COMPUTATIONAL STUDIES IN
TOKAMAK EQUILIBRIUM AND
TRANSPORT
1. The multigrid method can be modified in such a way that relaxation can meaningfully be done on several levels in parallel. Specifically: Consider the formal equation $Lu = f$. Using standard multigrid concepts and notation as in Ref. [1], let $r^h = f^h + r^h - L^h u^h$ denote the discrete residual on level $h$ at a particular stage of the multigrid iteration ($r^h$ is the estimated discretization error), and define the short-wavelength component of the residual by $r^h := r^h - I_h^H r^H$. (Level $h$ shall not be the coarsest level, and $I_h^H$ and $I_H^h$ denote, respectively, the fine-to-coarse and coarse-to-fine transfer operators between level $h$ and the next coarser level $H$.) Then a relaxation process on level $h$ that is intended to reduce $r^h$ can be done in parallel with the usual relaxation on level $H$ that is intended to reduce $I_h^H r^H$.


2. Using the previous modification of the multigrid method, the double discretization scheme (see Ref. [1], Sec. 10.2) can be equipped with a definition of algebraic convergence: If $r_0^h$ and $r_1^h$ are the residuals computed using, respectively, a stable, low-order discretization $L_0^h$, and a high-order, possibly unstable discretization $L_1^h$, then algebraic convergence is defined by the two equations $r_0^h = 0$ and $I_h^H r_1^H = 0$.

3. By employing a symmetrized compact discretization for the one-dimensional convection-conduction equation it is possible to attain formal fourth-order accuracy and uniform third-order accuracy without a stability limitation on the ratio between the strength of convection and the strength of conduction. When applied to the equation \( Lu = f \), where

\[
Lu = \frac{d}{dx} \left( b(x)u - \epsilon c(x) \frac{du}{dx} \right),
\]

this discretization has the structure

\[
L^h v^h = Q^h f^h, \quad u^h = Q^h v^h,
\]

in which \( L^h \) and \( Q^h \) are three-point operators on a mesh of spacing \( h \). The discretization is fourth-order accurate as \( h \to 0 \) for fixed \( \epsilon \), and is uniformly third-order accurate for all \( \epsilon > 0 \). The discretization of the adjoint of Eq. (1), which is in convection form instead of in conservation form, may be chosen as the adjoint of the discretization of Eq. (1), and then has the same order of accuracy. (This result may be compared with previous work [1], in which stable, formally fourth-order accurate compact discretizations of the standard structure \( L^h u^h = Q^h f^h \) have been derived that achieve uniform second-order accuracy for the convection form and uniform first-order accuracy for the conservation form.)


4. For the purpose of rapid numerical solution of the equations of fluid dynamics, the combination of the multigrid method with a generalized conjugate gradient acceleration is promising [1]. A major further improvement may be expected from applying relaxation to short-wavelength error components only and employing the conjugate gradient acceleration in a local manner.

5. In tokamaks that are equipped with strong additional heating, the phenomena of plasma rotation and pressure anisotropy will often occur together. To determine the plasma configuration on the basis of ideal MHD equilibrium theory can be very misleading under those circumstances. Furthermore, it is inconsistent to consider either one effect or the other, but not both. Previous analytical and numerical studies have not taken sufficient notice of this.

6. Let the functions \( F_m(\lambda, r, z) \) and \( G_m(\lambda, r, z) \) be defined for \( m \) integral, \( \lambda, r, \) and \( z \) real, \( m \geq 0, \) and \( r > 0, \) according to:

\[
F_m(\lambda, r, z) = \frac{\pi}{2} \left[ J_m(\lambda r)J_m(\lambda z) - J_m(\lambda r)J_m(\lambda z) \right] \exp(\lambda z),
\]

\[
G_m(\lambda, r, z) = \frac{\pi}{2} \left[ -J_m(\lambda r)Y_m(\lambda r) + J_m(\lambda r)Y_m(\lambda r) \right] \exp(\lambda z),
\]

where \( J \) and \( Y \) denote Bessel functions, and where for \( \lambda = 0 \) the limiting value as \( \lambda \to 0 \) is understood. Then:

\[
F_m(\lambda, r, z) = \sum_{n=0}^{\infty} \lambda^n D_{m,n}(z, r),
\]

\[
G_m(\lambda, r, z) = \sum_{n=0}^{\infty} \lambda^n N_{m,n}(z, r),
\]

in which \( D_{m,n} \) and \( N_{m,n} \) are the functions introduced by W. Dommaschk [1] and widely used in stellerator modelling work. (The interest of this result is that it leads easily to explicit representations for \( D_{m,n} \) and \( N_{m,n} \). Such representations were not given in Ref. [1].)


7. The magnetic control of the plasma equilibrium in NET/INTOR poses certain engineering problems, but does not present a serious computational problem: it will be possible to compute the plasma equilibrium and the appropriate response matrix within any timescale that is relevant for active control. It follows that the choice between a single-null and a double-null configuration can be made without concern for the simplicity of computational aspects of the feedback control system.
8. For the phenomenological study of tokamak discharges a further development of methods for the interpretation of instantaneous measurements is, at present, more important than a further development of transport simulation codes.

9. Point measurements of the equilibrium poloidal magnetic field around a tokamak are prone to errors due to nearby eddy currents and to misalignment of the coils, and should be abandoned in favour of measurements made using partial or variable-winding Rogowski coils.

\[\text{(this thesis, section 3.5.)}\]

10. Of the many generalizations of the conjugate gradient method that have been proposed for the solution of nonsymmetric systems of equations (see Ref. [1]), in general the truncated ORTHOMIN algorithm is to be preferred.


11. In view of the well-known inadequacies of Fortran it is unfortunate that more modern programming languages, such as Pascal, Modula-2, and C, were not designed with more attention to the requirements of scientific computation. This is particularly painful in the case of Modula-2, because the principal deficiency of that fine language (the lack of flexibility of the array types) could so easily have been avoided by the designer of the language.

12. An author's choice of a "cute" acronym for a particular numerical method (such as SIMPLE, QUICK, and their superlatives) is deplorable, and does not have to be followed.

