acceleration large for all angles $\theta$.

We show in Figs. 1 to 3 the soliton width $|\overline{\mu}|^{1/2}$, the potential divided by the ion temperature $c_T T_i$, the $E \cdot B$ drift speed $cE_i B$, and the ratio $R_\parallel$.

\[
R_\parallel = \frac{1}{\gamma^3 - \frac{m_e}{m_i} - \frac{\gamma \bar{m}_e}{\gamma T_i - \gamma T_i \cos^2 \theta}}.
\]  

as a function of the propagation angle $\theta$ for three cases: the low $\beta$ case (Fig. 1), the high $\beta$ case with $T_e > T_i$ (Fig. 2), and the high $\beta$ case with $T_e = T_i$ (Fig. 3). The numerator in $R_\parallel$ is essentially the potential due to the Boltzmann distribution, and is zero in the cold plasma model. The denominator in $R_\parallel$ remains finite even in the limit of zero $\beta$ plasmas. Since the potential is proportional to the sum of the numerator and denominator of $R_\parallel$ (Eq. (70)), the quantity $R_\parallel$ shows the relative importance of the potential due to the Boltzmann distribution in the total potential.

The magnetic field strength and the electron density are chosen to be $B = 10^4$ gauss, and $n = 1$/cm$^3$, respectively, for three cases. From the values of $B$ and $n$, we can estimate the electron and ion inertial lengths and the Alfven speed as $c/\omega_{pe} = 5.3 \times 10^5$ cm/s, $c/\omega_{pi} = 2.3 \times 10^7$ cm/s, and $v_A = 2.2 \times 10^7$ cm/sec, respectively, for plasmas with the ion-to-electron mass ratio $m_i/m_e = 1836$.

We change the electron and ion temperatures in Figs. 1 to 3 keeping the magnetic field strength and the plasma density unchanged. The Mach number $M$ and the specific heat ratios are set to $|M - 1| = 1$, and $\gamma_i = \gamma_e = 5.3$, respectively.
In the low $\beta$ case (Fig. 1), we put $T_e = T_i = 10$ eV, and hence the $\beta$ value is $\beta \sim 8 \cdot 10^{-2}$. The ion thermal speed is $v_i = T_i / m_i = 1.2 \cdot 3.1 \cdot 10^7$ cm/sec. The soliton width takes its minimum value at the critical angle $\theta_c$ (Fig. 1(a)); in the region $\theta \approx \theta_c \lesssim 2$, the characteristic soliton width is the electron inertial length $\lambda_{ee}$. As $\theta$ decreases from the critical angle $\theta_c$, the $|\mu|^2$ increases. For the angles such that $\tan \theta \sim 1$, the characteristic soliton width is the ion inertial length $\lambda_{ii}$. The potential monotonically increases as $\theta$ decreases from $\pi/2$. The drift speed $v \times B$ takes its maximum value at $\theta = \pi$, and minimum value at $\theta = \pi/2$ (Fig. 1(c)). The minimum value is $v \times B = 5 \cdot 10^7$ cm/sec, which is about $2v_i$. In the perpendicular propagation, the drift speed is $v \times B = 1 \cdot 10^7$ cm/sec $+ 5v_i$. The ratio $R_i$ is shown in Fig. 1(d); it is smaller than $3.4 \cdot 10^{-2}$ for all angles. This indicates that in this low $\beta$ case, the term $\frac{1}{\beta} \frac{1}{\mu} \frac{1}{\mu} \frac{1}{\mu}$ is not so important in Eq. (70). As seen in Fig. 1, in the vicinities of $\theta = 0$ and $\theta = \pi$, some physical quantities diverge. This suggests that in these small regions our perturbation method needs some modification. The same problem also arose in the cold plasma theory by Kakutani et al. At the critical angle $\theta_c$, they employed stretched coordinates slightly different from ours, Eqs. (29) and (30), and derived a generalized Korteweg-de Vries equation. A correct theory for the nonlinear magnetosonic waves in the vicinity of $\theta = 0$ has not been made even in the cold plasma model. We will not discuss the region in the vicinity of $\theta = 0$.

