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QUANTIZATION OF FERMIONS IN EXTERNAL SOLITON FIELDS AND INDEX CALCULATION*

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Abstract

We review recent results on the quantization of fermions in external fields, discuss equivalent and inequivalent representations of the canonical anticommutation relations, indicate how the requirement of implementability of gauge transformations leads to quantization conditions, determine the algebra of charges, identify the Schwinger term and remark finally how one may calculate a ground state charge.

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I. INTRODUCTION

One motivation of our studies goes back to 1976, when it was observed that half-integer charged states may occur in cetain field theoretical models [1]. There was renewed interest in such problems from solid state physics around 1980, since in long polymers, like polyacetylen, solitons may be formed and electrons hopping along the chain may feel a soliton like background [2].

Afterwards a number of people looked to such problems - the simplest one are the external field problems [3].

From a more mathematical point of view one treats representations of the canonical anticommutation relations (CAR) and one may ask a number of questions:

Compare two representations, for example, a Fock representation to another one connected to a Dirac operator with soliton potential: are they unitarily equivalent or not?

Consider gauge transformations: are they all implementable?

A one-parameter family of implementable gauge transformations yields smeared charges as generators: which algebra do they fulfill?

Finally we remark that applying chiral transformations with non-zero winding number allows to charge the vacuum. We obtain formulas for the charge difference, relate it to the η-invariant and indicate how Krein's spectral shift function helps in calculating the index.

According to the above questions we divide the talk into four parts:

II. FORMULATION - BOGOLIUBOV TRANSFORMATION - UNITARY EQUIVALENCE

The simplest situation is obtained by comparing two one-dimensional Dirac operators acting in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$. The free one, $H_0 = \alpha p + \beta$, with mass $m = 1$ has essential spectrum $\sigma_{ess}(H_0) = (-\infty,-1] \cup [1,\infty)$. In order to obtain the same essential spectrum for $H = \alpha p + \beta v(x)$, we may either choose trivial asymptotics $\lim_{x\to\pm\infty} v(x) = 1$ or a soliton (kink) asymptotics $\lim_{x\to\pm\infty} v(x) = \pm 1$. In the second case one obtains exactly one zero energy bound state.

Let $P^o_\pm$ be the spectral projection operators onto the positive and negative energy spectrum of H_o, and $P_\pm$ be similarly projections of H; if a zero energy bound state exists, we add it to P_+.

For second quantization we go over to the Fock space $F = \bigoplus_{n=0}^{\infty} H_n$ where $H_o = \mathbb{C}$ and H_n denotes the antisymmetric tensor-product of n copies of H. We introduce creation and annihilation operators $b^\dagger(f)$ and $b(f)$ for $f \in P^o_+ H$ fulfilling the CAR:

$$b^\dagger(f)\, f_1 \wedge \dots \wedge f_n = f \wedge f_1 \wedge \dots \wedge f_n , \quad \{b(f), b^\dagger(g)\} = \langle f, g\rangle , \quad (1)$$

where the r.h.s. bracket denotes the scalar product in H. $d(f)$, $d^\dagger(f)$ act similarly for $f \in P^o_- H$ and $B(f)$'s and $D(g)$'s denote annihilation operators for $f \in P_+ H$ and $g \in P_- H$.

Next we define a field operator $\psi(f)$ and decompose f into two ways:

$$\psi(f) = b(P^o_+ f) + d^\dagger(P^o_- f) = B(P_+ f) + D^\dagger(P_- f) . \quad (2)$$

This determines the relation between bare and dressed operators: choose orthonormal systems $\{\varphi^o_{n\pm}\}$ for $P^o_\pm H$, $\{\varphi_{n\pm}\}$ for $P_\pm H$ and denote $b(\varphi^o_{n+}) = b_n, \dots$. Equ. (2) determines the Bogoliubov transformation

$$\begin{pmatrix} B_n \\ D^\dagger_n \end{pmatrix} = \begin{pmatrix} W^1_{nm} & W^2_{nm} \\ W^3_{nm} & W^4_{nm} \end{pmatrix} \begin{pmatrix} b_m \\ d^\dagger_m \end{pmatrix} , \quad \begin{matrix} W^1 = P_+ P^o_+ , & W^2 = P_+ P^o_- , \\ W^3 = P_- P^o_+ , & W^4 = P_- P^o_- , \end{matrix} \quad (3)$$

where the W^i are the kernels of the product of projection operators and the W-matrix is unitary. We require that there exist vacua ω, Ω such that

$$b_n \omega = d_n \omega = 0 , \qquad B_n \Omega = D_n \Omega = 0 . \quad (4)$$

It is well-known [4] that finiteness of $\|\Omega\| < \infty$ is equivalent to the existence of a unitary operator U such that

$$U b_n U^\dagger = B_n , \quad U d_n U^\dagger = D_n \quad \text{and} \quad U\omega = \Omega . \quad (5)$$

