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FORMATION AND PROPAGATION OF SAND DUNES: A NONLINEAR TREATMENT *

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ABSTRACT

The nonlinear evolutionary equations previously derived for a plane with a rigid lid are here generalized to the free surface model. It is shown that similar equations are obtainable but the coefficients are strongly dependent on the Froude number, F, of the flow. (F is defined as $U/(gd))^{1/2}$, where U is the basic uniform flow, g the gravitational acceleration and d the mean depth of the layer.) When F vanishes, the evolutionary equations reduce to those derived previously for the rigid lid model.

The equations possess a dunetrain solution. The stability of this solution is analyzed and found to depend crucially on F. It is found, however, that for all values of F a dunetrain can develop into a solitary dune.

The above results apply only when the phase shift δ, originally introduced for the instability of the linear problem, vanishes. For other admissable values of δ, the analysis showed that the neutral solution of the linear theory prevails in the nonlinear regime.

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I. INTRODUCTION

The problem of desertification and its detrimental effects has been realized in Sudan for about half a century (El-Karouri 1986). This is because about half the country is covered with dry sand and the movement of sand from one place to another is a common daily occurrence. However, in the last fifteen years the problem of desertification has been particularly aggravated by a number of "civilization" acts, notably the settlement of nomads. In addition, the droughts of 1973-76 and 1982-84 have made the situation worse. The western part of Sudan borders the Sahara and when vegetation disappears, the sand moves in and the land is lost. The advancement of sand dunes has been calculated roughly by a number of researchers (see e.g. El-Baz, 1977). However, it is understood that the rate of movement depends on the strength and direction of the wind as well as on the presence of vegetation.

Another problem created by movement of sand in Northern Sudan is that sand blown into the River Nile has resulted in the erosion of the silted clay on either side of the river due to the side-flows produced by the presence of sand islands in the middle of the river (Eltayeb and Hassan, 1981). Consequently the annual yield of production has diminished in an area which has been self-sufficient for centuries.

It is our purpose to study the movement of sand here. It must be stated, however, that despite the numerous studies made on the movement of sand no agreement has been reached regarding the dependence of the sediment transport flux on the fluid velocity and on the sand parameters even for the much studied alluvial system (see e.g. Kennedy, 1980; Englund and Fredsøe, 1982). As regards aeolian bedforms the classical book by Bagnold (1954) has not been matched although some recent studies have improved our understanding of the subject (Owen, these proceedings). Here sand particles of diameter 0.1 mm form dust and move as a suspension while larger particles (of diameter $\sim$ 0.9 mm) usually move by saltation or by rolling (Abbott and Francis, 1977).

The two most authorative methods for studying erodible bedforms are (i) statistical methods and (ii) continuum theory methods. The first method is more suitable for the study of dust while the second is more appropriate for saltating particles. Here we shall adopt the second. A model due to Kennedy (1963) is used. The most controvesial part of the model, as indeed in all other models, is the relationship between the sediment transport flux and the fluid velocity and sand parameters. Such a relationship is not known with adequate accuracy and the use of empirical formulae, which show satisfactory agreement with experimental data, is unavoidable. The formula used here is

$$G = \tilde{m}[\tilde{u} - \tilde{U}_c]^n \quad , \tag{1.1}$$

where G is the flux of sediment in the direction of the prevailing wind $\tilde{u}$, $\tilde{U}_c$ the frictional velocity required before a particle can move and $\tilde{m}$ and n are parameters which can be adjusted to suit various situations. This is particularly relevant for theoretical models which are expected to study the problem in the span of the parameter space with a view to providing some insight into the problem. Other simplifying assumptions are made. First, the mean flow of the fluid above the erodible bedform is assumed uniform. However, this is not expected to affect the qualitative nature of the results although a non-uniform flow may have some quantitative effect. Secondly, the perturbation flow is taken to be solenoidal. This is most likely to be the case for the linear theory (Eltayeb and Hassan, 1986) but it may lead to some change in the non-linear theory results. However, it seems reasonable to make these assumptions at this stage of the development of the nonlinear theory.

The plan of the paper is as follows. In Sec. 2 the problem is formulated and in Sec. 3 the results of the linear stability are summarized. In Sec. 4 the evolutionary equations of the nonlinear problem are derived and in Sec. 5 some solutions of the evolutionary equations are discussed.

2. THE BASIC EQUATIONS AND BOUNDARY CONDITIONS

Consider an incompressible inviscid fluid of uniform density $\tilde{\rho}$ flowing uniformly with mean velocity $\tilde{\underset{\sim}{U}}$ over an erodible bed consisting of non-cohesive (sand) particles. Take a cartesian system of coordinates $O(\tilde{x},\tilde{y},\tilde{z})$ such that $O\tilde{x}$ is parallel to $\tilde{\underset{\sim}{U}}$, $O\tilde{z}$ is vertically upwards and $O\tilde{y}$ completes the triad. The bed is taken to be

$$\tilde{z} = \tilde{\eta}(\tilde{x},\tilde{y},\tilde{t}) \quad , \tag{2.1}$$

where the amplitude of $\tilde{\eta}$ is much smaller than the mean thickness $\tilde{d}$ of the fluid above the bed. The top surface of the fluid is expected to be disturbed from its mean value d by the presence of the undulations (2.1) so that it is defined by

$$\tilde{z} = \tilde{d} + \tilde{\xi}(\tilde{x},\tilde{y},t) \quad . \tag{2.2}$$

In the region between the values of z in (2.1) and (2.2) the fluid motion is governed by the equations of motion and continuity:

$$\frac{\partial \tilde{\underset{\sim}{u}}}{\partial \tilde{t}} + (\tilde{\underset{\sim}{u}} \cdot \nabla)\tilde{\underset{\sim}{u}} + \nabla(\frac{\tilde{p}}{\tilde{\rho}} + \tilde{g}\tilde{z}) = 0 \tag{2.3}$$

$$\nabla \cdot \tilde{\underset{\sim}{u}} = 0 \quad , \tag{2.4}$$

where $\tilde{\underset{\sim}{u}}$ is the velocity, $\tilde{t}$ the time, $\tilde{p}$ the pressure, $\tilde{g}$ the uniform gravitational acceleration and ∇ the usual gradient operator. As was pointed out by Eltayeb and Hassan (1986) the situation of practical importance and analytical simplicity is that for which the perturbated flow is irrotational. Then

$$\tilde{\underset{\sim}{u}} = \nabla\tilde{\phi} \tag{2.5}$$

and (2.4) gives

$$\nabla^2\tilde{\phi} = 0 \quad . \tag{2.6}$$

With the help of the vector identity

$$\nabla(\tfrac{1}{2}\tilde{\underset{\sim}{u}}^2) = \tilde{\underset{\sim}{u}} \cdot \nabla\tilde{\underset{\sim}{u}} + \tilde{\underset{\sim}{u}} \wedge \mathrm{curl}\ \tilde{\underset{\sim}{u}} \tag{2.7}$$

and the result

$$\mathrm{curl}\ (\nabla\tilde{\phi}) = 0 \tag{2.8}$$

we can write (2.3) as

$$\frac{\tilde{p}}{\tilde{\rho}} + \tilde{g}\tilde{z} + \frac{1}{2}(\nabla\tilde{\phi})^2 + \frac{\partial\tilde{\phi}}{\partial\tilde{t}} = \text{constant}. \tag{2.9}$$

Eq. (2.6) must be solved subject to the following boundary conditions:

i) The component of velocity normal to free surface (2.2) must vanish;
ii) The component of velocity normal to the surface of the erodible bed (2.1) must vanish;
iii) The pressure on the surface (2.2) must match the atmospheric pressure;
iv) The rate of change of $\tilde{\eta}$, the amplitude of the erodible bed must be due to the transport of sand. This is known as the transport equation or conservation of sediment equation.