Figure 2 shows $|\mu|^2$, $v \times T_i$, $v \times B$, and $R_i$ in the high $\beta$ case, $\beta = 2.1$, with $T_e = 5000$ eV and $T_i = 10$ eV ($v_i = 3.1 \cdot 10^7$ cm/sec). The soliton width has its minimum value at $\theta = \theta_c$ as in the low $\beta$ case. The critical angle $\theta_c$ is, however, shifted to a lower value in the high $\beta$ case from $0.493\pi$ to $0.488\pi$: the region of $\theta \approx 0$ has increased from $0.07\pi$ in Fig. 1 to $0.12\pi$ in Fig. 2. Unlike the low $\beta$ case, the $|\mu|^2$ takes the maximum value at
The maximum value ($\sim 6 \times 10^6 \text{cm}$) is smaller by a factor of four than the ion inertial length $c_\perp \approx 2.3 \times 10^7 \text{cm}$. The potential is rather a flat function of $\theta$. The minimum value of $cE_z B$ is at $\theta = 0.1 \pi$, and its minimum value is $cE_z B = 1.4 \times 10^5 \text{cm/sec} \approx 6.4 \mu_A$. At $\theta = 2$, the drift speed is $cE_z B = 5 \times 10^7 \text{cm/sec} \approx 230 \mu_A$. Comparing these values of $cE_z B$ with those in the low $\beta$ case, we note three important points. First, the critical angle is shifted to the lower value, hence the region of extremely strong acceleration is broadened. Second, the angle of the minimum $cE_z B$ is changed to the lower value from $\theta = 4$ in Fig. 1(c) to $\theta = 0.1 \pi$ in Fig. 2(c). Third, the minimum value of $cE_z B = 1.4 \times 10^5 \text{cm/sec}$ is about three times larger than the one in the low $\beta$ case. From the second and third points, it follows that even in the region $\theta = 0.1 \pi$, fairly strong $i_p \cdot B$ acceleration can occur for a large region of the propagation angles in the nonlinear magnetosonic fast wave in high $\beta$ plasmas. Figure 2(d) shows that $R_b > 1.7$ for all angles $\theta$. This indicates that the potential due to the Boltzmann distribution ($c_\perp^2$ in Eq. (70)) becomes important in the high $T_e$ plasma.

Plotted in Fig. 3 are $|\hat{p}|^{1/2}$, $c_\perp T_e$, $cE_z B$, and $R_b$ in the high $\beta$ plasma $\beta = 2.1$, with $T_e = 100 \text{eV}$ and $T_i = 500 \text{eV}$. In such a high $T_e$ plasma, kinetic effects (Landau damping and finite Larmor radius effects) may modify the wave properties from those described in the previous sections. Although the kinetic effects are not included in our theory, we here show such a high $T_e$ case to illustrate the basic features of the parameter dependence of the nonlinear fast wave and the associated $i_p \cdot B$ acceleration. The soliton width $|\hat{p}|^{1/2}$ has the same value as the one in Fig. 2(a). An interesting point is that, although the plasma $\beta$ value is the same as the one in Fig. 2, the $cE_z B$ in Fig. 3(c) is smaller than the one in the high $T_e$ case (Fig. 2(c)) by a factor of 2 to 10. This is because the term $(c_\perp^2 - m_e/m_i c_\perp^2)$ in the potential expression, Eq. (70), is small for low
T., Figure 3(d) verifies that this term is unimportant: the ratio \( R_\theta \) is smaller than 0.1 except the region near \( \theta = 0 \).

B. Slow Wave

The slow wave has the negative dispersion \( \mu < 0 \) for all angles \( \theta \), and hence unlike the fast wave, it has no critical angle where \( \mu \) becomes zero.

In a low \( \beta \) plasma, the phase velocity of the slow wave is of the order of the sound speed.

\[
\frac{v_s^2 - c_s^2 \cos^2 \theta}{v_i^2} = \frac{c_s^2 \sin^2 \theta \cos^2 \theta}{v_i^2 - c_s^2 \cos^2 \theta} \quad 0 \frac{v_i^2}{c_s^2} \approx 1. \tag{73}
\]

and the soliton width \( \mu^{1/2} \) is much smaller than the ion inertial length.

\[
\mu^{1/2} \approx \frac{\sqrt{2}}{2} \frac{c_s}{v_i} \frac{|v_s^2 - c_s^2|^{1/2}}{v_i} \approx \frac{c_s}{v_i}. \tag{74}
\]

Although this quantity is quite small, the \( E \parallel B \) velocity cannot be so large because the potential jump is also small: the potential jump is roughly proportional to the small quantity \( v_{s0}^2, c_s^2 \) in a low \( \beta \) plasma.