The Shale-Stinespring-Berezin-Friedrichs criterion implies the existence of U iff the conditions

$$\|P_\pm P^o_\mp\|_{HS} < \infty \iff \|P_\pm - P^o_\pm\|_{HS} < \infty \quad (6)$$

are fulfilled.

It is easy to obtain the explicit form of U [4]; note that from conditions (6) and the unitarity of the W-matrix one deduces that W^1 and W^4 are Fredholm operators and their index fulfills

$$i(W^1) = -\, i(W^4) = n - m , \quad n = \dim \ker W^1 , \quad m = \dim \ker W^4 . \quad (7)$$

U applied to the bare vacuum becomes

$$\Omega_{n,m} = \exp[b^\dagger_p A^\perp_{pq} d^\dagger_q]\, b^\dagger_1 \dots b^\dagger_n d^\dagger_1 \dots d^\dagger_m \omega , \quad (8)$$

where $A^\perp = -(W^1_\perp)^{-1} W^2$ and $(W^1_\perp)^{-1}$ is the inverse of W^1 on the subspace orthogonal to $\ker W^1$ and a suitable base has been chosen for $\ker W^1$ and $\ker W^{1\dagger} = \ker W^4$. Note that for $(n,m) \neq (0,0)$ the new vacuum becomes orthogonal to the old one and <u>charged</u>:

$$Q\,\Omega_{n,m} = \sum_p (b^\dagger_p b_p - d^\dagger_p d_p)\Omega_{n,m} = (n-m)\Omega_{n,m} ,$$
$$\Delta Q = \langle \Omega_{n,m}, Q\Omega_{n,m}\rangle = n - m . \quad (9)$$

The charge difference equals the Fredholm index $i(W^1)$.

<u>Remark</u>: For an equivalent but more abstract formulation one starts from an algebra of operators $a(f)$, $a^\dagger(g)$ fulfilling the CAR and studies states determined by projection operators P_+:

$$\omega_{P_+}(a(f_n)\dots a(f_1) a^\dagger(g_1) \dots a^\dagger(g_m)) = \delta_{nm} \det \langle f_i, P_+ g_j\rangle . \quad (10)$$

These pure quasifree states correspond to the "filling of the Dirac sea"; the GNS-construction allows to identify in a representation

$$\pi_{P_+}(a^{\dagger}(f)) = \begin{cases} B^{\dagger}(f) & \text{for} \quad f \in P_+ H \\ D(f^*) & \text{for} \quad f \in P_- H \,. \end{cases} \tag{11}$$

<u>EXAMPLE 1</u>: The kink potential $v(x) = \text{th}\, x$ leads to a solvable Dirac equation

$$H = \alpha p + \beta\, \text{th}\, x = \begin{pmatrix} 0 & A \\ A^{\dagger} & 0 \end{pmatrix}, \quad A\phi_2 = E\phi_1 \,, \quad A^{\dagger}\phi_1 = E\phi_2 \,. \tag{12}$$

Multiplying (12) times $A^{\dagger}$ and A gives two almost isospectral Schrödinger operators

$$AA^{\dagger}\phi_1 = (-\frac{d^2}{dx^2} + 1)\phi_1 = E^2\phi_1 \,, \quad A^{\dagger}A\phi_2 = (-\frac{d}{dx^2} + 1 - \frac{2}{\text{ch}^2 x})\phi_2 = E^2\phi_2 ; \tag{13}$$

ϕ_1 solves a free equation and ϕ_2 is obtained from (12)

$$\phi_{\pm}(k,x) = \begin{pmatrix} 1 \\ \frac{-ik + \text{th}\, x}{\pm E_k} \end{pmatrix} \frac{e^{ikx}}{\sqrt{4\pi}} \,, \quad E_k^2 = k^2 + 1 \,. \tag{14}$$

Note that the kink potential is reflectionless; $v(t - cx)$ gives a soliton solution of the modified Korteweg-de Vries equation. The spectrum of the [illegible] operator is left invariant under the MKdV-flow.