Before we formulate these boundary conditions it is convenient to cast the equations in non-dimensional form. We separate the basic state and perturbations by

$$\tilde{\Phi} = \tilde{U}\tilde{x} + \tilde{\phi} \tag{2.10}$$

where $\tilde{U}$ is the uniform horizontal speed of the flow and ϕ is the influence of the undulations of the erodible bed on the velocity potential. We use $\tilde{U}$ as a unit of speed, $\tilde{d}$ as a unit of distance and $\tilde{d}/\tilde{U}$ as a unit of time. Then the equations and boundary conditions reduce to (see Eltayeb and Hassan, 1981)

$$\nabla^2\phi = 0 \qquad \text{for} \qquad \eta \leq z \leq 1 + \xi \tag{2.11}$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x} + \frac{\partial\phi}{\partial x}\frac{\partial\xi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\xi}{\partial y} \qquad \text{at} \quad z = 1 + \xi, \tag{2.12}$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y} \qquad \text{at} \quad z = \eta, \tag{2.13}$$

$$\frac{\xi}{F^2} + \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] = 0 \tag{2.14}$$

at $z = 1 + \xi$,

$$\frac{\partial\eta}{\partial t} + H_c\frac{\partial T}{\partial x} = 0 \qquad \text{at} \quad z = \eta. \tag{2.15}$$

Here the non-dimensional numbers H_c and the Froude number F are defined by (see Eq. (1.1) above)

$$H_c = \frac{\tilde{m}(\tilde{U} - \tilde{U}_c)^n}{\tilde{U}\tilde{B}\tilde{d}} \quad , \qquad F = \tilde{U}/\tilde{g}\tilde{d} \tag{2.16}$$

The variable T is the dimensionless volume rate of sediment transport. Because the main flow of fluid is in the x-direction, the flux of sediment is taken to be in the x-direction only because the perturbation velocity $|\nabla\phi|$ is assumed less than the critical velocity U_c required for the initiation of sediment transport. The explicit dependence of T on U_c, d, U, ϕ and particle size is a subject of controversy and we shall follow the counsel of Kennedy (1969) and define T by (see Eq. (1.1) above)

$$T(x,y,z,t) = \left[1 + \frac{\frac{\partial\phi}{\partial x}(x-\delta,y,z,t)}{1 - U_c}\right]^n \quad . \tag{2.17}$$

Here δ is a phase shift introduced by Kennedy and we discuss its role later.

3. THE SOLUTION OF THE LINEAR PROBLEM

Eq. (2.11) is linear but the boundary conditions (2.12)-(2.15) have nonlinear parts. The simplest form of stability is the linear one and this form of stability is relevant for disturbances of small amplitude. Indeed when a small amplitude disturbance is made a linear stability analysis is required to see if the amplitude of the disturbance is growing or decaying. In the case of linear instability the small amplitude will grow. However when it grows to a reasonable size, the nonlinear terms will become potent and it is the nonlinear system that will determine the fate of the disturbance. It is then necessary to analyze the linear stability of the erodible bed first. The nonlinear problem will be considered in the next section.

The linearized form of Eqs. (2.11)-(2.15) is

$$\nabla^2\phi = 0 \qquad \text{in} \qquad 0 \leq z \leq 1 \tag{3.1}$$

and

$$\frac{\partial\phi}{\partial z} = \frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x} \qquad \text{at} \quad z = 1 \tag{3.2}$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x} \qquad \text{at} \quad z = 0 \tag{3.3}$$

$$\frac{\xi}{F^2} + \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x} = 0 \qquad \text{at} \quad z = 1 \tag{3.4}$$

$$\frac{\partial\eta}{\partial t} + \overline{H}_c\frac{\partial\phi^*}{\partial x} = 0 \qquad \text{at} \quad z = 0 \; . \tag{3.5}$$

Here ϕ^* is evaluated at $x = x - \delta$ and

$$\overline{H}_c = \frac{H_c n}{1 - U_c} \quad . \tag{3.6}$$

We assume that

$$\{\eta,\xi,\phi,p\} = \{\eta_{11},\xi_{11},\phi_{11}(z),p_{11}\}\exp\{ik(x-U_b t) + \gamma t\} \tag{3.7}$$

and note that (3.1) takes the form

$$\left(\frac{d^2}{dz^2} - k^2\right)\phi_{11}(z) = 0, \tag{3.8}$$

we have

$$\phi_{11}(z) = A\cosh(kz) + B\sinh(kz) \quad . \tag{3.9}$$

We now apply the conditions (3.2)-(3.5), using (3.7) and (3.9) to find that

$$\left.\begin{aligned}
\phi_{11} &= A\{\cosh(kz) - \lambda\sinh(kz)\}\ , \\
\eta_{11} &= \frac{k^2\bar{H}_c e^{-ik\delta}A}{\gamma-ikU_b}\ , \\
\xi_{11} &= -F^2A[\gamma+ik(1-U_b)](1-\lambda\sigma)\cosh(k), \\
p_{11} &= A\left\{[\gamma+ik(1-U_b)]\cdot\frac{k(1-\sigma^2)\cosh(k)}{k+F^2[\gamma+ik(1-U_b)]^2\sigma}\right. \\
&\qquad \left. - ik(1-U_b)[\cosh(kz) - \lambda\sinh(kz)]\right\},
\end{aligned}\right\} \tag{3.10}$$

and

$$U_b = \frac{k\bar{H}_c(1-U_b)\ [1-kF^2(1-U_b)^2\sigma]}{\sigma - k\ F^2(1-U_b)^2}\cos(k\delta), \tag{3.11}$$

$$\gamma = -\frac{k^2\bar{H}_c[1-kF^2(1-U_b)^2\sigma]}{\sigma - kF^2(1-U_b)^2}\sin(k\delta). \tag{3.12}$$

where

$$\sigma = \tanh(k) \quad \text{and} \quad \lambda = \frac{k\sigma + F^2[\gamma + ik(1-U_b)]^2}{k + F^2[\gamma + ik(1-U_b)]^2\sigma}$$

The expressions (3.11) and (3.12) are generalizations of the expressions obtained by Kennedy (1969) for the phase speed U_b and the growth rate γ; the difference here being due to the neglect of the assumption $\gamma \ll kU_b$ made by Kennedy. As a consequence no singularity can arise when $\sigma - kF^2 = 0$.

The linear stability of sand bed is then determined by the sign of γ. If γ is positive the disturbance grows exponentially with time and the bedform is unstable. If γ is negative it decays and consequently the bedform will return to its flat position. If, however, $\gamma = 0$ then the bedform is neutrally stable and any form of disturbance made will neither grow nor decay.

The expression (3.12) for γ is proportional to $\sin(k\delta)$ and hence it vanishes if

$$k\delta = \pi s \quad ; \quad s = 0,1,2,3,\ldots \tag{3.13}$$

These values of δ then correspond to neutral stability. For other values of γ the bedform is either stable $(\gamma < 0)$ or unstable $(\gamma > 0)$. However identification of the various regions of instability is complicated by the dependence on U_b. If we make the assumption that $\bar{H}_c \ll 1$ so that

$$\left.\begin{aligned}
U_b &= \frac{k\bar{H}_c[1-kF^2\sigma]}{\sigma-kF^2}\cos(k\delta) \\
\gamma &= -\frac{k^2\bar{H}_c(1-kF^2\sigma)}{\sigma-kF^2}\sin(k\delta)
\end{aligned}\right\} \tag{3.14}$$

the expressions for U_b and γ become identical to those studied by Kennedy. Restricting ourselves to $0 \le k\delta \le 2\pi$, we find that γ is positive for the following three cases:

(a) $\frac{\sigma}{k} < F^2 < \frac{1}{k\sigma}$ and $0 < k\delta < \pi$

(b) $F^2 < \frac{\sigma}{k}$ and $\pi < k\delta < 2\pi$

(c) $F^2 > \frac{1}{\sigma k}$ and $\pi < k\delta < 2\pi$.

This is illustrated in Fig. 2. It then follows that for every value of $k(>0)$ there is a value of F^2 and δ for which instability will occur.

The phase speed U_b is positive for the shaded region only if $\pi/2 < k\delta < 3\pi/2$. In this case the bedform drifts downstream. Outside the shaded region U_b is positive if $0 < k\delta < \pi/2$ or $3\pi/2 < k\delta < 2\pi$. Thus the unstable modes (a) above drift upstream if $0 < k\delta < \pi/2$ and downstream if $\pi/2 < k\delta < \pi$. For $F^2 < \sigma/k$ the unstable modes drift upstream if $\pi < k\delta < 3\pi/2$ and downstream if $3\pi/2 < k\delta < 2\pi$. Similar results hold for $F < 1/\sigma k$. The reader is referred to Kennedy (1969) for a table of these results.