On the other hand, as the plasma \( \beta \) value is increased, the phase velocity of the slow wave tends to the phase velocity of the shear Alfven wave, \( v_s^2 - c_s^2 \cos^2 \theta \), it readily follows from Eq.(26) that

\[
\frac{v_s^2 - c_s^2 \cos^2 \theta}{c_s^2} = \frac{v_{s0}^2 \sin^2 \theta \cos^2 \theta}{c_s^2 - v_{s0}^2 \cos^2 \theta} \quad 0 \frac{v_{s0}^2}{c_s^2} \approx 1. \tag{75}
\]

when \( c_s^2 < v_{s0}^2 \). Because of this effect, the potential jump becomes large in a high \( \beta \) plasma. The soliton width \( \mu^{1/2} \) is comparable to or larger than the
ion inertial length.

$$\mu^{-2} = \frac{C}{2^{1/2} \pi \gamma_{sp}} \frac{C_{2}}{v_{1} \sin \theta}$$

Since the potential jump and the characteristic length are roughly proportional to $u_{p0}^{-2} (v_{2} \cos \theta)^{-1}$ and $u_{p0}^{-2} (v_{2} \cos \theta)^{-2}$, respectively, the $E_{i} \cdot B$ drift velocity changes nearly proportionally to $u_{p0}^{-2} (v_{2} \cos \theta)^{-1}$ in the high $\beta$ regime and can be very large.

We show in Figs. 4 to 6 the parameter dependence of the soliton width $\mu^{-2}$, the potential $\kappa$, $T_{i}$, the drift speed $v_{1} E_{i} \cdot B$, and the ratio $R_{\delta}$. We choose the plasma parameters same as those in Figs. 1 to 3, i.e., $B = 10^{4}$ gauss and $n = 10^{37}$ cm$^{-3}$, and consequently, $c_{\omega_{pi}} = 5.3 \times 10^{4}$ cm/sec, $\Lambda_{0} = 2.3 \times 10^{5}$ cm, and $\chi_{i} = 2.2 \times 10^{7}$ cm/sec. The Mach number $\gamma$ is again set to 5.2.

In Fig. 4, $\kappa$ and $T_{i}$ are taken to be $\kappa = 10^{4}$ eV as in Fig. 1, hence $\mu = 3 \times 10^{-2}$ and $\nu = 3.1 \times 10^{6}$ cm/sec. Unlike the fast wave, there is no critical angle in the physical quantities except for the point $\theta = 0$. Since the theory may not be valid in the vicinity of $\theta = 0$, we will not discuss the behavior in this small region. The soliton width $\mu^{-2}$ ranges from $0 \leq \mu_{0}$ to $4 \times 10^{6}$ cm : $0.17 < \theta_{0}$). The potential does not change much as $\theta$ moves from 0 to $\pi$. For most of the region of $\theta$, $E_{i} \cdot B$ is in the range from $4 \times 10^{6}$ cm/sec to $2 \times 10^{7}$ cm/sec, which is not so large compared to $\nu_{i}$ and $\nu_{B}$. The slow wave will not give rise to the strong $v_{i} \cdot B$ acceleration in a low $\beta$ plasma. The ratio $R_{\delta}$ is rather large, $R_{\delta} \beta$: the potential is generated mostly by the Boltzmann distribution.

Figure 5 shows the high $\beta$ case, $\beta \gg 1$, with $T_{i}$, $500 eV$ and $T_{i}$, $100 eV$ ($\nu_{i} = 3.1 \times 10^{6}$ cm/sec). For most of the region, $\kappa = 0 \leq \pi$, the physical quantities are fairly flat functions of $\theta$. The width $\mu^{-2}$ is much larger
than the one in the low $\beta$ case; $\mu_{12}^2 \gtrsim c\gamma_{Bi} \left(2.3 \times 10^7 \text{ cm/sec}\right)$. In the region $\pi/2 \leq \theta \leq 2$, the drift speed $cE_{r}B$ is $8 \times 10^7 \text{ cm/sec}$, which is about thirty times as large as the ion thermal speed $\upsilon_{Ti}$, and four times as large as the Alfvén speed $\upsilon_{A}$. Hence the strong ion acceleration in the wide region of $\theta$ is expected in the slow wave in high $\beta$ plasmas. The ratio $R_{3}$ is 0.1 to 0.5 for most of the angles.

We show in Fig.6 the high $\beta$ case, $\beta=2.1$, with $T_{e}=10\text{eV}$ and $T_{i}=500\text{eV}$. The total $\beta$ value is the same as the one in Fig.5, however, the ion temperature is raised and the electron temperature is lowered. As in Fig.3, the kinetic effects of the ions may be important in the high $T_{i}$ plasma. Even though our results are obtained from the two-fluid model without kinetic effects, we show such a case in Fig.6 to see basic features of the parameter dependence of the nonlinear slow wave and the $\upsilon_{p} \cdot B$ acceleration. Comparing Figs.5 and 6, we find that the soliton width $\mu_{12}^2$ has the same values in two cases. Even if the ion and electron temperatures are changed, the value of $\mu_{12}^2$ remains unchanged as long as the total $\beta$ is kept fixed. (See Eq. (68); note that $\gamma_{i}=\gamma_{e}$ is assumed in the calculation.) The drift speed $cE_{r}B$ is about a half of that in Fig.5. This is because the $R_{3}$ is so small in the high $T_{i}$ and low $T_{e}$ plasma that the contribution of the potential due to the Boltzmann distribution is very small. However, since the value of $R_{3}$ in the slow wave is rather small even in the high $T_{e}$ case, the difference between the ion and electron temperatures is not as important as in the fast wave.