Starting from solvable problems allows to show [5]:

<u>THEOREM 1</u>: Define two classes of potentials; the first one with trivial asymptotics such that $|m - v_1(x)| \in L_p$ for some $1 < p \leq 2$, the second one with kink asymptotics and $|m\, \text{th}\, x - v_2(x)| \in L_p$ for $1 < p \leq 2$. Representations connected to potentials from each class are equivalent; all problems from the first class are <u>inequivalent</u> to those from the second class.

For the proof one represents the projection operators through resolvents, and estimates the resulting expression [5].

<u>III. IMPLEMENTABILITY OF GAUGE TRANSFORMATIONS</u>

Given a unitary transformation in the one-particle Hilbert space, like gauge, axial gauge or chiral transformations, we ask whether it is implementable in the given representation; whether one may reach charged vacua by introducing gauge fields is a related question.

Let G be a group, V_α, $\alpha \in G$, be a representation of G in H; an automorphism of the algebra is obtained through

$$\tau_\alpha\, \pi_{P_+}(a(f)) = \pi_{P_+}(a(V_\alpha f)) \,. \tag{15}$$

A second quantized form of V_α called $\Gamma(V_\alpha)$ exists and fulfills

$$\Gamma(V_\alpha)\pi_{P_+}(a(f))\Gamma(V_\alpha)^{\dagger} = \pi_{P_+}(\tau_\alpha(a(f))) \,, \tag{16}$$

iff ω_{P_+} is unitarily equivalent to $\omega_{V_\alpha P_+ V_\alpha^+}$ which holds iff

$$\|P_{\pm}V_\alpha P_{\mp}\|_{HS} < \infty \iff \|X_\alpha^{\pm}\|_{HS} < \infty \quad \text{for} \quad X_\alpha^{\pm} = V_\alpha P_{\pm} V_\alpha^{\dagger} - P_{\pm} \,. \tag{17}$$

Note that both $X_\alpha^{\pm}$ fulfill the cocycle condition [6]

$$V_\beta\, X_\alpha^{\pm}\, V_\beta^{-1} - X_{\alpha\beta}^{\pm} + X_\beta^{\pm} = 0 \,. \tag{18}$$

More specifically for Example 1 we obtain:

<u>THEOREM 2</u>: Let $V_{\underset{\sim}{A}} = \exp\,(iA(x) + i\gamma_5\, A_5(x))$ with $A, A_5 \in C^{\infty}$ and $A', A_5' \in C_0^{\infty}$; $\Gamma(V_{\underset{\sim}{A}})$ exists iff $A_5(\pm\infty) = N_{\pm}\pi$ with $N_{\pm} \in \mathbb{N}$.

The proof follows from simple estimates [7], since the kernels of $P_{\pm}$ are explicitly known.

Note that there is no restriction on $A(\pm\infty)$; the asymptotic values of $V_{\underset{\sim}{A}}$ correspond to <u>symmetries</u> for global transformations. This gives a further example of a system which "determines certain quantum numbers by themselves".

These integers should be connected to the charge difference of the two vacua; for chiral transformations we expect that the Fredholm index of W^1 equals the winding number of the gauge function. This is known to be true for the massless as well as the massive Dirac operator without external potentials [8,9] and holds also for

EXAMPLE 2: Dirac operator on a finite interval, all possible boundary conditions: start with the Hilbert space $H = L^2([0,1]) \otimes ¢^2$; the operator $h = \sigma_3 \frac{1}{i} \frac{d}{dx}$ on $C_0^\infty([0,1])$ functions has defect indices (2,2); all self-adjoint extensions are parametrized by a 2×2 matrix U depending on four angles α, β, γ, δ [10]. The spectrum of H_U consists of points $\epsilon_{n,\pm} = \alpha \pm \beta + 2\pi n$. We distinguish three cases:

a) $\delta = 0$ gives uncoupled spinor modes,

b) for $\delta = \pi/2$ spinors are coupled only at the ends of the interval,

c) $\delta \neq 0, \pi/2$ means mixed boundary conditions.