Now the case relevant to the nonlinear treatment of Sec. 4 below corresponds to neutral stability of the linear theory, i.e. those values of $k\delta$ given by (3.13). Here $\gamma = 0$ and λ reduces to

$$\lambda = \frac{\sigma - kF(1-U_b)^2}{1 - kF^2(1-U_b)^2} \quad . \tag{3.15}$$

Accordingly, the variables $\underset{\sim}{u}(=(u, v, w))$ and p have a vertical dependence given by

$$u = ikA[\cosh(kz) - \lambda\sinh(kz)]$$

$$\omega = kA[\sinh(kz) - \lambda\cosh(kz)]$$

$$p = ik(1-U_b)A\left\{\frac{(1-\sigma^2)\cosh(k)}{1-kF^2(1-U_b)^2\sigma} - \cosh(kz) + \lambda\sinh(kz)\right\}$$

These functions are illustrated in Fig. 3. Kennedy (1969) argued that maximum amplitude was attained when $k\delta = 2\pi$. However, the nonlinear treatment below indicated that the value of $k\delta$ has a profound effect on the evolution of sand dunes on long scale time.

4. DERIVATION OF THE NON-LINEAR EVOLUTION EQUATIONS

In this section we shall extend the theory developed in earlier publications (Eltayeb and Hassan, 1981, 1986) for the rigid lid model to the free top surface layer which is more relevant to natural systems in the desert and streams. The details of the derivation are not given and the reader who wishes to familiarize himself with the method of multiple scales is referred to the earlier papers and references therein. Moreover, readers who have no appetite for the derivation of the nonlinear equations may skip this section to the next where the possible solutions and their stability are discussed. Also we shall assume that $\delta = 0$. Later in this section we shall comment on other values of δ satisfying (3.13).

We are therefore required to solve the set of equaitons (2.11)-(2.15) in its entirety. This can be achieved only for amplitudes of the bed form much smaller than the height of the layer $\tilde{d}$. If $\tilde{h}$ is the dimensional amplitude of the bedform undulations, we define a parameter

$$\epsilon = \tilde{h}/\tilde{d} << 1 \tag{4.1}$$

and expand all variables in powers of ϵ thus

$$\phi = \sum_{j=n}^{\infty}\sum_{n=0}^{\infty} \epsilon^{j}(\phi_{nj}E^{n} + \bar{\phi}_{nj}E^{-n}) \quad , \tag{4.2}$$

$$p = \sum_{j=n}^{\infty}\sum_{n=0}^{\infty} \epsilon^{j}(p_{nj}E^{n} + \bar{p}_{nj}E^{-n}) \quad , \tag{4.3}$$

$$\xi = \sum_{j=n}^{\infty}\sum_{n=0}^{\infty} \epsilon^{j}(\xi_{nj}E^{n} + \bar{\xi}_{nj}E^{-n}) \quad , \tag{4.4}$$

$$\eta = \sum_{j=n}^{\infty}\sum_{n=0}^{\infty} \epsilon^{j}(\eta_{nj}E^{n} + \bar{\eta}_{nj}E^{-n}) \quad , \tag{4.5}$$

where ϕ_{nj}, p_{nj} are functions of X, Y, z, τ and ξ_{nj}, η_{nj} are functions of X, Y, τ only, and, because we have already separated the perturbations from the basic state, we must set

$$\eta_{00} = \phi_{00} = \xi_{00} = P_{00} = 0 \quad . \tag{4.6}$$

Here the "overbar" denotes the complex conjugate and

$$X = \epsilon(x-U_g t), \quad Y = \epsilon y, \quad \tau = \epsilon^2 t, \quad E = \exp ik(x-U_b t) \tag{4.7}$$

in which the group velocity U_g is defined by

$$U_g = \partial(kU_b)/\partial k \tag{4.8}$$

so that X is measured from the "centre" of the disturbance where energy is mainly concentrated.

The scaling (4.7) deserves a comment here. We note that the spatial and time variations are different. This is not an ad hoc assumption but it is dictated by the fact that the relevant equations derived below should be nontrivial and well-posed since the original problem is well-posed. This is the only scaling satisfying these two basic conditions.

We now substitute the expressions (4.2)-(4.5) using (4.7) and then equate the coefficients of $\epsilon^\ell E^r$ (ℓ, r = 0, 1, 2, 3,...) to zero to obtain a hierarchy of systems of equations which can be solved seriatim. To close the leading order problem we require to consider the systems (ℓ,r) = (0,1), (0,2), (1,1), (1,2), (2,2), (0,3), (1,3). We shall consider these problems (briefly) individually.

Problem (0,1)

Here we are required to solve the equation

$$\frac{\partial^2 \phi_{01}}{\partial z^2} = 0$$

subject to the conditions

$$\frac{\partial \phi_{01}}{\partial z} = 0 \quad \text{at} \quad z = 0,1 \quad .$$

The solution can be written down as

$$\phi_{01} = \phi_{01}(X,Y,\tau) \tag{4.9}$$

so that ϕ_{01} is independent of z. Also we find that

$$\xi_{01} = 0 \quad . \tag{4.10}$$

Problem (1,1)

The solution of

$$\frac{\partial^2 \phi_{11}}{\partial z^2} - k^2 \phi_{11} = 0 \tag{4.11}$$

subject to the conditions

$$\left.\begin{aligned} -\frac{\partial \phi_{11}}{\partial z} + ik(1-U_b)\xi_{11} &= 0 \quad \text{at} \quad z = 1 \\ -\frac{\partial \phi_{11}}{\partial z} + ik(1-U_b)\eta_{11} &= 0 \quad \text{at} \quad z = 0 \\ \xi_{11} + ikF^2(1-U_b)\phi_{11} &= 0 \quad \text{at} \quad z = 1 \\ ikU_b\eta_{11} + k^2\bar{H}_c\phi_{11} &= 0 \quad \text{at} \quad z = 0 \end{aligned}\right\} \tag{4.12}$$

This problem is identical with that solved for the linear problem in the previous section. The solution can then be written in the form

$$\phi_{11} = A \cosh(k)\, h(z) \quad ; \quad \eta_{11} = \frac{ik\bar{H}_c}{U_b} A \quad ;$$

$$\xi_{11} = -ikF^2(1 - U_b) \cosh(k) h A, \tag{4.13}$$

where

$$h(z) = [\cosh(kz) - \lambda \sinh(kz)]/\cosh(k),$$
$$h = h(1), \tag{4.14}$$

and A is an arbitrary function of X, Y, τ, provided that U_b is goverened by

$$U_b = \frac{k\bar{H}_c(1-U_b)[1 - kF^2(1-U_b)^2\sigma]}{\sigma - kF^2(1-U_b)^2} \left(= \frac{k\bar{H}_c(1-U_b)}{\lambda} \right) \quad . \tag{4.15}$$

Problem (0,2)

The problem is defined as

$$\frac{\partial^2 \phi_{02}}{\partial z^2} = 0 \tag{4.16}$$

together with

$$\left.\begin{aligned}
&-\frac{\partial \phi_{02}}{\partial z} = 0 \quad \text{at} \quad z = 1 \\
&-\frac{\partial \phi_{02}}{\partial z} + (1-U_g)\frac{\partial \eta_{01}}{\partial X} = 0 \quad \text{at} \quad z = 0 \\
&-\frac{2U_g}{\bar{H}_c}\frac{\partial \eta_{01}}{\partial X} - k^2\left\{\eta_{11}\frac{\partial \bar{\phi}_{11}}{\partial z} + \bar{\eta}_{11}\frac{\partial \phi_{11}}{\partial z}\right\} = 0 \quad \text{at} \quad z = 0 \\
&\frac{2\xi_{02}}{F^2} + 2(1-U_g)\frac{\partial \phi_{01}}{\partial X} + \gamma_1 \cosh^2(k)|A|^2 = 0 \quad \text{at} \quad z = 1
\end{aligned}\right\} \tag{4.17}$$