C. Unlimited Acceleration

Katsouleas and Dawson\textsuperscript{23} have analyzed the $\upsilon_{p} \cdot B$ acceleration of electrons in laser-plasma beat-waves. They pointed out that if the
electric field is so large that $eE/B > c$. Then the resonant electrons are not detrapped. As a result, the resonant electrons gain extremely high energy by the $\mathbf{v}_p \cdot \mathbf{B}$ acceleration (unlimited acceleration).

Before closing this section, we will briefly discuss the possibility of the unlimited acceleration of ions by the magnetosonic waves.

Using Eq. (67), we can write the condition that the drift speed exceeds the light speed as

$$
\left[ \frac{2}{3} \left( \frac{\gamma - 1}{\gamma + 1} \right)^{\frac{3}{2}} \right] \left[ 1 - \frac{m_e}{m_i} \left( \frac{\gamma}{\gamma - 1} - \frac{c_s^2}{c_i^2} \right) \right] \geq c. \tag{77}
$$

For the perpendicular magnetosonic fast wave, this condition is reduced to

$$
\left[ \frac{4}{3} \right] \left[ \frac{\gamma - 1}{\gamma + 1} \right] c_s^2 \left( \frac{\gamma}{\gamma - 1} - \frac{c_s^2}{c_i^2} \right) \left[ 1 - \frac{m_e}{m_i} \right] \geq c. \tag{78}
$$

This condition is satisfied, for instance, in a low $\beta$ plasma with a high Alfven speed.

$$
\frac{1}{\gamma} \geq \frac{m_e}{m_i}^{1/2}. \tag{79}
$$

unless the Mach number is too close to unity. Therefore, the unlimited acceleration of ions by the magnetosonic fast wave could occur in a plasma with a high Alfven speed. In a real plasma there may be some saturation mechanism for the acceleration. However, at least, it is very likely that relativistic ions are produced by magnetosonic shock waves in such a plasma.

For more detailed discussion, we need a relativistic theory for the waves and ion motions.
V. SUMMARY

On the basis of the two-fluid model with finite ion and electron pressures, we have derived the Korteweg-de Vries equation that is applicable to both the nonlinear magnetosonic fast and slow waves. Using the nonlinear theory, we have evaluated the magnitude of the $v_p \cdot B$ acceleration of ions in the magnetosonic fast and slow waves.

As in a cold plasma, the magnetosonic fast wave in a finite $\beta$ plasma has a positive soliton in the region $\theta_c < \theta < \pi/2$ and has a negative soliton in the region $\pi/2 < \theta < \pi$. The critical angle $\theta_c$ is shifted to lower values by finite $\beta$ effects, and hence the region of the positive soliton is broadened.

The slow wave has a positive soliton for all angles of propagation; the density perturbation in the soliton is positive, with the propagation speed greater than that of the linear slow wave. As is well known, the perturbation of the magnetic field $B_z$ is negative when the density perturbation is positive in the slow wave.

Finite $\beta$ effects enhance the $v_p \cdot B$ acceleration in the magnetosonic fast and slow waves. For the fast wave, the $\theta$ values that correspond to extremely strong acceleration almost coincide with the $\theta$ values for the positive soliton. Since the critical angle $\theta_c$ decreases as the total $\beta$ value is increased, the region for extremely strong acceleration increases with the total $\beta$ value. Moreover, the electron pressure raises the potential jump and also increases the magnitude of the acceleration for all propagation angles $\theta$.

The ion acceleration in the magnetosonic slow wave is rather weak.
when the $\beta$ value is small. However, when the $\beta$ value is large, $\sim 0.1$, fairly strong acceleration is expected. In the slow wave, the dependence of the acceleration on the propagation angle is not as strong as in the fast wave.

It is also shown that unlimited acceleration of ions by the magnetosonic fast wave is possible in a plasma with a high Alfvén speed, $v_A > C : \omega_A, m_i, \nu_i$. 