Both questions answered for the external field, Example 1, have been solved also for the second case:

THEOREM 3:

α) Representations connected to H_U and $H_{\tilde{U}}$ are equivalent iff $U = \tilde{U}$.

β) $V_{\mathcal{A}} = (\frac{1+\gamma_5}{2})e^{iA_+} + (\frac{1-\gamma_5}{2})e^{iA_-}$ is implementable iff

$A_\pm(1) = A_\pm(0) + 2\pi n_\pm$ for case a),

$A_+(1) = A_-(1) + 2\pi n_+$ and $A_-(0) = A_+(0) + 2\pi n_-$ for case b),

both conditions a) and b) hold for case c).

γ) If $V_{\mathcal{A}}$ is implementable, we obtain $n_+ - n_- = i(V_{\mathcal{A}})$, where $i(V_{\mathcal{A}})$ denotes the Fredholm index of the related Bogoliubov transformation; for the proof we follow [8] and use a deformation argument to connect the problem to a shift operator for which the index is easily calculated and related to the winding number.

It follows that chiral transformations allow to charge the vacuum. Like in the Schwinger model [11] there exist sectors labelled by two integers; the gauge invariant algebra will be represented irreducible on the various sectors and yields inequivalent representations.

IV. ALGEBRA OF CHARGES - SCHWINGER TERM

We ask for implementability of a unitary group of transformations $V_{t\mathcal{A}}$. If $\Gamma(V_{t\mathcal{A}})$ exists as a strongly continuous one-parameter group of unitaries, there exists according to Stone's theorem infinitesimal generators which are the smeared charges [12]. $\Gamma(V_{\mathcal{A}})$ exists, iff the generators $j_{\mathcal{A}}$ of $V_{t\mathcal{A}} = 1 + tj_{\mathcal{A}}$ obey the Hilbert-Schmidt criterion:

THEOREM 4 [10]: $\| P_\pm j_{\mathcal{A}} P_\mp \|_{HS} < \infty$, iff $A_5(\pm\infty) = 0$ for Example 1; iff $n_+ = n_- = 0$ for Example 2. The generators are given by smeared charges

$$Q_{\mathcal{A}} = \sum_{j=+,-} \int_0^1 dx\, A_j(x) :\psi^\dagger(x)\left(\frac{1 + j\gamma_5}{2}\right)\psi(x): , \tag{19}$$

which fulfill the algebra

$$i[Q_{\mathcal{A}}, Q_{\mu}] = -2 \operatorname{Im} \operatorname{Tr} P_-\, j_{\mathcal{A}}\, P_+\, j_{\mu} = S(\mathcal{A},\mu) . \tag{20}$$

The Schwinger term is explicitly given for Example 2 by

$$S(\mathcal{A},\mu) = - \int_0^1 \frac{dx}{2\pi} \left(A_+'(x)\mu_+(x) - A_-'(x)\mu_-(x)\right) , \tag{21}$$

and is a nontrivial two cocycle which determines a ray representation of the group $U(1)_{loc} \times U(1)_{loc}$.

For the proof [13] one starts with a quantization map $A \to Q(A) = \sum_{n,m} a_n^\dagger A_{nm} a_m$ for trace class operators A, determines the algebra of Q(A)'s, goes over to normal ordered operators and extends to A's fulfilling $\operatorname{Tr} P_- A P_+ A < \infty$. Only the charge, axial charge commutator is nonvanishing in (21) corresponding to a δ' Schwinger term; the Fourier coefficients of the currents fulfill a Kac-Moody algebra with central extension [10].

V. INDEX CALCULATION

First we discuss the question how to calculate the charge difference of two vacua for two equivalent representations. From unitarity of W we get

$$\mathrm{Tr}\, P_{\ker W_1^\dagger} = \mathrm{Tr}\, W_2 W_2^\dagger P_{\ker W_1^\dagger} = \mathrm{Tr}\, W_2 W_2^\dagger - \mathrm{Tr}\, W_2 W_2^\dagger W_1 P_{\mathrm{Im}\, W_1^\dagger} W_1^{-1} P_{\mathrm{Im}\, W_1} =$$
$$= \mathrm{Tr}\, W_2 W_2^\dagger - \mathrm{Tr}\, W_3^\dagger W_3 P_{\mathrm{Im}\, W_1^\dagger} \, , \tag{22}$$

