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DYNAMIC OPTIMIZATION OF COMBINED HARVESTING OF A TWO-SPECIES FISHERY *

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ABSTRACT

In the present paper, the author considers the problem of dynamic optimization of the exploitation policy connected with the combined harvesting of two competing fish species, each of which obeys the logistic growth law. The singular extremal trajectory in the phase plane is derived by taking the harvesting effort as a dynamic variable. Biological or bioeconomic interpretations of the constraints required for this singular extremal are also given.

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I. INTRODUCTION

The problem of combined harvesting of two ecologically independent fish populations, each of which obeys the logistic law of growth, was considered by Clark [1]. One of his main results is that "when two populations are exploited jointly, one population may be driven to extinction, whereas the other population continues to support the fishery in bionomic (one species) equilibrium. Populations with relatively low btp (biotechnical productivity) are subject to elimination under joint harvesting conditions, provided that the cost-price ratios of other species are sufficiently low". He also attempted "with limited success", to solve the optimal control problem associated with his model, and could actually derive only the equilibrium solution of the optimal control problem.

Gause [2] developed a model of interspecific competition between two species each of which is subject to the logistic growth, and discussed some aspects of the model both analyticlly and experimentally. However, he did not take harvesting into account. Clark [1] studied Gause's model with exploitation of one species out of the two.

Silvert and Smith [3] considered the problem of optimal (independent) exploitation of a multispecies community and they also could not go beyond the equilibrium solution of the optimal control problem.

The problem of combined harvesting of both the species in the Gause model [2] has recently been studied by Chaudhuri [4]. He has shown that

i) the open access fishery may possess a bionomic equilibrium which drives one species to extinction;
ii) the non-trivial critical point of the dynamical system describing the fishery is either an asymptotically stable node or an unstable saddle point depending on the values of the biological parameters:
iii) the nature of the trivial critical point (origin) is critically dependent on the biotechnical productivity of each species;
iv) the dynamical system does not possess any limit cycle;
v) the fishery attains a dynamic as well as a bioeconomic stability if the product of the coefficients of intra-specific competition between the species is greater than that of their coefficients of interspecific competition.

Lastly, Chaudhuri has given the mathematical formulation of the optimal harvest policy and has derived its solution in the equilibrium case only by using Pontryagin's maximum principle.

In the present paper, the author reconsiders the optimal harvest policy of the above problem and makes a modest attempt to achieve the solution for dynamic optimization. The singular extremal trajectory in the phase plane is derived by taking the effort to be a dynamic variable. Bioeconomic interpretations of the constraints required for this singular extremal are also given.

II. STATEMENT OF THE PROBLEM

We consider two competing fish species whose growths are governed by the system of simultaneous differential equations

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) - \alpha xy = F(x,y),$$
$$\frac{dy}{dt} = sy\left(1-\frac{y}{L}\right) - \beta xy = G(x,y) \qquad (1)$$

where r, s, α, β, K, L are all positive constants. These are biological parameters having explicit biological meanings: (r,s) are the biotic potentials and (K,L) are the carrying capacities of the two species. In this model, it is assumed that the species compete for an external resource which supports each species according to the logistic law of growth in the absence of the other species. α, β are their coefficients of inter-specific competition, while (r/K), (s/L) may be interpreted as the coefficients of intraspecific competition. Also αx may be looked upon as the trophic function or the functional response of the second species to the density of the first. A similar interpretation holds for βy also. The salient features of the dynamical system (1) may be found in Gause [2] and Clark [1].

Let us now suppose that both the fish species are subjected to a combined harvesting effort as reflected in the dynamical system:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) - \alpha xy - q_1 E x$$
$$\frac{dy}{dt} = sy\left(1-\frac{y}{L}\right) - \beta xy - q_2 E y \qquad (2)$$

where $E = E(t)$ denotes the combined harvesting effort and q_1, q_2 are the catchability coefficients of the species.

The dynamical aspects as well as the bionomic equilibrium of the system (2) have been studied in detail by Chaudhuri [4].

