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SOME CURVATURE PROPERTIES OF QUARTER SYMMETRIC METRIC CONNECTIONS *

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ABSTRACT

A linear connection Γ_{ji}^{h} with torsion tensor $T_j^h p_i - T_i^h p_j$, where T_j^h is an arbitrary (1,1) tensor field and p_i is a 1-form, has been called a quarter-symmetric connection by Golab [3]. Some properties of such connections have been studied by Rastogi [5], Mishra and Pandey [4] and Yano and Imai [6]. In this paper based on the curvature tensor of quarter-symmetric metric connection we define a tensor analogous to conformal curvature tensor [2] and study some properties of such a tensor.

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1. INTRODUCTION

Let ∇ be a linear connection in a Riemannian manifold M of dimensions n (>2) such that it is expressed locally as follows [6]:

$$\Gamma_{ji}^{h} = \left\{ {h \atop ji} \right\} + U_j^h p_i - p_j V_i^h - U_{ji} p^h, \tag{1.1}$$

where $\left\{ {h \atop ji} \right\}$ are Christoffel symbols, U_{ji} and V_{ji} are, respectively, symmetric and skew-symmetric parts of an arbitrary tensor T_{ji}, $U_j^h = U_{jt} g^{th}$, $V_j^h = V_{jt} g^{th}$ and p_j is a 1-form such that $g^{hj} p_j = p^h$.

Γ_{ji}^{h} given by (1.1) is a quarter-symmetric connection with torsion tensor $T_j^h p_i - T_i^h p_j$ and curvature tensor R_{kji}^{h} expressible as [6]

$$\begin{aligned} R_{kji}^{h} = K_{kji}^{h} - [& U_k^h p_{ji} + p_k^h U_{ji} - p_i \nabla_k U_j^h \\ & + p^h \nabla_k U_{ji} + V_i^h \nabla_k p_j + p_j \nabla_k V_i^h \\ & - V_t^h p_j p_i U_k^t + U_{jt} V_i^t p_k p^h \\ & + U_k^h V_i^t p_j p_t - p_k p^t V_t^h U_{ji}] - j|k, \end{aligned} \tag{1.2}$$

where $p_{ji} = \nabla_j p_i - U_{jt} p^t p_i + \frac{1}{2} U_{ji} p^t p_t$, $p_i^h = p_{it} g^{th}$ and $-j|k$ means interchange of indices j and k and subtraction.

2. SYMMETRIC T_{ji}

If T_{ji} is symmetric $V_{ji} = 0$ and (1.1) and (1.2), respectively, reduce to

$$\Gamma_{ji}^{h} = \left\{ {h \atop ji} \right\} + T_j^h p_i - T_{ji} p^h \tag{2.1}$$

and

$$R_{kji}^{h} = K_{kji}^{h} - [T_k^h p_{ji} + p_k^h T_{ji} - p_i \nabla_k T_j^h + p^h \nabla_k T_{ji}] - j|k. \tag{2.2}$$

From (2.2) it is easy to get

$$\begin{aligned} R_{ji} = K_{ji} - T p_{ji} - p T_{ji} + T_j^h p_{hi} + p_j^h T_{hi} \\ + p_i (\nabla_h T_j^h - \nabla_j T) - (\nabla_h T_{ji} - \nabla_j T_{hi}) p^h \end{aligned} \tag{2.3}$$

and

$$R = K - 2Tp + 2T^t_m p^m_t + 2(\nabla_t T^t_m - \nabla_m T)p^m, \tag{2.4}$$

where $T = T^t_t$, $p = p^t_t$.

Now on the basis of R_{kjih}, R_{ji} and R we define the following tensor:

$$\begin{aligned} {}^*C_{kjih} \overset{\text{def.}}{=} & R_{kjih} - (n-2)^{-1}[g_{kh}R_{ji} - R_{ki}g_{hj} + g_{ji}R_{kh} \\ & - R_{jh}g_{ki} - (n-1)^{-1}R(g_{kh}g_{ji} - g_{jh}g_{ki})], \end{aligned} \tag{2.5}$$

which vanishes if $R_{kjih} = 0$.

${}^*C_{kjih}$ satisfies (i) ${}^*C_{kjih}\, g^{kh} = 0$, (ii) For $R_{ji} = 0$, ${}^*C_{kjih} = R_{kjih}$.