where P_D denote projection operators onto domains D, and we may formulate

THEOREM 5: If the Bogoliubov transformation (3) is implementable, we get equivalence of

$$i(W^1) = \Delta Q = \mathrm{Tr}\, W_3 W_3^\dagger - \mathrm{Tr}\, W_2 W_2^\dagger = \mathrm{Tr}\, P_- P_+^0 - \mathrm{Tr}\, P_+ P_-^0 =$$
$$= -\frac{1}{2}\mathrm{Tr}\,[(P_+ - P_-) - (P_+^0 - P_-^0)] = -\frac{1}{2}(\eta - \eta^0) \, , \tag{23}$$

where we introduced the APS-η-invariant at the end of the chain of equalities.

A regularized form of η [14] may serve as a definition of the charge of the ground state, even if one compares two inequivalent representations, which we illustrate through

EXAMPLE 1': Define a Dirac operator

$$H_m = \begin{pmatrix} m & A \\ A^\dagger & -m \end{pmatrix} , \qquad A = \frac{d}{dx} + \phi(x) \, , \tag{24}$$

and the η_m-invariant through

$$\eta_m = \lim_{t \downarrow 0} \mathrm{Tr}\, H_m |H_m|^{-1} e^{-t|H_m|^2} \, . \tag{25}$$

η_m may be studied with the help of Krein's spectral shift function $\xi_{12}(.)$ [15,16] and turns out to be equal to the Witten index

$$\Delta(A) = \lim_{z \to 0} (-z)\, \mathrm{Tr} \left(\frac{1}{A^\dagger A - z} - \frac{1}{AA^\dagger - z}\right) \tag{26}$$

in the $m \downarrow 0$ limit for suitable operators A. Relative scattering theory methods leading also to Levinson's theorem allow to determine a number of relations like

$$\eta_m = m \int_0^\infty \frac{d\lambda\, \xi_{12}(\lambda)}{(\lambda + m^2)^{3/2}} \xrightarrow[m \to 0]{} \Delta = -\xi_{12}(0_+) \, . \tag{27}$$

In [15,16] specific one- and two-dimensional models are studied and the topological invariance of the Witten index is proven for suitable perturbations. In one dimension Δ becomes the ratio of Fredholm determinants which yields the Fredholm index of A if A is Fredholm; more generally we obtain

THEOREM 6: Let $\phi_\pm = \lim_{x \to \pm\infty} \phi(x)$, then

	# E = 0 bound states of		# E = 0 resonances of		$\Delta(A)$	$i(A)$
	$A^\dagger A$	$AA^\dagger$	$A^\dagger A$	$AA^\dagger$		
$\phi_+ > 0 > \phi_-$	1	0	0	0	1	1
$\phi_\pm > 0$	0	0	0	0	0	0
$\phi_+ > 0 = \phi_-$	0	0	1	0	1/2	-
$\phi_\pm = 0$	0	0	1	1	0	-

Applying (23) yields "effective charges" (= spectral asymmetry) 0, ±1/2, ±1/4. In Example 2 one gets a continuously varying ΔQ. The same holds for a generalization of $H = \alpha p + \beta \text{ th } x$ (Example 3) where we studied all reflectionless potentials of $h = \alpha p + \beta v + \gamma w$ (α, β, γ are the three σ-matrices), which are also solitons of coupled MKdV equations [17], by solving the GLM equation explicitly [18]. Besides a study of inequivalent representations and implementability of gauge transformations we determined the effective charge for a N-soliton solution which turns out to be a sum of individual contributions from one soliton solutions [18]

$$\Delta Q = \frac{1}{\pi} \sum_{i=1}^{N} \alpha_i , \qquad \epsilon_i = m \cos \alpha_i , \qquad \kappa_i = m \sin \alpha_i , \tag{28}$$

where ϵ_i are the energies in the gap which determine the N-soliton solution.

VI. CONCLUSION

In the study of one-dimensional Dirac operators with external fields one obtains many inequivalent representations of the CAR depending on the asymptotic behaviour and asymptotic values of the potentials. There exist different sectors distinguished by different charge quantum numbers; the charge difference is related to the winding number of chiral transformations. The algebra of charges yields the Schwinger term.

For implementable transformations one obtains a relation between the index of the Bogoliubov transformation and the spectral asymmetry, which is related to the Witten index; non-implementable transformations may yield any value of n.

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