III. THE OPTIMAL HARVEST POLICY

Our problem is to maximize the present value J of a continuous time-stream of revenues as given by the relation

$$J = \int_{t_0}^{T} e^{-\delta t}\,\pi(x,y,E,t)\,dt$$
$$= \int_{t_0}^{T} e^{-\delta t}\left[p_1 q_1 x + p_2 q_2 y - c\right]E(t)\,dt \qquad (3)$$

where δ = instantaneous annual rate of discount,
p_1 = price per unit biomass of the x-species,
p_2 = price per unit biomass of the y-species,
c = fishing cost per unit effort.

We assume that

$$x(t_0) = x_0, \quad y(t_0) = y_0 \qquad (4)$$

at the initial time t_0.

The fish populations remain at constant levels at the steady state $P(\bar{x},\bar{y})$, say, of the uncontrolled system (1).

Assuming that P is a desirable target of our control programme and that T is its completion time, we must have

$$x(T) = \bar{x}, \quad y(T) = \bar{y}. \qquad (5)$$

The final time T may be specified or unspecified. Specifying T means imposing an additional constraint on our control programme. We keep T unspecified.

Here $E(t)$ is the control variable subject to the constraints

$$0 \le E(t) \le E_{max} \qquad (6)$$

which define the control set $U_t = [0, E_{max}]$. $E(t)$ is subjected to an upper limit because the technical facilities and manpower available are limited. E_{max} may be some function of time reflecting the changing capabilities of harvesting. For our purpose, it is assumed to be a constant.

Our problem is to choose an optimal control which will drive the system from the initial state (x_0, y_0) to the final state $(\bar{x},\bar{y})$ and at the same time, will maximize the objective functional J subject to the state equations (2).

As pointed out by CLark [1], Silvert and Smith [3], it is a formidable task to solve this dynamic optimization problem and it remains unsolved so far. Chaudhuri [4] derived an optimal equilibrium solution of this problem.

We make here a modest attempt to solve the dynamic optimization problem stated above. However, the calculations turn out to be lengthy and tedious.

It can be easily shown that the steady state of the uncontrolled system (1) is given by

$$
\begin{aligned}
x = \bar{x} &= \frac{Ks(r-\alpha L)}{rs-\alpha\beta KL}, \\
y = \bar{y} &= \frac{Lr(s-\beta K)}{rs-\alpha\beta KL}.
\end{aligned} \tag{7}
$$

IV. THE DYNAMIC OPTIMIZATION

The Hamiltonian function for the dynamic optimization problem is

$$
\begin{aligned}
H = {} & \lambda_0 e^{-\delta t}\left[p_1 q_1 x + p_2 q_2 y - c\right]E + \lambda_1\left[-\frac{r}{K}x^2 - \alpha x y + x(r - q_1 E)\right] \\
& + \lambda_2\left[-\frac{s}{L}y^2 - \beta x y + y(s - q_2 E)\right].
\end{aligned} \tag{8}
$$

It has been pointed out by Goh [5] that the harvesting policy for $\delta = 0$ is more robust than that which incorporates a non-zero discount rate. Moreover, $\delta = 0$ makes the calculation slightly simpler. We, therefore, proceed with $\delta = 0$ in the subsequent steps.

For optimal control $E = E^*(t)$ and optimal population levels $x = x^*(t)$, $y = y^*(t)$, $t \in [t_0, T^*]$, all of the adjoint (or costate) variables λ_0, λ_1, λ_2 must not be zero where λ_0 is a non-negative constant.

The adjoint equations are

$$
\dot{\lambda}_1 = -\frac{\partial H}{\partial x} = -\lambda_0 p_1 q_1 E - \lambda_1\left(-\frac{2rx}{K} - \alpha y + r - q_1 E\right) + \lambda_2 \beta y, \tag{9}
$$

$$
\dot{\lambda}_2 = -\frac{\partial H}{\partial y} = -\lambda_0 p_2 q_2 E + \lambda_1 \alpha x - \lambda_2\left(-\frac{2sy}{L} - \beta x + s - q_2 E\right) \tag{10}
$$

where the dot $(\cdot)$ denotes total differentiation with respect to t. The optimal control theory [5] requires that

$$
H\left(x^*(t), y^*(t), E^*(t), \lambda_1(t), \lambda_2(t)\right) = 0. \tag{11}
$$

Moreover, the optimal control $E^*(t)$ maximizes $H(x^*(t), y^*(t), E, \lambda_1(t), \lambda_2(t))$ along an optimal trajectory with respect to all admissible controls. Therefore

$$
H_E > 0 \Rightarrow E^*(t) = E_{max}, \tag{12}
$$

$$
H_E < 0 \Rightarrow E^*(t) = 0 \tag{13}
$$

where the subscript denotes partial differentiation.