If in (2.5) we substitute from (2.2), (2.3) and (2.4) we obtain

$${}^*C_{kjih} = C_{kjih} - M_{kjih} \tag{2.6}$$

where C_{kjih} is conformal curvature tensor(Eisenhart [2])and

$$\begin{aligned} M_{kjih} \overset{\text{def.}}{=} & [p_{ji}(T_{kh} - (n-2)^{-1}Tg_{kh}) + p_{kh}(T_{ji} - (n-2)^{-1}Tg_{ji}) \\ & - (n-2)^{-1}g_{ji}\{pT_{kh} - T^t_k p_{th} - T_{th}p^t_k - p^t(\nabla_t T_{kh} \\ & - \nabla_k T_{th})\} - (n-2)^{-1}g_{kh}\{pT_{ji} - T^t_j p_{ti} \\ & - T_{ti}p^t_j - p^t(\nabla_t T_{ji} - \nabla_j T_{ti})\} \\ & + 2(n-1)^{-1}(n-2)^{-1}g_{ji}g_{kh}\{Tp - T^t_m p^m_t \\ & - (\nabla_t T^t_m - \nabla_m T)p^m\}] - k|j. \end{aligned} \tag{2.7}$$

If $C_{kjih} = 0$, Eq.(2.6) implies

$${}^*C_{kjih} + M_{kjih} = 0. \tag{2.8}$$

Conversely, if (2.8) is satisfied, (2.6) gives $C_{kjih} = 0$. Hence we have

<u>Theorem 2.1</u>

A necessary and sufficient condition for a Riemannian manifold M admitting a quarter-symmetric metric connection (2.1) to be conformally flat is given by (2.8).

From (2.6) we can see that if ${}^*C_{kjih} = C_{kjih}$, $M_{kjih} = 0$ and conversely if $M_{kjih} = 0$, we get ${}^*C_{kjih} = C_{kjih}$. Hence we have

<u>Theorem 2.2</u>

A necessary and sufficient condition for ${}^*C_{kjih}$ to be equal to conformal curvature tensor is that M_{kjih} vanishes identically.

If $M_{kjih} = 0$, by multiplying (2.7) by $g^{ji}g^{kh}$ we obtain on simplification

$$p^t(\nabla_k T^k_t - \nabla_t T) = n\, T^t_k p^k_t. \tag{2.9}$$

Now we shall consider some special cases.

<u>Case I</u> - If we assume $T_{kh} = C_{kh} = -\frac{K_{kh}}{(n-2)} + \frac{Kg_{kh}}{2(n-1)(n-2)}$ Eq. (2.9) gives

$$p^t \nabla_t K + nKp = 2n(n-1)K^t_k p^k_t. \tag{2.10}$$

From (2.10) we obtain the following Corollary to Theorem 2.2.

<u>Corollary 1</u>

If a Riemannian manifold M admits a quarter-symmetric metric connection (2.1) for $T_{kh} = C_{kh}$ and satisfies ${}^*C_{kjih} = C_{kjih}$, it also satisfies (2.10).

If in (2.10), $K^t_k = \alpha\delta^t_k$, for some constant $\alpha \neq 0$, we get $(n-2)\,p = 0$. Hence we have

<u>Corollary 2</u>

If a Riemannian manifold M admits a quarter-symmetric metric connection (2.1) for $T_{kh} = C_{kh}$, $K^t_k = \alpha\delta^t_k$ and also satisfies ${}^*C_{kjih} = C_{kjih}$, then p vanishes identically.

Case II - If we assume $T_{kh} = G_{kh} = K_{kh} - \frac{1}{2} K g_{kh}$, similar to (2.10) we get

$$(n-2)p^t \nabla_t K + nKp = 2nK^t_k p^k_t , \tag{2.11}$$

which implies a result similar to Corollary 1.

Case III - If $T_{kh} = K_{kh}$, (2.9) gives

$$nK^t_k p^k_t + \frac{1}{2} p^t \nabla_t K = 0. \tag{2.12}$$

Now leaving the trivial case of $p^t = 0$, from (2.12) we can obtain

Corollary 3

If a Riemannian manifold M admits a quarter-symmetric metric connection (2.1) for $T_{kh} = K_{kh}$ and if it satisfies $^*C_{kjih} = C_{kjih}$ also, a necessary and sufficient condition for it to be of constant scalar curvature K is that it satisfies $K^t_k p^k_t = 0$.