13. Contrary to popular belief, there does not exist a simple positive relationship between the quality of a computer program and the number of hours of Cray-1 CPU-time that may be required for its execution.

\[\text{Princeton, May 15 1986}\]
COMPUTATIONAL STUDIES IN TOKAMAK EQUILIBRIUM AND TRANSPORT

NUMERIEKE STUDIES VAN TOKAMAK EVENWICHT EN TRANSPORT
(MET EEN SAMENVATTING IN HET NEDERLANDS)

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE WISKUNDE EN NATUURWETENSCHAPPEN AAN DE RIJKSUNIVERSITEIT TE UTRECHT, OP GEZAG VAN DE RECTOR MAGNIFICUS PROF. DR. O.J. DE JONG, VOLGENS BESLUIT VAN HET COLLEGE VAN DEKANEN IN HET OPENBAAR TE VERDEDIGEN OP WOENSDAG 25 JUNI 1986 DESNAMIDDAGSTE0.45 UUR

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Aan mijn ouders
# TABLE OF CONTENTS

Table of Contents vii

## Chapter 1. General Introduction 1

1.1. Basic principles of controlled fusion 1
1.2. Magnetic plasma confinement—the tokamak 7
1.3. Tokamak equilibrium 11
1.4. Boundary plasma 14
1.5. Outline of the thesis 17
Acknowledgement 20
References 20

## Part I: Determination of the Magnetohydrodynamic Equilibrium in Tokamaks 23

## Chapter 2. Magnetohydrodynamic Equilibrium Calculations Using Multigrid 25

*Presented at the "2nd European Conference on Multigrid Methods", Cologne, October 1-4, 1985 (Proceedings to be published by Springer Verlag in the series "Lecture Notes in Mathematics").*

Abstract 25

2.1. Introduction 25
2.2. A multigrid code for axisymmetric equilibrium 27
2.3. Axisymmetric equilibrium in inverse coordinates 30
Chapter 3. The Interpretation of Tokamak Magnetic Diagnostics 39

Report IPP 5/2, Max-Planck-Institut für Plasmaphysik, 1985.

Abstract 39

3.1. Introduction 40

3.2. Fundamental relations for axisymmetric confinement 42

3.3. Moments of the toroidal current density 51

3.4. Moments involving a generalized pressure 54

3.5. Evaluation of the current moments from the measured data 63

3.6. Full equilibrium determination from magnetic measurements 68

3.7. Fast identification of the plasma boundary 82

3.8. Fast determination of characteristic parameters 95

3.9. Conclusions 100

Acknowledgements 101

Appendix 3.A. Transformation of free-field boundary conditions 101

Appendix 3.B. The homogeneous equilibrium equation 103

Appendix 3.C. Determination of the current distribution from the flux surface structure 114

References 117

Chapter 4. Fast Determination of Plasma Parameters through Function Parametrization 123

Based on (a) B.J. Braams, W. Jilge and K. Lackner, Report IPP 5/3, Max-Planck-Institut für Plasmaphysik, 1985 (submitted to Nuclear Fusion), and (b) B.J. Braams and K. Lackner, Report IPP 1/228, 1984.

Abstract 123

4.1. Introduction 124

4.2. Mathematical description 125

4.3. Application to the ASDEX magnetic data analysis 130

4.4. Discussion 139

Table of Contents
Part II: Modelling of the Edge Plasma

Chapter 5. Modelling of the Boundary Plasma of Large Tokamaks

B. J. Braams, P. J. Harbour, M. F. A. Harrison, E. S. Hotston and J. G. Morgan,

Abstract

5.1. Introduction

5.2. Application of the simple, one-dimensional model

5.3. The two-fluid, one-dimensional model and its applications

5.4. Calculations with the two-dimensional model

5.5. Conclusions

Acknowledgements

References


Abstract

6.1. Introduction

6.2. A two-dimensional model of the edge plasma

6.3. Numerical treatment

6.4. Example calculations

Acknowledgements

References
Chapter 7. Low Temperature Plasma near a Tokamak Reactor Limiter 181


Abstract 181
7.1. Introduction 181
7.2. Two-point model 182
7.3. Two-dimensional flows 186
7.4. Conclusions 191
Appendix. Equations for the two-dimensional model 193
Acknowledgements 194
References 194

Chapter 8. A Multi-Fluid Code for the Study of Helium Transport in
the Edge Plasma 197

Abstract 197
8.1. Introduction 197
8.2. Equations for describing the multi-fluid edge plasma 198
8.3. Numerical solution 201
8.4. Example calculations 202
8.5. Conclusions 208
Acknowledgements 209
References 209

Samenvatting 211

Curriculum Vitae 215

Postscript 217

Table of Contents
1. GENERAL INTRODUCTION

Nuclear fusion has the potential of providing a long-term supply of energy. In order to realize this potential it is necessary to find a way of confining a mixture of deuterium and tritium (or other light elements), heat it to a sufficiently high temperature, and keep it confined for a long enough time and at a high enough density for net power to be produced. Research towards this goal is being carried out throughout the industrialized world.

This thesis is concerned with some problems arising in the magnetic confinement approach to controlled thermonuclear fusion. The work addresses the numerical modelling of equilibrium and transport properties of a confined plasma and the interpretation of experimental data. The motivation for carrying out this research is to improve the understanding of present experiments and to help optimize the design of future major experimental facilities.

1.1. Basic Principles of Controlled Fusion

Some fundamental concepts and principles of controlled fusion research are outlined in this Section. The material presented here and the supporting data can be found in much of the introductory fusion literature and elsewhere, e.g. Refs. [1]–[12]. Specific references are therefore kept to a minimum here.
Fusion as a source of energy. Fusion energy is generated when two light atomic nuclei react and the total rest mass of the reaction products is less than that of the initial nuclei. The principal reactions involve the hydrogen isotopes deuterium (D) and tritium (T):

\[
\begin{align*}
D + T & \rightarrow {}^4\text{He} + n + 17.6 \text{ MeV}, \quad (1a) \\
D + D & \rightarrow {}^3\text{T} + H + 4.0 \text{ MeV}, \quad (1b) \\
D + D & \rightarrow {}^3\text{He} + n + 3.3 \text{ MeV}, \quad (1c) \\
D + {}^3\text{He} & \rightarrow {}^4\text{He} + H + 18.3 \text{ MeV}. \quad (1d)
\end{align*}
\]

The energy is released in the form of kinetic energy of the reaction products, e.g. for reaction (1a) the helium nucleon carries 3.5 MeV and the neutron carries 14.1 MeV. The energy released by reaction (1a) amounts to 337 MJ per mg of fuel, and a similar amount of energy is released by the complex consisting of (1b), (1c), and the follow-up reactions involving D + T and D + {}^3\text{He}. For comparison, the combustion of fuel oil provides 46 J/mg.