The solution of this system is

$$\phi_{02} = \phi_{02}(X,Y,\tau) \quad , \quad \eta_{02} = \eta_{02}(Y,\tau) \quad ,$$

$$\xi_{02} = -F^2(1-U_g)\frac{\partial \phi_{01}}{\partial X} - \frac{1}{2}\gamma_1 F^2|A|^2 \quad , \tag{4.18}$$

in which

$$\gamma_1 = k^2[h^2 + f^2 - 2kF^2(1-U_b)^2 hf]\cosh^2(k), \tag{4.19}$$

where

$$f = \sigma - \lambda \tag{4.20}$$

Problem (1,2)

Here it is necessary to solve the equation

$$\frac{\partial^2 \phi_{12}}{\partial z^2} - k^2\phi_{12} = -2ik\frac{\partial \phi_{11}}{\partial X} \tag{4.21}$$

subject to the four conditions

$$\left.\begin{aligned}
&-\frac{\partial \phi_{12}}{\partial z} + ik(1-U_b)\xi_{12} + (1-U_g)\frac{\partial \xi_{11}}{\partial X} = 0 \\
&\xi_{12} + ikF^2(1-U_b)\phi_{12} + F^2(1-U_g)\frac{\partial \phi_{11}}{\partial X} = 0
\end{aligned}\right\} \text{at} \quad z = 1 \tag{4.22}$$

$$\left.\begin{aligned}
&-\frac{\partial \phi_{12}}{\partial z} + ik(1-U_b)\eta_{12} + (1-U_g)\frac{\partial \eta_{11}}{\partial X} - 2\eta_{01}\frac{\partial^2 \phi_{11}}{\partial z^2} = 0 \\
&-\frac{ikU_b}{\bar{H}_c}\eta_{12} - \frac{U_g}{\bar{H}_c}\frac{\partial \eta_{11}}{\partial X} - k^2\phi_{12} + 2ik\frac{\partial \phi_{11}}{\partial X} - 2k^2\eta_{01}\frac{\partial \phi_{11}}{\partial z} = 0
\end{aligned}\right\} \text{at} \quad z = 0 \tag{4.23}$$

This problem is different from Problem (1,1) only in that it has a non-homogeneous part. It can then be solved in the form of a complementary function, similar to that of Problem (1,1), and a particular integral. However, because the homogeneous part is identical to that of Problem (1,1) a consistency condition must be satisfied by the solution to this problem before a solution can be obtained. We find

$$\phi_{12} = Dh(z) + (i\bar{H}_c/U_b)(2-U_b-U_g/U_b)\frac{\partial A}{\partial X}\sinh(kz)$$
$$+ iz\frac{\partial A}{\partial X}[\lambda\cosh(kz) - \sinh(kz)] \tag{4.24}$$

$$\eta_{12} = \frac{ik\bar{H}_c}{U_b}D + (\bar{H}_c/U_b)(2-U_g/U_b)\partial A/\partial X \quad ,$$

$$\xi_{12} = \frac{-i}{1-U_b}fD + \{(k\bar{H}_c/U_b)(2-U_b-U_g/U_b-f-h+kF^2(1-U_b)(1-U_g)h\}/k(1-U_b) \tag{4.25}$$

Here D is an arbitrary function of X, Y and τ.

The consistency condition here demands that

$$\eta_{01} = 0 \quad . \tag{4.26}$$

Problem (2,2)

Here we need to solve the equation

$$\frac{\partial^2\phi_{22}}{\partial z^2} - 4k^2\phi_{22} = 0 \tag{4.27}$$

subject to the conditions

$$\left.\begin{aligned} &-\frac{\partial\phi_{22}}{\partial z} + 2ik(1-U_b)\xi_{22} - 2k^2\xi_{11}\phi_{11} = 0 \\ &F^{-2}\xi_{22} + 2ik(1-U_b)\phi_{22} + ik(1-U_b)\xi_{11}\frac{\partial\phi_{11}}{\partial z} - \frac{1}{2}k^2\phi_{11}^2 + \frac{1}{2}\left(\frac{\partial\phi_{11}}{\partial z}\right)^2 = 0 \end{aligned}\right\} \text{ at } z = 1$$

$$\left.\begin{aligned} &-\frac{\partial\phi_{22}}{\partial z} + 2ik(1-U_b)\eta_{22} - 2k^2\eta_{11}\phi_{11} = 0 \\ &-\frac{2ikU_b}{\bar{H}_c}\eta_{22} - 4k^2\phi_{22} - k^2\eta_{11}\frac{\partial\phi_{11}}{\partial z} - i\alpha_2k^3\phi_{11}^2 = 0 \end{aligned}\right\} \text{ at } z = 0$$

This problem is solved in a way similar to Problem (1,1) to get

$$\phi_{22} = A^2[M\cosh(2kz) + N\sinh(2kz)], \tag{4.28}$$

$$\eta_{22} = (1-U_b)^{-1}[k^2\bar{H}_c/U_b - iN]A^2, \tag{4.29}$$

$$\xi_{22} = -F^2\{2ik(1-U_b)\phi_{22}(1) + k^3F^2(1-U_b)^2hf - \tfrac{1}{2}k^2h^2 + \tfrac{1}{2}k^2f^2\}, \tag{4.30}$$

in which

$$iM = \frac{-1}{4k\delta_3}\{\delta_1(1-\sigma^2) - \delta_2[1+\sigma^2-4kF^2(1-U_b)^2\sigma]\} \quad ,$$

$$iN = \frac{-1}{2k\delta_3}\{\delta_2[\sigma - kF^2(1-U_b)^2(1+\sigma^2)] - \delta_1\lambda(1-\sigma^2)\} \ ,$$

$$\delta_1 = k^3F^2(1-U_b)[2kF^2(1-U_b)^2hf + f^2 - 3h^2] \ ,$$

$$\delta_2 = (k^3\bar{H}_c/U_b)[2 - \lambda^2 + \alpha_2(1-U_b)] \ ,$$

$$\delta_3 = \lambda[1+\sigma^2-4kF^2(1-U_b)^2\sigma] - \sigma + kF^2(1-U_b)^2(1+\sigma^2) \ . \tag{4.31}$$

Problem (0,3)

Here ϕ_{03} must satisfy the equation

$$\frac{\partial^2\phi_{03}}{\partial z^2} = -\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)\phi_{01} \tag{4.32}$$

together with the boundary conditions

$$-\frac{\partial\phi_{03}}{\partial z} + (1 - U_g)\frac{\partial\xi_{02}}{\partial X} - k^2F^2(1 - U_b)\left\{\phi_{11}\frac{\partial\bar{\phi}_{11}}{\partial X} + \bar{\phi}_{11}\frac{\partial\phi_{11}}{\partial X}\right\} = 0 \quad \text{at } z = 1 \tag{4.33}$$

$$-\frac{\partial\phi_{03}}{\partial z} + (1 - U_g)\frac{\partial\eta_{02}}{\partial X} + (k^2\bar{H}_c/U_b)\left\{\phi_{11}\frac{\partial\bar{\phi}_{11}}{\partial X} + \bar{\phi}_{11}\frac{\partial\phi_{11}}{\partial X}\right\} = 0 \quad \text{at } z = 0 \tag{4.34}$$

$$\begin{aligned} &-(2U_g/\bar{H}_c)\frac{\partial\eta_{02}}{\partial X} + 2\frac{\partial^2\phi_{01}}{\partial X^2} - \left(\frac{ik^3\bar{H}_c}{U_b}\right)\left(\phi_{11}\frac{\partial\bar{\phi}_{12}}{\partial z} - \bar{\phi}_{11}\frac{\partial\phi_{12}}{\partial z}\right) \\ &+(2k^2\bar{H}_c/U_b)\left\{\phi_{11}\frac{\partial^2\bar{\phi}_{11}}{\partial z\partial X} + \bar{\phi}_{11}\frac{\partial^2\phi_{11}}{\partial z\partial X}\right\} - k^2\left\{\eta_{12}\frac{\partial\bar{\phi}_{11}}{\partial z} + \bar{\eta}_{12}\frac{\partial\phi_{11}}{\partial z}\right\} + k^2\alpha_2\left\{\phi_{11}\frac{\partial\bar{\phi}_{11}}{\partial X} + \bar{\phi}_{11}\frac{\partial\phi_{11}}{\partial X}\right\} = 0 \\ &\text{at } z = 0 \end{aligned} \tag{4.35}$$

and fourth boundary condition at $z = 1$ which merely provides an expression for ξ_{03} and since it is not required here it is not given.