Case IV - If $T_{kh} = \alpha g_{kh}$, for some scalar $\alpha \neq 0$, and if we also have $p_h \nabla_k \alpha = 0$, from equation (2.6) we get $^*C_{kjih} = C_{kjih}$. Conversely, if $^*C_{kjih} = C_{kjih}$, we obtain from (2.7)

$$(p_h g_{ji} \nabla_k \alpha + p_i g_{kh} \nabla_j \alpha) - j|k = 0, \tag{2.13}$$

which implies $p^j \nabla_j \alpha = 0$, $p_h \nabla_k \alpha = 0$. Hence, we have

Corollary 4

If a Riemannian manifold M admits a quarter-symmetric metric connection (2.1) for $T_{kh} = \alpha g_{kh}$, a necessary and sufficient condition for $^*C_{kjih}$ to be identically equal to conformal curvature tensor C_{kjih} is given by $p_h \nabla_k \alpha = 0$.

Case V - If $T_{kh} = g_{kh}$, Corollary 4 easily gives $^*C_{kjih} = C_{kjih}$ as obtained for semi-symmetric case by Amur and Pujar [1].

3. SKEW-SYMMETRIC T_{ji}

If T_{ji} is skew-symmetric $U_{ji} = 0$ and (1.1), (1.2) respectively reduce to [6]

$$\Gamma_{ji}^h = \left\{ {h \atop ji} \right\} - p_j T^h_i \tag{3.1}$$

and

$$R_{kji}{}^h = K_{kji}{}^h - [T^h_i \nabla_k p_j + p_j \nabla_k T^h_i] - j|k. \tag{3.2}$$

From (3.2) it is easy to obtain

$$R_{ji} = K_{ji} - T^h_i (\nabla_h p_j - \nabla_j p_h) - p_j \nabla_h T^h_i + p_h \nabla_j T^h_i \tag{3.3}$$

and

$$R = K + 2 \nabla_h (T^{hj} p_j). \tag{3.4}$$

If we substitute from (3.2), (3.3) and (3.4) in (2.5) we get

$$^*C_{kjih} = C_{kjih} + N_{kjih} , \tag{3.5}$$

where

$$\begin{aligned} N_{kjih} \overset{\text{df.}}{=} \big[& T_{ih} \nabla_j p_k - p_j \nabla_k T_{ih} + (n-2)^{-1} \{ (\nabla_a p_j - \nabla_j p_a) \\ & (g_{kh} T^a_i - g_{ki} T^a_h) + g_{kh} (p_j \nabla_a T^a_i \\ & - p_a \nabla_j T^a_i + (n-1)^{-1} g_{ji} \nabla_a (T^{ab} p_b)) \\ & + g_{ji} (p_k \nabla_a T^a_h - p_a \nabla_k T^a_h + (n-1)^{-1} g_{kh} \cdot \\ & \cdot \nabla_a (T^{ab} p_b)) \} \big] - j|k \end{aligned} \tag{3.6}$$

If $C_{kjih} = 0$ equation (3.5) gives $^*C_{kjih} = N_{kjih}$, conversely if $^*C_{kjih} = N_{kjih}$, we get $C_{kjih} = 0$. Hence we have

Theorem 3.1

A necessary and sufficient condition for a Riemannian manifold M admitting a quarter-symmetric metric connection (3.1) to be conformally flat is given by $^*C_{kjih} = N_{kjih}$.

Further if $^*C_{kjih} = C_{kjih}$, $N_{kjih} = 0$ and conversely if $N_{kjih} = 0$, $^*C_{kjih} = C_{kjih}$. Hence we have

Theorem 3.2

A necessary and sufficient condition for $^*C_{kjih}$ to be equal to conformal curvature tensor is that N_{kjih} vanishes identically.

If we multiply (3.6) by $g^{ji}g^{kh}$ and put $N_{kjih} = 0$, we get

$$\nabla_k (T^{ak} p_a) = 0. \tag{3.7}$$

Hence we have

Corollary 5

If a Riemannian manifold M admits (3.1) and satisfies $^*C_{kjih} = C_{kjih}$ it also satisfies (3.7).

4. KAEHLERIAN MANIFOLD

We now assume that M is a Kaehlerian manifold of $2n$ $(n \geq 2)$ dimension with metric tensor g_{ji} and almost complex structure tensor F_i^h satisfying $F_j^t F_t^h = -\delta_j^h$, $F_j^t F_i^s g_{ts} = g_{ji}$, $F_{ji} = -F_{ij}$, $F_{ji} = F_j^t g_{ti}$, $\nabla_k F_j^h = 0$. Now we shall consider some special cases.