Deuterium can easily and cheaply be extracted from water, and the amount of deuterium in the oceans is essentially unlimited (the natural abundance is about one atom D per 6670 atoms H). Tritium undergoes radioactive decay with a half-life of 12.3 yr and is therefore not available in nature, while {}^3\text{He} is stable but extremely scarce. On the other hand, the physical requirements for obtaining net energy from the D + T reaction are less stringent than those for pure deuterium fusion or for the D + {}^3\text{He} reaction. Therefore the process (1a) is the favoured candidate for first generation fusion power plants. The necessary tritium can be produced from lithium by the reactions

\[
\begin{align*}
{}^7\text{Li} + n & \rightarrow {}^4\text{He} + T + n - 2.5 \text{ MeV}, \quad (2a) \\
{}^6\text{Li} + n & \rightarrow {}^4\text{He} + T + 4.8 \text{ MeV}, \quad (2b)
\end{align*}
\]

using neutrons from the reaction (1a). The amount of lithium that can be economically retrieved from land deposits is estimated to correspond to an energy reserve of \(\simeq 8 \times 10^{24} \text{ J}\), which may be compared with estimates of \(\simeq 8 \times 10^{22} \text{ J}\) in ultimately recoverable fossil fuels (mainly coal), and \(\simeq 9 \times 10^{24} \text{ J}\) in mineable \(^{238}\text{U}\) for fast breeder reactors. These numbers are from Ref. [5] (Table II-X), which should be consulted for an assessment of the uncertainties in the data.
Nuclear fusion based on the deuterium–tritium reaction therefore has the potential of providing a long-term supply of energy, and if pure deuterium fusion can be achieved the fuel supply will be essentially inexhaustable.

Reaction rates. Because the reactants in (1) are charged nuclei, they must collide with a relative velocity that is sufficiently high to overcome their electrostatic repulsion. Figure 1 shows the cross-sections $\sigma$ of the various reactions listed above as a function of the incoming deuteron energy, assuming the second reactant to be stationary. The maximum cross-section for the $D + T$ reaction (1a) is $\sigma_{DT} \simeq 5 \times 10^{-28} \text{m}^2$, and it occurs for a deuteron energy near 100 keV if the tritium ion is stationary, or for a total energy near 60 keV in the centre-of-mass frame. For the $D + D$ reactions (1b) and (1c) together the maximum cross-section is $\sigma_{DD} \simeq 2 \times 10^{-29} \text{m}^2$ for an incoming deuteron energy of 2 MeV or a centre-of-mass energy of 1 MeV. For comparison, the cross-section for Coulomb interaction between deuterons is about $3 \times 10^{-28} \text{m}^2$ at 1 MeV incoming ion energy and about $3 \times 10^{-24} \text{m}^2$ at 10 keV incoming ion energy, while atomic cross-sections have values around $10^{-20} \text{m}^2$.

Most of the present fusion energy research is dedicated to achieving a sufficiently
high reaction rate by confining and heating a deuterium-tritium plasma. (A plasma is a mixture of electrons and positive ions that is charge-neutral on average.) For the fusion reaction to proceed at an appreciable rate, the density and the thermal energy of the fuel ions must be high enough. The rate $R_{\alpha\beta}$ at which a reaction proceeds in a uniform mixture of species $\alpha$ and $\beta$ can be expressed in terms of the reactivity $\langle \sigma v \rangle$ and of the number densities of the reactants, $n_\alpha$ and $n_\beta$, as

$$R_{\alpha\beta} = c_{\alpha\beta} n_\alpha n_\beta \langle \sigma v \rangle,$$

where $c_{\alpha\beta} = 1$ for distinct species and $c_{\alpha\beta} = 1/2$ for similar species. The reactivity $\langle \sigma v \rangle$ in a thermal plasma is obtained from the product of the cross-section of the reaction and the relative velocity of the reactants by averaging over a Maxwellian velocity distribution. Figure 2 displays $\langle \sigma v \rangle$ as a function of temperature for the D + T and the D + D reactions. Here, and throughout this thesis, temperatures are expressed in energy units, 1 eV corresponding to $1.16 \times 10^4$ K approximately.

The fusion power density, in the case of the D + T reaction, is the product of the reaction rate and the energy released per reaction.

$$P_{\text{fus}(\text{DT})} = R_{\text{DT}} \mathcal{E}_{\text{DT}},$$

where $\mathcal{E}_{\text{DT}} = 17.6$ MeV. In the case of the D + D reaction the situation is somewhat more complicated because the D + T and D + $^3$He follow-up reactions must be taken into account. Most of the tritium would be expected to fuse, but because of the lower cross-section a significant fraction of the $^3$He might not react before diffusing out of the burn region. Nevertheless one has a relation

$$P_{\text{fus}(\text{DD})} = R_{\text{DD}} \mathcal{E}_{\text{DD}},$$

where $12.5$ MeV $\approx \mathcal{E}_{\text{DD}} < 21.6$ MeV; the lower value corresponds to the situation where all of the T but none of the $^3$He reacts, and the upper value is correct when all the T and $^3$He is burnt.

In any reactor the neutrons will immediately leave the region in which fusion takes place, and will deposit their energy in a ‘blanket’ surrounding the reactor core, leading to the tritium breeding reactions (2) or other nuclear reactions. The charged reaction product, on the other hand, will lose most of its energy through Coulomb collisions within the fusion fuel, thereby heating the fuel and helping to maintain the conditions.

1. General Introduction
under which subsequent fusion reactions can take place. The fusion power density redeposited in the plasma is therefore approximately given by

\[
P_{\text{fus(DT)}} = R_{\text{DT}}E_{\text{DT}}' \\
P_{\text{fus(DD)}} = R_{\text{DD}}E_{\text{DD}}'
\]

where \(E_{\text{DT}}' = 3.5 \text{ MeV}\) and \(4.2 \text{ MeV} \leq E_{\text{DD}}' \leq 13.4 \text{ MeV}\), the latter value again depending upon how much of the generated \(^3\text{He}\) is burnt in the follow-up reaction (1d).