Now Eq. (4.35) does not involve ϕ_{03} and can therefore be simplified using the solutions obtained for Problems (1,1) and (1,2) to

$$\frac{\partial}{\partial X}\left\{\frac{U_g}{\bar{H}_c}\eta_{02} - \frac{\partial\phi_{01}}{\partial X} + b_1|A|^2\right\} = 0 \tag{4.36}$$

in which

$$b_1 = \frac{k^4 \bar{H}_c}{2U_b^2} [1 - U_g + 2(1 - U_b) - \frac{1}{2} k^2 \alpha_2 . \qquad (4.37)$$

Integration of (4.32) once and the use of the result in (4.33) and (4.34) lead to the equation

$$(1 - U_g) \frac{\partial \xi_{02}}{\partial X} - (1 - U_g) \frac{\partial \eta_{02}}{\partial X} + \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)\phi_{01} - b_2 \frac{\partial}{\partial X} |A|^2 = 0 \qquad (4.38)$$

where

$$b_2 = \frac{k^2 \bar{H}_c}{U_b} + k^2 F^2 (1 - U_b) h^2 \cosh^2(k) \ . \qquad (4.39)$$

<u>Problem (1,3)</u>

The last problem required here is goverened by the equation

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\phi_{13} = -2ik \frac{\partial \phi_{12}}{\partial X} - \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right) \phi_{11} \qquad (4.40)$$

subject to the boundary conditions

$$\left.\begin{aligned} -\frac{\partial \phi_{13}}{\partial z} + ik(1-U_b)\xi_{13} + h_1 &= 0 \\ F^{-2}\xi_{13} + ik(1-U_b)\phi_{13} + h_2 &= 0 \end{aligned}\right\} \text{ at } z = 1 \qquad (4.41)$$

$$\left.\begin{aligned} -\frac{\partial \phi_{13}}{\partial z} + ik(1-U_b)\eta_{13} + h_3 &= 0 \\ -\frac{ikU_b}{\bar{H}_c}\eta_{13} - k^2\phi_{13} + h_4 &= 0 \end{aligned}\right\} \text{ at } z = 0 \qquad (4.42)$$

Here we have defined

$$h_1 = - ikF^2(1-U_b)(1-U_g) \frac{\partial \phi_{12}}{\partial X} - F^2(1-U_g)^2 \frac{\partial^2 \phi_{11}}{\partial X^2} - ikF^2(1-U_b) \frac{\partial \phi_{11}}{\partial \tau}$$
$$- 4ik^3F^2(1-U_b)\phi_{11}\phi_{22} - 2k^2\xi_{02}\phi_{11} - k^4F^4(1-U_b)^2\phi_{11}\bar{\phi}_{11}\frac{\partial \phi_{11}}{\partial z}$$
$$+ \frac{1}{2} k^4F^2\phi_{11}^2 \bar{\phi}_{11} - \frac{1}{2} k^2F^2\bar{\phi}_{11}\left(\frac{\partial \phi_{11}}{\partial z}\right)^2 + 2k^2F^2(1-U_b)\phi_{11}\frac{\partial \phi_{01}}{\partial X}$$
$$- \frac{1}{2} k^4F^4(1-U_b)^2 \phi_{11}^2 \frac{\partial \bar{\phi}_{11}}{\partial z} - k^4F^4(1-U_b)^2 \phi_{11}\bar{\phi}_{11}\frac{\partial \phi_{11}}{\partial z} \quad ,$$

$$h_2 = (1-U_g) \frac{\partial \phi_{12}}{\partial X} + \frac{\partial \phi_{11}}{\partial \tau} - 2k^2F^2(1-U_b)^2 \bar{\phi}_{11}\frac{\partial \phi_{22}}{\partial z} + 2ik(1-U_b)\xi_{02}\frac{\partial \phi_{11}}{\partial z}$$
$$- 2k^2F^2(1-U_b)^2\frac{\partial \bar{\phi}_{11}}{\partial z}\phi_{22} + ik^3F^4(1-U_b)^3\phi_{11}\frac{\partial \phi_{11}}{\partial z}\frac{\partial \phi_{11}}{\partial z} - \frac{i}{2} k^3F^2(1-U_b)\phi_{11}^2\frac{\partial \phi_{11}}{\partial z}$$
$$+ \frac{1}{2} ik(1-U_b)F^2\frac{\partial \bar{\phi}_{11}}{\partial z} \cdot \left(\frac{\partial \phi_{11}}{\partial z}\right)^2 + 2ik\,\phi_{11}\frac{\partial \phi_{01}}{\partial X} + 2k^2\bar{\phi}_{11}\phi_{22} + \frac{\partial \bar{\phi}_{11}}{\partial z} \cdot \frac{\partial \phi_{22}}{\partial z}$$
$$+ \frac{3}{2} ik^5F^4(1-U_b)^3\phi_{11}^2\bar{\phi}_{11} - 2ik^3F^2(1-U_b)\phi_{11}\left(\bar{\phi}_{11}\frac{\partial \phi_{11}}{\partial z} + \phi_{11}\frac{\partial \bar{\phi}_{11}}{\partial z}\right) ,$$

$$h_3 = \frac{ik\bar{H}_c}{U_b}(1-U_g)\frac{\partial \phi_{12}}{\partial X} + \frac{\bar{H}_c}{U_b}(1-U_g)(2 - \frac{U_g}{U_b})\frac{\partial^2 \phi_{11}}{\partial z^2} + \frac{2ik^3\bar{H}_c}{U_b}\bar{\phi}_{11}\phi_{22}$$
$$- 2k^2\eta_{02}\phi_{11} + k^2\bar{\phi}_{11}\eta_{22} - \frac{2k^2\bar{H}_c}{U_b}\phi_{11}\frac{\partial \phi_{01}}{\partial X} - \frac{k^4\bar{H}_c^2}{2U_b^2}\phi_{11}\left(\frac{1}{2}\phi_{11}\frac{\partial \bar{\phi}_{11}}{\partial z} + \bar{\phi}_{11}\frac{\partial \phi_{11}}{\partial z}\right) ,$$

$$h_4 = -\frac{ikU_g}{U_b}\frac{\partial \phi_{12}}{\partial X} - \frac{U_g}{U_b}(2-U_g/U_b)\frac{\partial^2 \phi_{01}}{\partial X^2} + \frac{ik}{U_b}\frac{\partial \phi_{11}}{\partial \tau} + 2ik\frac{\partial \phi_{12}}{\partial X} + \frac{\partial^2 \phi_{11}}{\partial X^2}$$
$$+ \frac{4ik^3\bar{H}_c}{U_b}\bar{\phi}_{11}\frac{\partial \phi_{22}}{\partial z} - 2k^2\eta_{02}\frac{\partial \phi_{11}}{\partial z} - k^2\eta_{22}\frac{\partial \bar{\phi}_{11}}{\partial z} + 2i\alpha_2k^3\bar{\phi}_{11}\phi_{22}$$
$$- 2\alpha_2k^2\phi_{11}\frac{\partial \phi_{01}}{\partial X} - \frac{k^6\bar{H}_c^2}{2U_b^2}\bar{\phi}_{11}\phi_{11}^2 - \frac{2\alpha_2k^4\bar{H}_c}{U_b}\bar{\phi}_{11}\phi_{11}\frac{\partial \phi_{11}}{\partial z} - \alpha_3k^4\phi_{11}^2\bar{\phi}_{11} .$$

$$(4.43)$$

This problem is therefore a non-homogeneous form of Problem (1,1) and as in the case of Problem (1,2) we must satisfy a consistency condition before a solution can be legitimately found. We shall omit the details since they are identical to those adopted for Problem (1,2). The condition of solvability is

$$[1-kF^2(1-U_b)^2\sigma]\left\{-\frac{\partial\psi_p}{\partial z} - \lambda\psi_p + \frac{\lambda}{k}h_4 + h_3\right\}_{z=1}$$

$$+\left\{\frac{\partial\psi_p}{\partial z} - kF^2(1-U_b)^2\,\psi_p - h_1 + ikF^2(1-U_b)h_2\right\}_{z=0} = 0 \tag{4.44}$$

in which ψ_p is a particular solution of (4.40).