Since E appears linearly in the Hamiltonian function, we get singular control if

$$
H_E = \lambda_0\left[p_1 q_1 x + p_2 q_2 y - c\right] - \lambda_1 q_1 x - \lambda_2 q_2 y = 0 \tag{14}
$$

on the sub-interval $[t_1, t_2]$ of $[t_0, T]$.

Writing $D \equiv \frac{d}{dt}$, we get from (14),

$$
\begin{aligned}
DH_E &= \frac{\lambda_0}{KL}N - \frac{\lambda_1 a_1 x}{KL} - \frac{\lambda_2 a_2 y}{KL} \\
&= \frac{1}{KL}\left[\lambda_0 N - \lambda_1 a_1 x - \lambda_2 a_2 y\right]
\end{aligned} \tag{15}
$$

where

$$
a_1 = L(q_1 r x + q_2 K \alpha y), \tag{16}
$$

$$
a_2 = K(q_1 L \beta x + q_2 s y), \tag{17}
$$

$$
N = L p_1 q_1 x\{r(K-x) - K\alpha y\} + K p_2 q_2 y\{s(L-y) - L\beta x\}. \tag{18}
$$

By virtue of (14),

$$
0 = DH_E \Rightarrow \lambda_1 a_1 x + \lambda_2 a_2 y - \lambda_0 N = 0. \tag{19}
$$

Obviously we are interested in the states which yield

$$
x > 0, \quad y > 0 \tag{20}
$$

and hence

$$
a_1 > 0, \quad a_2 > 0. \tag{21}
$$

If $\lambda_0 = 0$, the condition (19) is satisfied iff $\lambda_1 = \lambda_2 = 0$. Hence there is no singular extremal for $\lambda_0 = 0$. The case $\lambda_0 = 0$ corresponds to the abnormal singular extremal [6]. However, here λ_0, λ_1, λ_2 being all zero, there is no abnormal singular trajectory. We, therefore, take $\lambda_0 > 0$ and without any loss of generality, we proceed with $\lambda_0 = 1$.

Eqs.(14) and (19) may now be rewritten as

$$\lambda_1 q_1 x + \lambda_2 q_2 y - M = 0 \tag{22}$$

and

$$\lambda_1 a_1 x + \lambda_2 a_2 y - N = 0 \tag{23}$$

where

$$M = p_1 q_1 x + p_2 q_2 y - c. \tag{24}$$

After a little calculation, we find that

$$\Delta = (q_1 a_2 - q_2 a_1) x y \neq 0 \tag{25}$$

provided at least one of the conditions

(A) $\quad \dfrac{x}{y} \neq \dfrac{(s/L)}{(r/K)}$

and

(B) $\quad \left(\dfrac{q_1 x}{q_2 y}\right)^2 \neq \dfrac{\alpha x}{\beta y}$

holds.

When condition (25) holds, we can solve Eqs.(22) and (23) simultaneously for λ_1, λ_2 and get the results

$$\lambda_1(t) = \frac{M a_2 - N q_2}{x(q_1 a_2 - q_2 a_1)}, \tag{26}$$

$$\lambda_2(t) = \frac{N q_1 - M a_1}{y(q_1 a_2 - q_2 a_1)}, \tag{27}$$

$$\lambda_2(t) = \lambda_1(t) \frac{x(N q_1 - M a_1)}{y(M a_2 - N q_2)}. \tag{28}$$

The biological interpretation of condition (A) runs as follows:

The ratio of the optimal x-population to that of the y-population must not equal the ratio of the intraspecific competition of the y-species to that of the x-species. Condition (B) also may be interpreted as follows:

The square of the ratio of the catches per unit effort must not be equal to the ratio of the trophic functions of the two species. Similarly, we get

$$D^2 H_E = \frac{1}{KL}\left[\lambda_1 b_1 + \lambda_2 b_2 + \left(L p_1 q_1 c_1 x + K p_2 q_2 c_2 y\right) E\right] \tag{29}$$

where

$$\begin{aligned} b_1 = {} & L\alpha x^2 y (r q_1 + K\beta q_2) + LEx(r q_1^2 x + K\alpha q_2^2 y) \\ & - L r x^2 (\alpha q_2 y + r q_1) - KL\alpha x y(\beta q_1 x + s q_2), \end{aligned} \tag{30}$$

$$\begin{aligned} b_2 = {} & K\beta x y^2 (s q_2 + L\alpha q_1) + KEy(s q_2^2 y + L\beta q_1^2 x) \\ & - K s y^2 (\beta q_1 x + s q_2) - KL\beta x y(\alpha q_2 y + r q_1), \end{aligned} \tag{31}$$

$$c_1 = 3 r q_1 x + (2 q_2 + q_1) K\alpha y - K r q_1, \tag{32}$$

$$c_2 = 3 s q_2 y + (2 q_1 + q_2) L\beta x - L s q_2. \tag{33}$$

After some complicated calculations, we obtain

$$\begin{aligned} \left(D^2 H_E\right)_E = {} & \lambda_1 x\left[L r q_1^2 x + KL\alpha q_2^2 y\right] + \lambda_2 y\left[K s q_2^2 y + KL\beta q_1^2 x\right] \\ & + \lambda_1 a_1 q_1 x + \lambda_2 a_2 q_2 y + a_1 q_1 x(p_1 - \lambda_1) + a_2 q_2 y(p_2 - \lambda_2) \\ & + L p_1 q_1 x\left[r q_1 (2x - K) + K\alpha y(q_1 + q_2)\right] \\ & + K p_2 q_2 y\left[s q_2 (2y - L) + L\beta x(q_1 + q_2)\right]. \end{aligned} \tag{34}$$

For the existence of an optimal singular control, it is required to satisfy the generalized Legendre condition [5]:

$$\left(D^2 H_E\right)_E \geq 0. \tag{35}$$

A careful look at the expression of $(D^2H_E)_E$ in (34) reveals that the condition (35) is certainly satisfied if we impose the following constraints:

i) $0 < \lambda_1 \leq p_1$

ii) $0 < \lambda_2 \leq p_2$

iii) $x \geq \frac{K}{2}$

iv) $y \geq L/2$.

It is to be noted that $\frac{K}{2}$ represents the MSY level of the x-species alone under harvesting at a constant rate when it does not interact at all with any other species. We briefly refer to it as "the x_{MSY}". Similarly, $\frac{L}{2}$ stands for the y_{MSY} bearing the same meaning.

We may thus conclude that the existence of an optimal singular control $E^*(t)$ is ensured iff

(a) the shadow price of each species be less than or equal to its actual price

and

(b) the optimal population size of each species is greater than or equal to its MSY size.

Utilizing (14) in (11), we find that the trajectory for the optimal singular extremal is given by

$$\lambda_1\left[rx\left(1-\frac{x}{K}\right)-\alpha xy\right]+\lambda_2\left[sy\left(1-\frac{y}{L}\right)-\beta xy\right]=0. \tag{36}$$

By virtue of (26)-(28), the singular extremal trajectory (36) becomes

$$(Ma_2-Nq_2)\left[r\left(1-\frac{x}{K}\right)-\alpha y\right]-(Ma_1-Nq_1)\left[s\left(1-\frac{y}{L}\right)-\beta x\right]=0. \tag{37}$$

Again, $D^2H_E = 0$ gives

$$\lambda_1 b_1+\lambda_2 b_2+(Lp_1q_1c_1x+Kp_2q_2c_2y)E=0. \tag{38}$$

Utilizing the values of λ_1 and λ_2 from (26) and (27) respectively, we can determine the optimal singular control $E^*(t)$ from (38) in terms of the optimal population levels of the two species and other biological, economic and technical parameters.

The quantities M, N, a_1, a_2 being all complicated expressions involving x and y, it is too difficult to discuss analytically the nature of the optimal trajectory given by (37). By assigning relevant numerical values to the biological parameters (r, s, K, L, α, β), economic parameters (p_1, p_2, c) and the technical parameters (q_1,q_2), one can study the nature of the optimal trajectory using computers. The author is unable to do it due to non-availability of information regarding such data as are used in fisheries.

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