Physics requirements for fusion energy production. The most direct way of obtaining fusion reactions is to fire a beam of deuterons at an energy of about 100 keV onto a target of solid tritium, and it is instructive to consider why this procedure is not suitable for fusion energy production. In fact, high-intensity ion beams at the indicated energy can be produced with an efficiency of about 70%, and beam-target interaction is indeed the standard means for obtaining large numbers of fusion neutrons, e.g. for radiation damage studies. The reason why this procedure fails to produce net power lies in the large difference between the cross-section for Coulomb collisions with the cold electrons and the cross-section for fusion reactions. The great majority of the incoming ions will gradually slow down in the medium without undergoing fusion, and the fraction of ions that does react is not large enough to compensate for the energy spent in accelerating the beam and in thermo-electric power conversion. In order to eliminate the degradation of energy by Coulomb collisions, it thus appears necessary to confine the fuel and heat it to a high enough temperature for fusion reactions to occur spontaneously at a sufficient rate. Figure 2 already indicates that the required temperature is many times higher than the ionization energy of hydrogen, which is only 13.6 eV. Under these circumstances the fuel will be a fully ionized plasma.

Some basic conditions for achieving net fusion energy production will now be discussed. A first condition comes from balancing the rate of energy production by fusion reactions against the inevitable loss due to collision-induced radiative processes, and gives a minimum requirement for the temperature of the plasma. A more accurate value for the desired plasma temperature follows from optimizing the fusion power density, e.g. at fixed plasma pressure. A requirement on the plasma confinement is then derived by balancing the fusion energy production against all losses. In this context the concepts scientific breakeven, engineering breakeven, and ignition are employed. Scientific breakeven is the weakest concept, and it is attained when the fusion power produced in the plasma exceeds the energy input into the plasma. Engineering breakeven is attained
when the electrical power produced from the fusion reactions exceeds the energy input into the plant, taking into account all losses due to thermo-electric power conversion and inefficient plasma heating methods. Ignition is attained when the fusion power directly captured in the plasma is sufficient to sustain the plasma temperature without using any external power input.

One inevitable energy loss term comes from the dipole radiation due to collisions between electrons and ions (bremsstrahlung). Assuming that the ions have a charge state \( Z_i \) and assuming \( T_e \approx T_i \), then the radiation power density is given by

\[
\frac{P_{\text{rad}}}{[W m^{-3}]} = 1.69 \times 10^{-38} Z_i^2 (T_e/[eV])^{1/2} n_e n_i / [m^{-6}]
\]

where \( n_e = Z_i n_i \). A necessary condition for scientific breakeven is \( P_{\text{fus}} > P_{\text{rad}} \), and for a self-sustained reaction the stronger condition \( P'_{\text{fus}} > P_{\text{rad}} \) must be met. As \( P_{\text{fus}}, P'_{\text{fus}}, \) and \( P_{\text{rad}} \) all scale as the density squared, it follows that the conditions for breakeven and ignition depend only on the temperature and on the reaction considered. The ignition condition is \( T_i > 4.5 \text{ keV} \) for D-T fusion, while \( T_i > 25 \text{ keV} \) is required for ignited D-D fusion.

The optimum temperature for fusion depends on the reactor design. Recall that the fusion power density scales as \( n^2 \langle \sigma v \rangle \), where \( \langle \sigma v \rangle \) is a function of the ion temperature alone. Therefore, if temperature and density are independent, the optimum temperature is the one at which \( \langle \sigma v \rangle \) attains its maximum, viz. about 65 keV for D-T fusion. On the other hand, if the plasma parameters are restricted by a limit on the pressure, then \( n \sim T^{-1} \), and one will select a temperature that maximizes \( \langle \sigma v \rangle / T^2 \). For D-T fusion this temperature is about 12 keV.

Besides having a sufficiently high temperature, the plasma must have a high enough density and remain confined for a long enough time so that the power produced by the fusion reactions at least exceeds the power required to heat the plasma: \( P_{\text{fus}} > \alpha P_{\text{in}} \), where \( P_{\text{in}} \) is the density of the heating power absorbed in the plasma, and \( \alpha \geq 1 \) is a factor to account for losses in power conversion and inefficiency of heating methods. In the case of steady-state operation the energy confinement time \( \tau_E \) is defined as the ratio between stored energy density and power input density. Therefore, \( \tau_E = 3 n_i T_i / P_{\text{in}} \) for a plasma of hydrogenic ions in which \( T_e = T_i \). The condition \( P_{\text{fus}} > \alpha P_{\text{in}} \) may then be rewritten in the form

\[
n_i \tau_E \geq L.
\]

1. General Introduction
The Lawson parameter $L$ [13] depends on the reaction considered, on the efficiency of power conversion, and on the temperature $T_i$. For the D-T reaction and $T_i \simeq 10\text{ keV}$ its numerical value is $L \simeq 1.5 \times 10^{20} \text{ m}^{-3}\text{ sec}$ for engineering breakeven, while $n_i \tau_E \simeq 3 \times 10^{20} \text{ m}^{-3}\text{ sec}$ is required for ignition. For D-D fusion the required value of $n_i \tau_E$ is one to two orders of magnitude larger, depending on the operating temperature. Therefore the D-T reaction is the most likely candidate for near-term demonstration of the feasibility of fusion power and is the focus of present research.

1.2. Magnetic Plasma Confinement—The Tokamak

The present fusion research programme attempts two different main lines of approach towards meeting the criteria for $T_i$ and $n_i \tau_E$. In the magnetic confinement approach (e.g. [6], [11], [14]) a low-density plasma is to be contained for a relatively long time by the use of a strong magnetic field. The target parameters for fusion in a magnetically confined D-T plasma are typically $T_i \gtrsim 10\text{ keV}$, $n_i \gtrsim 10^{20} \text{ m}^{-3}$, and $\tau_E \gtrsim 1\text{ s}$. In the inertial confinement approach (e.g. [8]), on the other hand, the required density is in the range $10^3$ to $10^4$ times normal solid-state density (which is $4.5 \times 10^{28} \text{ m}^{-3}$ for hydrogen), and the confinement time is between 10 and 100 ps. These conditions are to be attained by rapid compression and heating of a small pellet of frozen D-T mixture, using either lasers or particle-beams. The work elaborated in this thesis concerns magnetic plasma confinement, and in particular confinement in devices of the tokamak type.

Magnetic confinement. A particle of charge $q$ and velocity $v$ in a magnetic field $B$ experiences a force $\mathbf{F} = qv \times B$. In the absence of collisions and inhomogeneities this force causes the particle to gyrate about the direction of the field, so that in a magnetic field the plasma particles are constrained, in lowest order, to follow helical orbits ‘tied’ to a field line. The gyroradius of a particle is given by $r_c = m v_\perp / (qB)$, where $m$ is the particle mass and $v_\perp$ is the velocity perpendicular to the field. As an example, for a field strength of 5 T and a plasma temperature of 10 keV the thermal deuteron gyroradius is about 3 mm, the thermal electron gyroradius is about 0.05 mm, and the typical gyroradius of a 3.5 MeV fusion alpha particle is about 40 mm. Confinement of the plasma with regard also to its motion along the field may be achieved by choosing a configuration in which the magnetic field lines are enclosed in a topologically toroidal region, or alternatively by the use of magnetic mirrors at the two ends of an open configuration.