Now it can be shown directly by substitution in (4.40) that

$$\psi_p = z(\alpha z + \beta)\cosh(kz) + z(\gamma z + \delta)\sinh(kz) \tag{4.45}$$

where

$$\alpha = -\frac{1}{2}\frac{\partial^2 A}{\partial X^2},$$

$$\beta = i\lambda\frac{\partial D}{\partial X} + (\bar{H}_c/U_b)(2-U_b-U_g/U_b)\frac{\partial^2 A}{\partial X^2} + (\lambda/2k)\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)A,$$

$$\gamma = \frac{1}{2}\lambda\frac{\partial^2 A}{\partial X^2}, \quad \delta = -i\frac{\partial D}{\partial X} - \frac{1}{2k}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)A. \tag{4.46}$$

When we substitute from (4.43), (4.46) and use solutions of problems (1,1), (1,2) and (2,2) we can, after lengthy laborious algebra, reduce (4.44) to

$$i\frac{\partial A}{\partial T} + a_1\frac{\partial^2 A}{\partial X^2} - a_2 A\frac{\partial\phi_{01}}{\partial X} - a_3\eta_{02}A - a_4\frac{\partial^2 A}{\partial Y^2} + a_5 A|A|^2 = 0 \tag{4.47}$$

in which

$$T = \tau/a_6 \tag{4.48}$$

and

$$a_1 = -(k\bar{H}_c/U_b)(2-U_b-U_g/U_b)[-\sigma+2F^2(1-U_b)\ (U_b-U_g)\sigma+F^2(1-U_b)^2\sigma+kF^2(1-U_b)^2]$$

$$+\left(\frac{\lambda}{k}\right)\left\{1 - kF^2(1-U_b)^2\sigma]\left[1 + \frac{(1-U_g/U_b)(2-U_g/U_b)}{(1-U_b)}\right]\right.$$

$$+\frac{1}{2}k^2 + \frac{3}{2}k\sigma - \frac{1}{2}k^3F^2(1-U_b)^2\sigma - 2k^2F^2(1-U_b)(1-U_g) - kF^2(1-U_g)^2\sigma$$

$$\left.+\frac{1}{2}kF^2(1-U_b)^2\sigma\right\}$$

$$-\frac{1}{2}k\sigma - \sigma/2k - 3/2 + \frac{1}{2}k^2F^2(1-U_b)^2 + 2kF^2(1-U_b)(1-U_g)\sigma + F^2(1-U_g)^2,$$

$$a_2 = 2k\left(\alpha_2 + \frac{1}{1-U_b}\right)[\sigma - kF^2(1-U_b)^2] + 4k^2F^2(1-U_b)h$$

$$+ 2k^2F^2(1-U_g)(1-\sigma^2)[1-k^2F^4(1-U_b)^4][1-kF^2(1-U_b)^2\sigma]^{-1}$$

$$a_3 = 2k^2(1-\lambda^2)[1-kF^2(1-U_b)^2\sigma]$$

$$a_4 = -\frac{1}{2}\lambda\sigma[1 + F^2(1-U_b)^2] + \frac{1}{2}(1 + \sigma/k)$$

$$a_5 = [1 - kF^2(1-U_b)^2\sigma]\left\{\frac{k^3\lambda}{(1-U_b)^2}(1 + 2\lambda^2) + \frac{k^2(iN)}{1 - U_b}(7\lambda^2 - 1)\right.$$

$$\left.+\frac{2k^2\lambda(iM)}{(1 - U_b)}[1 + \alpha_2(1-U_b)] + \frac{2\alpha_2 k^3\lambda^3}{1-U_b} - \alpha_3 k^3\lambda\right\}$$

$$-\gamma_1 k^2F^2(1-\sigma^2)[1-k^2F^4(1-U_b)^4][1-kF^2(1-U_b)^2\sigma]^{-1}$$

$$+ 7k^5F^4(1-U_b)^2h^2f + \frac{1}{2}k^4F^2h(f^2-h^2) - \frac{1}{2}k^6F^6(1-U_b)^4h(2f^2 + 3h^2)$$

$$-\frac{1}{2}k^5F^4(1-U_b)^2f^3 + 4k^3F^2(1-U_b)h(iM)$$

$$+ 2k^3F^2(1-U_b)[h-kF^2(1-U_b)^2f](iK)+2k^3F^2(1-U_b)[f-2kF^2(1-U_b)^2h](iL),$$

$$a_6 = \frac{k\bar{H}_c}{U_b^2}[1 + F^2U_b^2(1-U_b)][2 - \lambda(1+2U_b)\sigma/U_b]; \tag{4.49}$$

and

$$(ik) = \frac{(1+\sigma^2)(iM) + 2\sigma(iN)}{1-\sigma^2}\ , \quad iL = \frac{2\sigma(iM) + (iN)(1+\sigma^2)}{1-\sigma^2}\ . \tag{4.50}$$

Thus we are required to solve Eqs. (4.36), (4.38) and (4.47) together with (4.18) for A, ϕ_{01}, η_{02}, ξ_{02} as functions of X, Y, T subject to some prescribed boundary conditions. This is discussed in Sec. 5 below.

Before we proceed to the next section, we comment on the nonlinear equations for non-zero values of δ satisfying (3.13). It was found that the application of the scaling (4.7) leads to a hierarchy of nested sets of equations which do not close for any finite value of ℓ or r. This of course means that the scaling is inappropriate and another one must be found. Indeed it was found that the appropriate scaling is

$$X = \epsilon^3(x - U_g t),\ \ Y = \epsilon^3 y,\ \ \tau = \epsilon^2 t. \tag{4.51}$$

However, with this case the problem closes with an equation devoid of spatial derivatives. It is discussed in Sec. 5 below.

5. SOME SOLUTIONS OF THE NON-LINEAR EQUATIONS

In the previous section we derived equations for the evolution of the amplitude $A(X,Y,T)$ of the bedform and the concomitant mean flow and bedform amplitude ϕ_0, η_{02}, respectively. For the benefit of the reader we summarize these equations here.

$$\frac{\partial}{\partial X}\left\{\frac{U_g}{\bar{H}_c}\eta_{02} - \frac{\partial\phi_{01}}{\partial X} + b_1|A|^2\right\} = 0, \tag{5.1}$$

$$(1-U_g)\frac{\partial\xi_{02}}{\partial X} - (1-U_g)\frac{\partial\eta_{02}}{\partial X} + \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)\phi_{01} - b_2\frac{\partial}{\partial X}|A|^2 = 0, \tag{5.2}$$

$$i\frac{\partial A}{\partial T} + a_1\frac{\partial^2 A}{\partial X^2} - a_2 A\frac{\partial\phi_{01}}{\partial X} - a_3\eta_{02}A - a_4\frac{\partial^2 A}{\partial Y^2} + a_5 A|A|^2 = 0 \tag{5.3}$$

$$\xi_{02} = -F^2(1-U_g)\frac{\partial\phi_{01}}{\partial X} - \frac{1}{2}\gamma_1 F^2|A|^2 \tag{5.4}$$

$$\xi_{01} = \eta_{01} = 0 \tag{5.5}$$

We first simplify these equations. Using (5.1) and (5.4) in (5.2) we get

$$\left[1-F^2(1-U_g)^2 - \frac{\bar{H}_c(1-U_g)}{U_g}\right]\frac{\partial^2\phi_{01}}{\partial X^2} + \frac{\partial^2\phi_{01}}{\partial Y^2} + \left[\frac{b_1\bar{H}_c(1-U_g)}{U_g} - b_2 - \frac{1}{2}\gamma_1(1-U_g)F^2\right]\frac{\partial}{\partial X}|A|^2 = 0 \tag{5.6}$$

Also (5.1) can be integrated to give

$$\eta_{02} = \frac{\bar{H}_c}{U_g}\left[\frac{\partial\phi_{01}}{\partial X} - b_1|A|^2\right] + Q(Y,T) \tag{5.7}$$

The utilization of (5.7) in (5.3) yields

$$i\frac{\partial A}{\partial T} + a_1\frac{\partial^2 A}{\partial X^2} - \left(a_2 + \frac{\bar{H}_c a_3}{U_g}\right)A\frac{\partial\phi_{01}}{\partial X} - a_4\frac{\partial^2 A}{\partial Y^2} + \left(a_5 + \frac{a_3 b_1\bar{H}_c}{U_g}\right)A|A|^2 - a_3 AQ = 0\ . \tag{5.8}$$

We shall consider two cases: (i) the dunetrain and (ii) the solitary dune solutions.

