THE FORCED NONLINEAR SCHRÖDINGER EQUATION

D.J. Kaup and P.J. Hansen

Institute for Theoretical Physics
University of California
Santa Barbara, CA 93106

It is of interest to understand the time evolution of systems governed by nonlinear partial differential equations, to which an external force is applied. For example, in ionospheric modification experiments, one directs a radio frequency wave at the ionosphere. At the reflection point of the wave, a sufficient level of electron plasma waves is excited to make nonlinear behavior important.[1] The nonlinear Schrodinger equation (NLSE) may describe such behavior.[2] As a simple model, consider the eletric field specified for all time at the reflection point, and ask for the time development of the nonlinear phenomena in the neighborhood of that point. We may thus think of a forced nonlinear system in terms of a nonlinear boundary value problem.

As a step toward the development of general techniques for handling such problems[3,4] we present preliminary numerical results for the forced NLSE. A WKB analysis on the time evolution equations for the NLSE in the inverse scattering transform (IST) formalism[5] is shown to be a crude order of magnitude estimate of the qualitative behavior of the forced NLSE.

Numerical Results

The time evolution of $q(x,t)$, given by

$$i\partial_t q + \partial_x^2 q + 2|q|^2 q = 0 \quad , \tag{1}$$

on $x>0$, with $q(x,0)=0$, $q(0,t)=Q_o \exp(-a(t-to)^2)$ $a=10$, $to = 0.4$ is studied numerically using the Crank-Nicolson method. Q_o is kept real, a and t_o fixed.

The number of solitons produced, and their amplitudes, as a function of Q_o, is shown in Fig. I. In terms of the IST eigen-

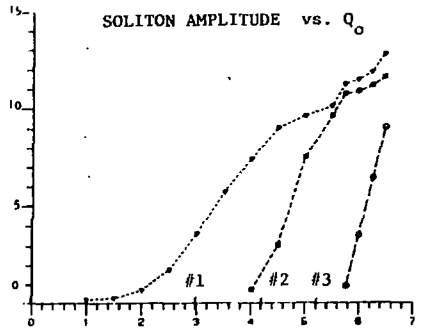

Fig. I

value $\zeta=\xi+i\eta$, the one soliton solution for the NLSE is[6]

$$q(x\ t)=\frac{2\eta \exp\left\{-4i(\xi^2-\eta^2)t-2i\xi x\right\}}{\cosh\left\{2\eta(x-x_o)+8\eta\right\}t} \tag{2}$$

so that the soliton velocity is -4ξ and its amplitude 2η. In figure II, the eigenvalue

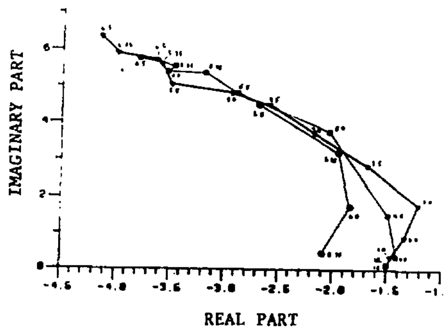

Fig. II

spectrum of these solitons is presented, as obtained numerically from their amplitudes and velocities. Finally, the real and imaginary parts of $\partial_x q(o,t)$ are given in Figs. III, IV, for a run.

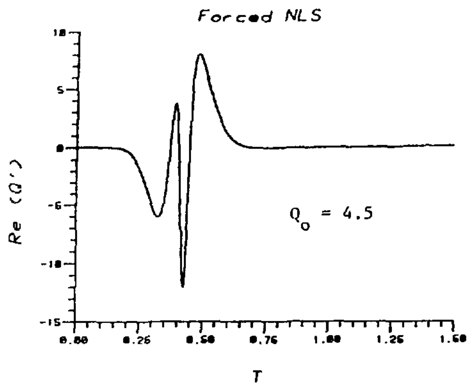

Fig. III

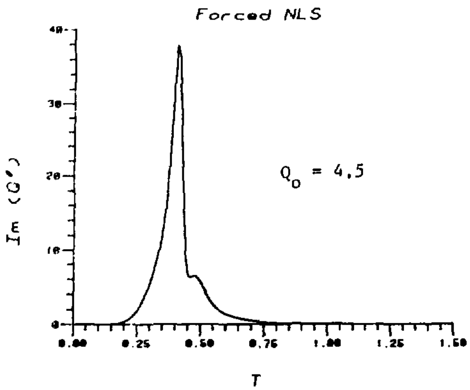

Fig. IV

A WKB Result

For the NLSE, the time evolution of the scattering data in IST, at x=0, is given by

$$\partial_t v_1 = -(2i\zeta^2 - i|q|^2)v_1 + (2\zeta q + i\partial_x q)v_2$$
$$\partial_t v_2 = (2i\zeta^2 - i|q|^2)v_2 + (-2\zeta q^* + i\partial_x q^*)v_1 \qquad (3)$$

with v_1, v_2 the scattering functions, ζ the eigenvalue, and $\partial_x q$ the slope at x=0. For the driven initial value problem, we have q(x,o) and q(o,t). Giving $\partial_x q(o,t)$ would over specify the problem. If we knew it, we could obtain the ζ-spectrum from (3). The bound states of such a spectrum correspond to the solitons produced by the forcing function q(o,t) in (3). For large ζ, we may ignore $\partial_x q$ as compared to $2\zeta q$, as shown by comparison of q and $\partial_x q$ in Figs. I, III, IV. Since σ_x is zero or oscillatory, we ignore it completely. At small ζ, the WKB is inaccurate at any rate. Ignoring $\partial_x q$ and considering $\partial_t q(o,t)/q(o,t)$ small compared to q (it vanishes at t=0), we obtain

$$-\partial_t^2 v_2 + (-E + V)v_2 = 0 \quad , \qquad (4)$$

where $E=4\zeta^4$, $v=-|Q|^4$, i.e. the usual Schrodinger equation. We estimate E with the WKB formula

$$\int_{t_1}^{t_2} \sqrt{-|E| + |Q(t)|^4}\, dt = \pi(n+\tfrac{1}{2}) \ . \qquad (5)$$

Since q(o,t) is a Gaussian, we approximate the left side of (5) by

$$W\sqrt{-|E| + |Q_o|^4} = \pi(n + \tfrac{1}{2}) \qquad (6)$$

where W is the width of the Gaussian ($\sqrt{\pi}$ /10). The spectrum obtained from (6) is shown in Fig. V. Qualitatively the WKB prediction

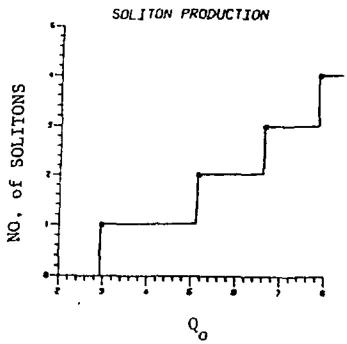

Fig. V

follows that of Fig. II. The WKB threshholds for soliton creation are not far from those seen in Fig. I. Recalling the drastic approximations made to get these results, this result is remarkable. Recall, too, that the IST has previously been a tool primarily for initial value problems.

We have thus a simple way of estimating the qualitative behavior of forced nonlinear systems governed by an IST. We hypothesize that it may be possible to extend more rigorously the realm of the IST, from that of initial value problems, to that of boundary value problems.

References

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