1.2. Magnetic Plasma Confinement—The Tokamak
Some closed (toroidal) confinement devices that are under investigation are the tokamak, the stellarator, and the reversed field pinch. The tokamak and the reversed field pinch are axisymmetric devices in which the magnetic field is a superposition of a toroidal field produced by external coils and a poloidal field produced by a current flowing in the plasma. In the tokamak the poloidal field is relatively weak compared to the toroidal field, while in the reversed field pinch the two are of comparable magnitude. The stellarator is a non-axisymmetric device in which the confining magnetic field is essentially produced by external currents.

Magnetic confinement in open systems relies on a mirror effect associated with variations in the field strength $B$. The collisionless gyromotion of a charged particle of mass $m$ and velocity $v$ in a magnetic field $B$ has an exact invariant $\epsilon = m(v^2 + v_\parallel^2)/2$ and an adiabatic invariant $\mu = mv^2/2B$, where the subscripts on $v$ denote the orientation with respect to the magnetic field. As $v_\parallel = \sqrt{2(\epsilon - \mu B)/m}$, it is seen that conservation of $\epsilon$ and $\mu$, and $\mu \neq 0$, implies that reflection of the particle occurs if $B$ increases to the value $B = \epsilon/\mu$. Confinement is imperfect because for small enough values of $\mu/\epsilon$ the particle is not reflected, and particles are continuously fed into the associated ‘loss cone’ in velocity space as a result of collisions and microturbulence. Open magnetic confinement systems are not further considered in this thesis.

**The tokamak.** The tokamak was first described by Sakharov and Tamm [15], and developed in a series of experiments in the Soviet Union [16]. From around 1970 it has been the leading contender in the international fusion programme, and it has come closest to meeting the reactor requirements for the temperature and the Lawson parameter $n_eT_E$. A recent review has been given by Furth [17]. The tokamak is an axisymmetric configuration that is characterized by a relatively strong toroidal field and a weaker poloidal field. The toroidal field is created by currents in external coils, while the poloidal field is due to a current flowing in the plasma. In present experiments this plasma current is driven by an induced electric field, the plasma acting as the secondary winding of a transformer. Besides creating the poloidal magnetic field, the plasma current also serves to heat the plasma through ohmic dissipation. The tokamak configuration is illustrated in Fig. 3.

An important property of the magnetic field configuration in axisymmetric devices such as the tokamak, is the existence of a set of nested, toroidal ‘flux’ surfaces, as illustrated in Fig. 4. In the region where the plasma is held confined, each magnetic field line lies on one such flux surface, either covering the surface densely in infinitely
many windings, or else closing in on itself after a finite number of toroidal and poloidal turns. Due to the very large ratio of parallel to perpendicular transport coefficients in a magnetized plasma, the density and temperature quickly equilibrate over flux surfaces, and the radial transport of particles and energy proceeds on a much slower timescale. The radial transport is induced by collisions and by small scale turbulence, but is dominated by the latter. The precise mechanisms are not well understood at present.

For each flux surface the 'safety factor' \( q \) is defined as the number of toroidal turns of a field line for each poloidal turn. The rational surfaces are those on which \( q \) is a rational number, so that the field lines are closed. These rational surfaces play an important part in tokamak stability theory, and stability considerations restrict the possible operating regime of the tokamak. For macroscopic stability it is required that \( q_b \gtrsim 2 \), where \( q_b \) is the \( q \) value on the plasma boundary. The ratio of poloidal to toroidal field on the plasma boundary can be expressed approximately as \( B_p/B_t \approx a/(Rq_b) \), where \( a \) is the plasma minor radius and \( R \) the major radius. Therefore the ratio \( B_p^2/B_t^2 \) in a tokamak is always much less than unity. Plasma equilibrium in a tokamak is only
possible if the average kinetic pressure of the plasma satisfies \( \langle p \rangle \lesssim (R/a)(B_p^2)/2\mu_0 \). Therefore the ratio of plasma pressure to magnetic pressure, \( \beta = 2\mu_0\langle p \rangle/(B^2) \), is also much less than unity.

Accurate values for the stability limit on the tokamak \( \beta \) can only be obtained with the aid of large-scale numerical calculations, e.g. [18]. Such calculations show a limit

\[
\beta \lesssim 3.5 \frac{I/[\text{MA}]}{(a/[\text{m}])(B/[\text{T}])},
\]

where \( I/[\text{MA}] \) is the plasma current in Mega-Ampères, \( a/[\text{m}] \) is the plasma minor radius.
in meters, and $B/[T]$ is the magnetic field strength in Tesla. This limit is consistent with experimental results [19], [20].

**Typical plasma parameters.** For toroidal systems, reactor economics will require the use of superconducting field coils, in which case the average magnetic field strength in the plasma chamber is limited to about 5 T at present technology. The stored energy density in such a field is $\simeq 1 \cdot 10^7 \text{ J m}^{-3}$, or $\simeq 100 \text{ atm}$. The tokamak $\beta$ limit appears to restrict the plasma pressure to about 5% of the magnetic pressure, or about $5 \times 10^5 \text{ J m}^{-3}$. At a temperature of 10 keV this implies an ion density near $n_i \simeq 1.5 \times 10^{20} \text{ m}^{-3}$. The $n_i\tau_E$ ignition criterion then demands an energy confinement time $\tau_E \simeq 2 \text{ sec}$. The parameters given above are close to those envisaged in the conceptual design of the international tokamak reactor experiment (INTOR) [21]. Geometric parameters for INTOR are: major radius 4.9 m, plasma minor radius 1.2 m (horizontal) and 1.9 m (vertical).

Present tokamak experiments have achieved temperatures of 7 – 8 keV [22;], and a Lawson parameter of $n_i\tau_E > 0.7 \times 10^{20} \text{ m}^{-3} \text{ s}$ has also been reported [23], but these parameters were not achieved at the same time or even in the same experiment. The best overall plasma parameters to-date have been achieved on the Doublet III experiment [19], where temperatures of 5 – 6 keV at an average density of $\simeq 6.5 \times 10^{19} \text{ m}^{-3}$ and an energy confinement time of $\simeq 70 \text{ msec}$ were attained. In that case the characteristic product $n_i\tau_E T_i$ is still roughly two orders of magnitude below the requirements of a reactor. Several new large experiments have recently commenced operation: the European experiment JET in England, TFTR and DIII-D in the U.S.A., and JT-60 in Japan. These devices are expected to make substantial progress towards reactor-relevant plasma parameters.