Case I - If $T_i^h = F_i^h$, Eqs. (3.1) and (3.2) respectively reduce to

$$\Gamma_{ji}^h = \left\{ {h \atop ji} \right\} - p_j F_i^h \tag{4.1}$$

and

$$R_{kji}{}^h = K_{kji}{}^h - (\nabla_k p_j - \nabla_j p_k) F_i^h . \tag{4.2}$$

From (4.2) it is easy to get

$$R_{ji} = K_{ji} - (\nabla_k p_j - \nabla_j p_k) F_i^k \tag{4.3}$$

and

$$R = K - 2 F^{jk} \nabla_k p_j . \tag{4.4}$$

For this case from (3.6) we shall have

$$N_{kjih} = \left[F_{ih} \nabla_j p_k + (2n-2)^{-1} \{ (\nabla_a p_j - \nabla_j p_a)(g_{kh} F_i^a - g_{ki} F_h^a) + 2(2n-1)^{-1} g_{kh} g_{ji} F^{ab} \nabla_a p_b \} \right] - j|k . \tag{4.5}$$

If in a special case p_i is a gradient vector $R_{kji}{}^h = K_{kji}{}^h$, $R_{ji} = K_{ji}$ and $R = K$, therefore, $N_{kjih} = 0$ and $^*C_{kjih} = C_{kjih}$. Hence we have

Theorem 4.1

If a Kaehlerian manifold M admits a quarter-symmetric metric connection (4.1) and p_j is a gradient vector the tensor $^*C_{kjih}$ is identically equal to conformal curvature tensor C_{kjih}.

Case II - If $U_{ji} = g_{ji}$ and $V_i^h = F_i^h$, Eq.(1.1) gives

$$\Gamma_{ji}^h = \left\{ {h \atop ji} \right\} + \delta_j^h p_i - p_j F_i^h - g_{ji} p^h , \tag{4.6}$$

while (1.2) gives

$$R_{kji}{}^h = K_{kji}{}^h - [\delta_k^h p_{ji} + p_k^h g_{ji} + F_i^h \nabla_k p_j - F_k^h p_j p_i + F_{ij} p_k p^h + \delta_k^h F_i^t p_j p_t - p_k p^t F_t^h g_{ji}] - j|k . \tag{4.7}$$

From (4.7) we can easily obtain

$$R_{ji} = K_{ji} - [2(n-1) p_{ji} + p g_{ji} + F_i^h (\nabla_h p_j - \nabla_j p_h) + F_j^h p_h p_i - F_{ji} p_h p^h + (2n-3) F_i^t p_t p_j] \tag{4.8}$$

and

$$R = K - 2\left[F^{jh}\nabla_h p_j + (2n-1)p\right] \qquad (4.9)$$

In analogy to (2.6) we can obtain

$$^*C_{kjih} = C_{kjih} - L_{kjih}, \qquad (4.10)$$

where

$$\begin{aligned} L_{kjih} \overset{\text{def.}}{=} \Big[& g_{kh}M_{ji} + g_{ki}M_{jh} - F_{kh}\left(p_j p_i + (2n-2)^{-1} g_{ji} p^t p_t\right) \\ & + F_{ki}\left(p_j p_h + (2n-2)^{-1} g_{jh} p^t p_t\right) + F_{ih}\nabla_k p_j \\ & - (2n-3)(2n-2)^{-1}\left(g_{kh} F_i^t p_t p_j\right. \\ & \left. + g_{ji} F_h^t p_t p_k\right) + (n-1)^{-1}(2n-1)^{-1} g_{kh} g_{ji} \cdot \\ & \cdot \left(F^{tm}\nabla_m p_t + (2n-1)p\right)\Big] - j|k, \end{aligned} \qquad (4.11)$$

and

$$M_{ji} = F_i^t p_t p_j - (2n-2)^{-1}\left(p g_{ji} + F_i^t(\nabla_t p_j - \nabla_j p_t) + F_j^t p_t p_i\right).$$

From (4.10) we can obtain

Theorem 4.2

If a Kaehlerian manifold M admits a quarter-symmetric metric connection (4.6), a necessary and sufficient condition for $^*C_{kjih}$ to be equal to conformal curvature tensor C_{kjih} is that L_{kijh} vanishes identically

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