The dunetrain is the solution in which the amplitude depends on T only. Here $\partial/\partial X, \partial/\partial Y \equiv 0$ in (5.6), (5.7) and (5.8). It is then clear that (5.8) gives

$$A = A_0 e^{ipt}, \qquad p = \left(a_5 + \frac{a_3 b_1\bar{H}_c}{U_g}\right)A_0^2 - a_3 A_0 Q_0 \tag{5.9}$$

in which Q_0 is an arbitrary constant. This is a solution in which a perturbaion of order ϵ produces a mean elevation $\epsilon^2\eta_{02}$ in the bedform and mean elevation $\epsilon^2\xi_{02}$ in the surface undulations, where

$$\epsilon^2\eta_{02} = \frac{-\epsilon^2\bar{H}_c b_1}{U_g}A_0^2\ , \quad \epsilon^2\xi_{02} = -\frac{1}{2}\epsilon^2\gamma_1 F^2 A^2\ . \tag{5.10}$$

This solution is parallel to that known as the Stoke's wavetrain in the theory of waterwaves. It is then informative to draw parallel results with the well-studied case of the Stoke's wavetrain. First we discuss the stability of this nonlinear solution. This is carried out simply by assuming that A, ϕ_{01}, η_{02}, ξ_{02} are functions of $\ell X + mY$ and T only (see Davey and Stewartson, 1974) so that (5.6) and (5.8) give

$$\left\{\ell^2\left[1 - F^2(1-U_g)^2 - \frac{\bar{H}_c(1-U_g)}{U_g}\right] + m^2\right\} \frac{\partial^2\phi_{01}}{\partial\chi^2} + \ell\left[\frac{b_1\bar{H}_c(1-U_g)}{U_g} - b_2 - \frac{1}{2}\gamma_1(1-U_g)F^2\right]\frac{\partial|A|^2}{\partial\chi^2} = 0$$

$$i\,\frac{\partial A}{\partial T} + (a_1\ell^2 - a_4 m^2)\,\frac{\partial^2 A}{\partial\chi^2} - \ell\left(a_2 + \frac{\bar{H}_c a_3}{U_g}\right) A\,\frac{\partial\phi_{01}}{\partial\chi}$$

$$+ \left(a_5 + \frac{a_3 b_1 \bar{H}_c}{U_g}\right) A|A|^2 = 0, \qquad (5.11)$$

where the term involving Q, which merely represents a phase change, has been absorbed in the first term, and

$$\chi = \ell X + mY \qquad (5.12)$$

We next eliminate $\partial\phi_{01}/\partial X$ from (5.11) to obtain a single equation

$$i\,\frac{\partial A}{\partial T} + \mu\,\frac{\partial^2 A}{\partial\chi^2} + \nu A|A|^2 = 0 \qquad (5.13)$$

where

$$n = \ell^2 a_1 - m^2 a_4,$$

$$\nu = a_5 + \frac{a_3 b_1 \bar{H}_c}{U_g} - \frac{\ell^2(a_2+\bar{H}_c a_3/U_g)[b_2+\frac{1}{2}\gamma_1(1-U_g)F^2 - b_1\bar{H}_c(1-U_g)/U_g]}{m^2 + \ell^2[1-F^2(1-U_g)^2 - \bar{H}_c(1-U_g)/U_g]} \qquad (5.14)$$

Eq. (5.13) is a Schrödinger equation identical to that studied by Hasimota and Ono (1972), Davey and Stewartson (1974), Eltayeb (1977) and Eltayeb and Hassan (1981). It is well-known that the solution (5.9) is unstable if

$$\mu\nu > 0 \qquad (5.15)$$

Numerical computations carried out on (5.15) showed that the dependence of the stability on the dimensionless parameters k, $\bar{H}_c$, α_2, α_3, F is extremely complicated and we shall not discuss it in detail here. It may suffice to say that the condition (5.15) is satisfied by some values of ℓ and m for almost all values of the parameters. For the benefit of the reader, however, an example of the results is shown in Fig. 3. It can be seen that for $F = 0$, i.e. the rigid lid case, the region (shaded) of instability is least complicated.

Variations in α_2, α_3 tend to shift the null curves (i.e. the curves $\mu = 0$, $\nu = 0$) slightly but the general shape of the region of instability remains the same. This is maintained for values of $F \lesssim 0.4$. However, for higher values of F the region of instability is more complicated and some values of k, depending on F, are associated with stability.

Localized solutions

When localized solutions are sought, the boundary conditions

$$\eta_{02}\,,\quad |\nabla\phi_{01}| \to 0 \quad \text{as} \quad X^2 + Y^2 \to \infty \qquad (5.16)$$

are imposed. Then Q of Eq. (5.7) vanishes. Further, if we look for two-dimensional solutions (i.e. set $\partial/\partial Y \equiv 0$) we can reduce Eqs. (5.6) and (5.8) to a single equation

$$i\,\frac{\partial A}{\partial T} + a_1\,\frac{\partial^2 A}{\partial X^2} + \zeta\, A|A|^2 = 0, \qquad (5.17)$$

where

$$\zeta = a_5 + \frac{a_3 b_1 \bar{H}_c}{U_g} - \frac{[a_2 + \bar{H}_c a_3/U_g][b_2 + \frac{1}{2}\gamma_1(1-U_g)F^2 - b_1\bar{H}_c(1-U_g)/Ug]}{1 - F^2(1-U_g)^2 - \bar{H}_c(1 - U_g)/U_g}\,. \qquad (5.18)$$

This equation is known to possess solitary solutions in the parameter space in which $a_1\zeta > 0$. This solution can be written as

$$A = \left(\frac{2p}{\zeta}\right)^{1/2} \operatorname{sech}\left[\left(\frac{p}{a_1}\right)^{1/2} X\right] e^{ipT} \qquad (5.19)$$

provided $a_1 > 0$ and consequently $\zeta > 0$ also. Comparing a_1, ζ with μ, ν we see that they are closely related and if $\mu\nu > 0$ then it is possible to satisfy $a_1\zeta > 0$. It then follows that the solution (5.19) occurs when the dunetrain is unstable. This indicates that the dunetrain (5.9) can degenerate into a solitary dune. This is the main result of the present study. However it remains to be shown how the dunetrain evolves into a solitary dune. This can only be done by numerical integration of the equations. Moreover, the solitary dune (5.19) is a two-dimensional phenomenon and the three-dimensional character of Eqs. (5.6) and (5.8) has not been explored. Indeed it is of great importance to identify the role of Y in the solutions and thereby clarify the lateral movement of sand in a unidirectional velocity field.

Now the above analysis applies when the phase shift δ is zero. When $\delta = 2\pi r/k$ with $r = 1, 2, \ldots$, we find that the problem is goverened by the scaling (4. 7) and then the evolutionary equations take the form

$$i \frac{\partial A}{\partial T} - a_3 \eta_{02} A + a_5 A|A|^2 = 0 \qquad (5.20)$$

$$\xi_{02} = -\frac{1}{2} \gamma_1 F^2 |A|^2 \qquad (5.21)$$

so that no spatial derivatives are present and therefore A, η_{02}, ξ_{02} are functions of time T with any dependence on X and Y being parametrized.