Acknowledgment

The author is grateful to Prof. T. Taniuti for invaluable discussions.
References


Figure Captions

Fig. 1 Dependence of various quantities in a magnetosonic fast wave on propagation angle \( \theta \). (a) Soliton width \( |\tilde{\mu}|^{1/2} \), (b) electric potential \( |e\phi/T_i| \), (c) \( E \times B \) velocity \( |cE_x/B| \), and (d) the ratio \( R_\theta \) defined by Eq. (72). Total \( \beta \) value is \( \beta=0.08 \), with \( T_e=T_i=10\text{eV} \).

Fig. 2 Dependence of various quantities in a magnetosonic fast wave on propagation angle \( \theta \). (a) Soliton width \( |\tilde{\mu}|^{1/2} \), (b) electric potential \( |e\phi/T_i| \), (c) \( E \times B \) velocity \( |cE_x/B| \), and (d) the ratio \( R_\theta \) defined by Eq. (72). Total \( \beta \) value is \( \beta=2.1 \), with \( T_e=500\text{eV} \) and \( T_i=10\text{eV} \).

Fig. 3 Dependence of various quantities in a magnetosonic fast wave on propagation angle \( \theta \). (a) Soliton width \( |\tilde{\mu}|^{1/2} \), (b) electric potential \( |e\phi/T_i| \), (c) \( E \times B \) velocity \( |cE_x/B| \), and (d) the ratio \( R_\theta \) defined by Eq. (72). Total \( \beta \) value is \( \beta=2.1 \), with \( T_e=10\text{eV} \) and \( T_i=500\text{eV} \).

Fig. 4 Dependence of various quantities in a magnetosonic slow wave on propagation angle \( \theta \). (a) Soliton width \( |\tilde{\mu}|^{1/2} \), (b) electric potential \( |e\phi/T_i| \), (c) \( E \times B \) velocity \( |cE_x/B| \), and (d) the ratio \( R_\theta \) defined by Eq. (72). Total \( \beta \) value is \( \beta=0.08 \), with \( T_e=T_i=10\text{eV} \).

Fig. 5 Dependence of various quantities in a magnetosonic slow wave on propagation angle \( \theta \). (a) Soliton width \( |\tilde{\mu}|^{1/2} \), (b) electric potential \( |e\phi/T_i| \), (c) \( E \times B \) velocity \( |cE_x/B| \), and (d) the ratio \( R_\theta \) defined by Eq. (72). Total \( \beta \) value is \( \beta=2.1 \), with \( T_e=500\text{eV} \) and \( T_i=10\text{eV} \).

Fig. 6 Dependence of various quantities in a magnetosonic slow wave on propagation angle \( \theta \). (a) Soliton width \( |\tilde{\mu}|^{1/2} \), (b) electric potential \( |e\phi/T_i| \), (c) \( E \times B \) velocity \( |cE_x/B| \), and (d) the ratio \( R_\theta \)
defined by Eq. (72). Total $\beta$ value is $\beta = 2.1$, with $T_e = 10\text{eV}$ and $T_1 = 500\text{eV}$. 
Fig. 1
(a) \( \frac{C}{\omega_{pe}} \)

\( \theta = 10 \text{ eV} \)

\( T_i = 500 \text{ eV} \)

\( \beta = 2.1 \)

(b) \( \frac{|e \gamma|}{T_i} \)

(c) \( U_A = 2.2 \times 10^7 \)

\( U_{Ti} = 2.2 \times 10^7 \)

(d) \( \frac{|CE_x|}{B} \)

\( R_B \)

Fig. 3
\[ |\tilde{u}| \frac{1}{2} = \frac{C}{\omega_{pe}} \]

\[ T_e = 10 \text{eV} \]
\[ T_i = 10 \text{eV} \]
\[ \beta = 0.08 \]

\[ \frac{|e\gamma|}{T_i} \]

\[ \frac{U_A}{B} \]
\[ U_A = 2.2 \times 10^7 \]
\[ U_{Ti} = 3.1 \times 10^6 \]

\[ R_B \]

**Fig. 4**
(a) $T_e = 500\,\text{eV}$  
$T_i = 10\,\text{eV}$  
$\beta = 2.1$

(b) 

(c) $V_A = 2.2 \times 10^7$  
$V_{Ti} = 3.1 \times 10^6$

(d) 

Fig. 5
Fig. 6

(a) $T_e = 10\text{eV}$
$T_i = 500\text{eV}$
$\beta = 2.1$

(b) $\frac{|e\varphi|}{T_i}$

(c) $\mathcal{U}_A = 2.2 \times 10^7$
$\mathcal{U}_{Ti} = 2.2 \times 10^7$

(d) $\frac{|CE_{xi}|}{B}$