1.3. Tokamak Equilibrium

Chapters 2-4 of this thesis are concerned with the determination of the magnetohydrodynamic (MHD) equilibrium in tokamaks.
The magnetohydrodynamic equations. For many purposes, including tokamak equilibrium theory and a substantial body of stability theory, a magnetically confined plasma can be considered as an ideal MHD fluid. The governing system of equations is then [24]–[28]:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0, \\
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p - \mathbf{J} \times \mathbf{B} &= 0, \\
\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \frac{5}{3} \rho \nabla \cdot \mathbf{v} &= 0, \\
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} &= 0, \\
\mu_0 \mathbf{J} &= \nabla \times \mathbf{B}, \\
\mathbf{E} + \mathbf{v} \times \mathbf{B} &= 0, \\
\nabla \cdot \mathbf{B} &= 0.
\end{align*}
\]

Here, \( \rho \) is the mass density, \( \mathbf{v} \) the flow velocity, \( p \) the pressure, \( \mathbf{J} \) the current, \( \mathbf{B} \) the magnetic field, \( \mathbf{E} \) the electric field, and \( \mu_0 \) the vacuum magnetic permeability. SI units are employed. The first three equations describe mass, momentum, and energy conservation for an ideal fluid having an adiabatic index \( \gamma = 5/3 \). The next four equations are Maxwell’s equations for an ideally conducting medium with the displacement current neglected. This system of equations provides a single-fluid description of macroscopic plasma behaviour.

The MHD model neglects many effects that are important in tokamak physics in one context or another. There is no heat conduction, particle diffusion, resistivity, or viscosity in the model. There are no electromagnetic waves, no displacement current or space charge, and also all particle kinetic effects are ignored. Therefore, other mathematical models are employed in order to represent either diffusive processes on the longer timescales, or a variety of waves and instabilities on short timescales. The ideal MHD model occupies a middle ground, in that the long-term plasma evolution proceeds by slow transport through a sequence of ideal MHD stable equilibria, whereas microscopic theory is needed in order to supply the transport coefficients.

1. General Introduction
Axisymmetric equilibrium. By setting \( \mathbf{v} = 0 \) and \( \partial/\partial t = 0 \) in the system of equations (3) one sees that the static equilibrium of plasma and magnetic field is governed by the equations

\[
\nabla \cdot \mathbf{B} = 0, \quad (4a)
\]
\[
\mu_0^{-1} (\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla p. \quad (4b)
\]

In the case of axisymmetry the system (4) can be further simplified by employing the representation

\[
\mathbf{B} = F \nabla \phi + \nabla \psi \times \nabla \phi,
\]

where \( \phi \) is the ignorable angle in the cylindrical \( (r, \phi, z) \) coordinate system, and where \( F \) and \( \psi \) are axisymmetric scalar functions. It can then be shown that \( \psi \) satisfies the 'almost linear' elliptic equation

\[
r \frac{\partial}{\partial r} \left( r^{-1} \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2} = -\mu_0 r^2 \frac{dp}{d\psi} - F \frac{dF}{d\psi}, \quad (5)
\]

in which \( p \) and \( F \) are both functions of \( \psi \) alone. This equation is the basis for the study of tokamak equilibrium.

Contributions in this thesis. An important problem for tokamak experiments is to determine the plasma equilibrium or certain characteristic parameters of the equilibrium from external measurements, and in particular from measurements of the magnetic field and flux made at the plasma edge. This is one of the problems addressed here. The main contributions are a demonstration of a fast numerical method for solving the equilibrium equation (which is important not only for the interpretation of diagnostics), a comprehensive review and further development of methods for the analysis of magnetic measurements, and a very effective procedure for determining characteristic parameters of the equilibrium on the basis of experimental data. This latter work also has applications to the problem of real-time control of a tokamak experiment.
1.4. Boundary Plasma

Chapters 5–8 of this thesis are concerned with numerical modelling of the boundary plasma in tokamaks.

**Plasma-wall interaction.** As plasma confinement is not perfect there is a flux of particles and energy incident upon the walls of any confinement system. Plasma ions that hit the wall are neutralized and then either absorbed in the wall or returned into the plasma as warm gas; an incident ion may furthermore cause the release of a previously trapped neutral. This ‘recycling’ process has the effect of extracting energy from the plasma edge. In a quasi-steady state the particle recycling coefficient is of order unity, so that particle exhaust and energy exhaust are uncoupled.

The ions hitting the wall also cause the release of wall material, which enters into the plasma as impurities. Oxygen and carbon are usually present on the vessel surface in the form of H$_2$O, CO$_2$ and other compounds, and these are easily desorbed. Metal ions may be sputtered from the wall when the incident ion energy exceeds 30–100 eV, depending upon the wall material. Over a wide range of plasma parameters these impurities are not fully stripped of their electrons and thus radiate strongly, causing a serious reduction of energy confinement. Maximum impurity concentrations allowed in the interior of a tokamak plasma are several percent for light impurities (C, O), about $10^{-3}$ for medium-Z impurities (Al, Ti, Fe), and as low as $10^{-4}$ for heavy metals such as Ta and W. Minimizing the impurity release is therefore a major concern for tokamak operation, and this the more so as the plasma heating power and the pulse length is increased.

References [29–31] provide good introductions to the physics of the plasma edge and of plasma-wall interaction.

**Edge plasma control.** Various methods have been applied on tokamaks in order to minimize the damaging effects of plasma-wall interaction. One method of edge plasma control involves a material ‘limiter’ protruding into the edge plasma, as illustrated in Fig. 5. The heat and particle load is thereby concentrated on a small surface area at the edge of the hot plasma column, which is rapidly desorbed of light impurities. Limiter operation has been successful in experiments with modest heating power, where
sputtering of wall and limiter material was not much of a problem, but it is not clear whether this scheme is adequate for fusion reactors.

The most successful technique to-date for edge plasma control in large tokamaks involves a **magnetic divertor**. This consists of an arrangement of coils such that the field lines near the vessel wall are diverted onto a special plate or wall area away from the core plasma. By pumping gas out of the divertor region particle exhaust can be provided. Figure 6 shows the flux surface topology of two typical divertor tokamaks, one having a closed divertor configuration, in which the divertor region is physically shielded from the main plasma, and the other having an open configuration.