Let

$$A = \tilde{A} \exp ia_3 \int \eta_{02} dT \qquad (5.22)$$

so that

$$i \frac{\partial \tilde{A}}{\partial T} + a_5 \tilde{A}|\tilde{A}|^2 = 0 \qquad (5.23)$$

This has a solution

$$\tilde{A} = A_0 e^{ipT} \quad , \quad p = a_5 |A_0|^2 \quad .$$

This is a dunetrain and because of the absence of the spatial derivatives it represents a neutral solution.

We may then conclude that $\delta = 0$ provides the situation leading to the formation of solitary dunes while the case $\delta = 2\pi/k$ refers to the case of the stable dunetrain.

6. SOME CONCLUDING REMARKS

The non-linear evolutionary equations for the amplitude of sand dunes using a model due to Kennedy (1963), in which the sediment transport flux varies as a power of the effective velocity of the wind are derived for a layer of thickness d bounded above by a free surface. The presence of the free surface is manifest by the introduction of the Froude number F into the problem. When $F = 0$, the results of the model in which the top surface is rigid are obtained. This result provides an alternative to the previous study (Eltayeb and Hassan, 1986) in which many misprints are present.

The nonlinear equations show a marked resemblance to those used in the study of water waves and the known general properties of the equations are used to predict the formation of solitary dunes from an initial duntrain train.

The role played by the phase shift δ is also investigated. While the presence of non-zero δ is essential for instability according to the linear theory, it is found that solutions representing the nonlinear growth and propagation of sand dunes are only obtainable for the case of zero phase shift. The inclusion of non-zero phase shift in the analysis predicts neutral solutions only.

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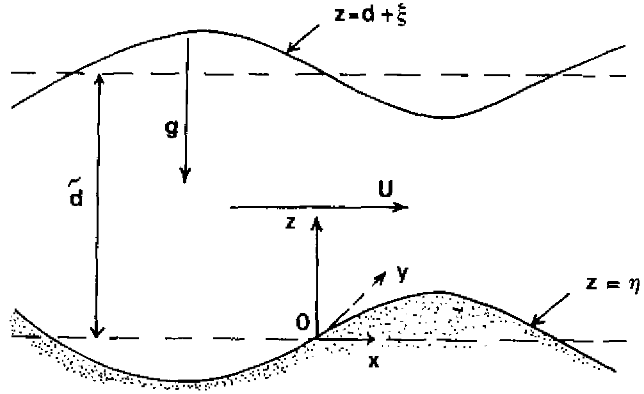
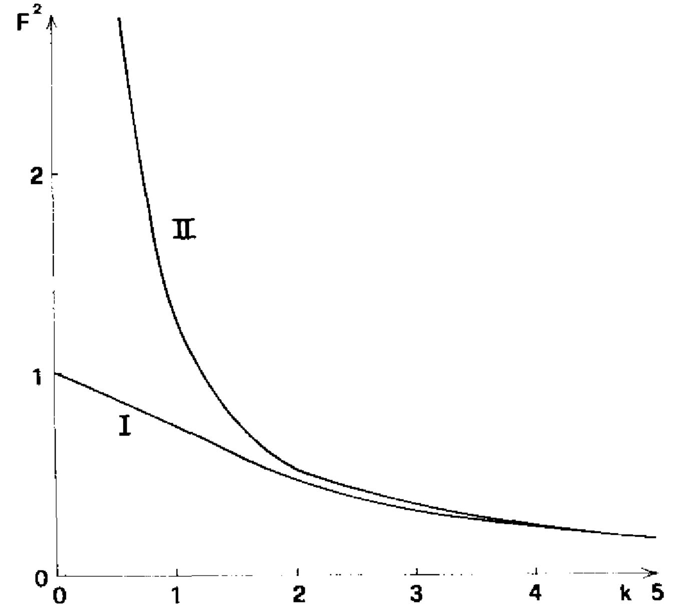
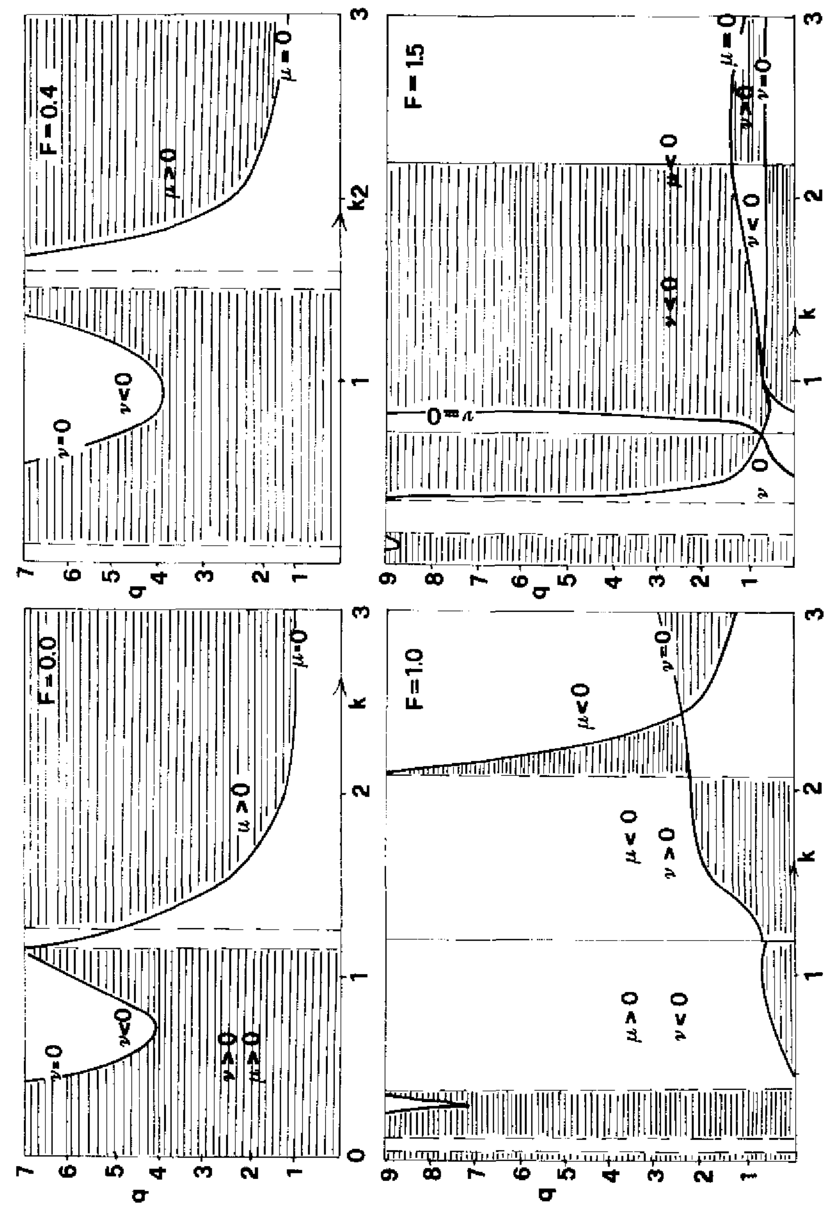




F²
k
I
II



F=0.0
F=0.4
F=1.0
F=1.5
q
k

FIGURE CAPTIONS

Fig. 1 Definition of the model.

Fig. 2 Curve I represents $F^2 = \tanh(k)/k$ and II refers to $F^2 = 1/k\tanh(k)$. The two curves are almost coincident for $k \gtrsim 3$. The two curves are symmetric about the F^2 axis and therefore only positive values of k are shown. The area between the two curves represents instability if $0 < k\delta < \pi$ and the area below I and above II refer to instability if $\pi < k\delta < 2\pi$. Therefore if these conditions are violated in the respective regions then a flat bed results. If $\delta = 0, \pi, 2\pi$ the system is neutrally stable.

Fig. 3 A sample of the stability results for the dunetrain when $\alpha_2 = 2.1$, $\alpha_3 = 1.0$, $\bar{\Pi}_c = 0.005$ for various values of F. The shaded areas refer to instability in k, q $(=\ell^2/m^2)$ plane.