A detailed understanding of edge plasma transport and of plasma-wall interaction is required in order to optimize methods of edge plasma control. The main issues

1.4. Boundary Plasma
are discussed in the introductory papers referred to earlier, and in greater detail in Refs. [32]-[36].

**Contributions in this thesis.** Edge plasma phenomena in a tokamak are essentially two-dimensional. The relevant transport processes include the fast flow and conduction along the magnetic field lines onto material boundaries, as well as the slow radial diffusion of particles and energy. By contrast, for describing the transport in the interior plasma it suffices to consider only radial transport processes, as the fast transport along the field rapidly eliminates any variations within flux surfaces. The two-dimensional modelling of the edge plasma is a principal topic of this thesis. A numerical code incorporating a complete set of two-dimensional fluid equations was de-
developed for the first time, and was applied to the study of the edge plasma in divertor and limiter tokamaks. This work is particularly relevant to reactor design, and indeed has been used as input to the International Tokamak Reactor (INTOR) design studies (summarized in Ref. [21]).

1.5. Outline of the Thesis

The thesis is divided into two parts. Part I (Chapters 2-4) is concerned with the determination of tokamak equilibrium, and Part II (Chapters 5-8) with two-dimensional modelling of the edge plasma. The linkage between the evolution of the plasma equilibrium and two-dimensional transport processes is a difficult subject, which is not explicitly considered here. Each chapter is a self-contained paper, and therefore some overlap between the introductory sections could not be avoided.

Determination of the MHD equilibrium in tokamaks. Part I of the thesis is devoted to some aspects of the MHD equilibrium problem, both in the 'direct' formulation (given an equation for the plasma current, the corresponding equilibrium is to be determined), and in the 'inverse' formulation (the interpretation of measurements made at the plasma edge).

The first paper, *Magnetohydrodynamic Equilibrium Calculations using Multigrid*, concerns the application of the multigrid numerical method to the direct MHD equilibrium problem. This numerical method was developed during the 1970's [37], and has come to be recognized as the most efficient procedure for solving general elliptic partial differential equations. Applications to other p.d.e.’s, in particular the equations of fluid mechanics, and to more general grid equations are evolving, e.g. [38], [39]. We have applied the method to solve the axisymmetric MHD equilibrium equation on an Eulerian grid, and achieve full multigrid efficiency. The possibility of applying the multigrid approach to the computation of axisymmetric equilibria in the 'inverse coordinates' formulation and to three-dimensional equilibrium and evolution calculations is investigated. The three-dimensional equilibrium problem is the most challenging application. It appears that a fully satisfactory multigrid code will require use of an adaptive grid, approximately tied to the unknown magnetic field configuration.

The second paper of Part I, *The Interpretation of Tokamak Magnetic Diagnostics*, concerns the inverse tokamak equilibrium problem: the determination of the MHD equilibrium configuration or of some characteristic equilibrium parameters from external
magnetic measurements. The paper provides a comprehensive review and comparison of the existing analytical and computational methods, and also contains a number of original contributions. The interpretation of these measurements relies to a large extent on two classes of integral relations that were first discussed by Zakharov and Shafrenov [40]. We have extended their work by including the contributions due to pressure anisotropy and plasma rotation, and by providing some explicit analytical forms. It is described how methods that are familiar from multivariate statistical analysis and from the treatment of ill-posed linear equations can be employed to evaluate these integrals from imperfect measurements, and an approach to the robust treatment of failing or erroneous signals is initiated. A novel fast algorithm for the determination of the complete equilibrium is proposed. We discuss different formulations of the boundary conditions for the MHD equilibrium problem with given external magnetic field. A published method of determining the plasma current distribution from knowledge of only the shape of the flux surfaces is corrected.

The third paper of Part I, Fast Determination of Plasma Parameters through Function Parametrization (based on joint work with K. Lackner and W. Jilge), demonstrates the usefulness of Wind's method of function parametrization [41], [42] for very rapid interpretation of magnetic measurements. This method relies on analysis of a large data base of simulated experiments in order to obtain an accurate functional representation for intrinsic physical parameters of a system in terms of the values of the measurements. Statistical techniques for dimension reduction and regression analysis are involved. The method is now used routinely on the ASDEX experiment to determine the time evolution of the plasma geometry and of its basic electromagnetic properties. Function parametrization is accurate and fast enough to be suitable for real-time evaluation of plasma parameters, and it potentially has important applications to the dynamic control of tokamak operation.

Modelling of the edge plasma. Part II of the thesis is devoted to numerical studies of the edge plasma. The appropriate Navier-Stokes system of fluid equations is solved in a two-dimensional geometry. The main interest of this work is to develop an understanding of particle and energy transport in the scrape-off layer and onto material boundaries, and also to contribute to the conceptual design of the NET/INTOR tokamak reactor experiment.

The first paper of Part II, Modelling of the Boundary Plasma of Large Tokamaks (joint work with P.J. Harbour, M.F.A. Harrison, E.S. Hotston and J.G. Morgan), surveys the modelling work done in the Exhaust Physics Group at Culham Laboratory
up to 1983. Within the thesis, this paper provides a context for the author's two-dimensional modelling work. The simplest model that has been employed describes a single fluid \( T_i = T_e \) in which radial variations are ignored except in prescribing the characteristic e-folding thickness of the scrape-off layer. The transport of energy parallel to the magnetic field is by electron conduction with a convective sheath boundary condition. A one-dimensional, two-fluid model based on the Braginskii equations has been used to model a divertor of INTOR size; it has also been used to study transport of helium and of impurities. Furthermore, the two-dimensional, two-fluid model of the edge plasma region is presented. This model permits a self-consistent treatment of the coupling between parallel and radial transport. It has been used to demonstrate the need for a high edge density if the temperature at the plate in an INTOR divertor is to be kept acceptably low.

The second paper, *Modelling of a Transport Problem in Plasma Physics*, describes the numerical methods that are employed for solving the equations of the two-dimensional, two-fluid model. The discretization scheme follows the ideas of D.B. Spalding and co-workers [43]. The code solves a finite-volume discretization of the conservation equations on a topologically rectangular mesh. The discrete coefficients depend continuously on the local cell Péclet number, and give central differencing and pure convective upwind differencing in the appropriate limits. The time stepping is fully implicit. A distributive relaxation method, leading to an elliptic equation, is employed to obtain the pressure correction at each iteration. The equations are solved by the strongly implicit procedure, which is based on the incomplete L*T decomposition. These concepts are clarified and their effectiveness is demonstrated.

The third paper of Part II, *Low Temperature Plasma near a Tokamak Reactor Limiter* (joint work with C.E. Singer), describes analytical and numerical studies of the plasma near a toroidally symmetric limiter for the anticipated geometry and energy flux in the conceptual Tokamak Fusion Core Experiment (TFCX). The two-dimensional, two-fluid model is employed to study the dependence of the plasma parameters near the limiter plate on the assumptions about the outer main plasma density and the thermal diffusivity. A temperature below 20 eV is found except when the density 10 cm inside the limiter contact is \( 8 \times 10^{-3} \) cm\(^{-3} \) or less and the thermal diffusivity in the edge region is \( 2 \times 10^4 \) cm\(^2\)/s or less.

The final paper, *A Multi-Fluid Code for the Study of Helium Transport in the Edge Plasma*, is concerned with two-dimensional multispecies modelling of the edge plasma, with specific interest in problems of helium ash removal from a fusion reactor and of

1.5. Outline of the Thesis
impurity penetration through the scrape-off layer. The previous single-ion code has been extended in order to treat in principle an arbitrary number of ion fluids that are coupled through ionization and recombination processes, interspecies friction, and electric and thermal forces. Each ion fluid has finite inertia and finite pressure, and all fluids are treated in a symmetric fashion. Classical transport theory has been used as a guide to obtain a simplified set of equations for the parallel forces. These equations are consistent with the standard classical theory in the limit when one fluid is dominant and all others are trace impurities, but they remain mathematically sound also at finite relative concentrations. In this sense the parallel transport model extends the one that has been used in previous one-dimensional simulations of impurity flow in the tokamak scrape-off layer. The numerical methods that are employed are well able to handle the strong frictional coupling between the different fluids. One example calculation concerns differential transport of hydrogen and deuterium in an ASDEX-like scrape-off layer, and another models the transport of helium in the edge plasma of the conceptual TFCX experiment.

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References


1. General Introduction

References


PART I: DETERMINATION OF THE MAGNETOHYDRODYNAMIC EQUILIBRIUM IN TOKAMAKS
2. MAGNETOHYDRODYNAMIC EQUILIBRIUM CALCULATIONS USING MULTIGRID

Abstract

The multigrid method has been applied to the solution of the two-dimensional elliptic equation that governs axisymmetric ideal magnetohydrodynamic equilibrium. The possibility of applying multigrid to the computation of axisymmetric equilibria in the 'inverse coordinates' formulation and to three-dimensional equilibrium and evolution calculations is investigated.

2.1. Introduction

In the magnetic confinement approach to controlled thermonuclear fusion, the hot plasma is prevented from escaping by the use of a strong magnetic field. The $v \times B$ force (velocity times magnetic field) inhibits motion of electrons and ions perpendicular to the magnetic field, and the plasma particles are constrained, in lowest order, to follow helical orbits 'tied' to a magnetic field line. In order to confine the plasma also with regard to its motion along the field, configurations are employed in which the magnetic field lines are enclosed in a toroidal region. The two main contenders in magnetic confinement research are the tokamak, which is an axisymmetric device, and the stellarator, which does not have a continuous symmetry. In either device, the magnetic field is generated by a combination of external currents and currents flowing in the plasma. The equilibrium of plasma and magnetic field is therefore governed by a nonlinear system of equations.
We are concerned here with the application of multigrid methods to the computation of magnetic confinement configurations. In this context the plasma may be considered as an ideal magnetohydrodynamic (MHD) fluid, and the static equilibrium of plasma and field is described by the system of equations,

\begin{align}
\nabla \cdot \mathbf{B} &= 0 \\
\mu_0^{-1}(\nabla \times \mathbf{B}) \times \mathbf{B} &= \nabla p
\end{align}

where \( \mathbf{B} \) is the magnetic field and \( p \) is the kinetic pressure. SI units are employed. Despite its simple appearance the system (1) is extremely difficult to solve numerically for general three-dimensional configurations, and a substantial fraction of the production time of the Cray-1 computer at Garching is spent on this problem. Typical calculations require ~ \( 10^4 \) iterations and take between 2 and 4 hours of CPU-time in order to compute a single equilibrium to acceptable accuracy. To find efficient methods for the computation of three-dimensional solutions to Eq. (1) is thus a major challenge for computational plasma physics. In the axisymmetric case the system (1) may be reduced to a single elliptic partial differential equation in the plane, which can be solved efficiently by a variety of methods.

Specifically, we consider in this paper the application of multigrid to three classes of problems in computational MHD: (1) Computation of axisymmetric equilibria on an Eulerian grid. (2) Axisymmetric equilibria in the inverse coordinates formulation. (3) Three-dimensional equilibrium and evolution calculations. Although fast numerical methods for these three problems are of considerable interest, no previous investigation into the use of multigrid for computations in magnetic confinement theory seems to exist.

Problem (1) requires the solution of an almost linear, uniformly elliptic equation (a nonlinearity occurs only in the right hand side), and application of multigrid is straightforward. Our code achieves full multigrid efficiency, and is several times faster than codes based on direct rapid elliptic solvers.

In regard to problem (2) we discuss the relationship between the inverse coordinates approach on one hand and grid generation through elliptic systems on the other. This leads to a formulation of the inverse coordinates equilibrium problem as a quasilinear elliptic system, which is suitable for multigrid treatment. (A code has not been written).

The discussion of problem (3) is also of an analytical nature. Local mode analysis is employed in order to develop a relaxation procedure, rather than for a posteriori
validation only. The best local relaxation scheme will still be slowly converging for
two classes of disturbances, viz. the slow magnetosonic and the shear Alfvén modes,
in both cases with the wavevector nearly transverse to the magnetic field. These are
the lowest frequency modes in the MHD spectrum. A satisfactory treatment of these
troublesome modes requires distributive line relaxation along the magnetic field and
plane relaxation or semi-coarsening on magnetic surfaces; this emphasizes the need for
an adaptive, field-tied grid for 3-D MHD calculations. The proposed simple relaxation
scheme can be suitable for implicit ideal MHD evolution calculations on the slowest
timescale.

The reader is assumed to be familiar with multigrid methods, as presented in par-
ticular in Refs. [1] and [2]. Useful reviews of the relevant MHD theory may be found
in Refs. [3]–[6].

2.2. A Multigrid Code for Axisymmetric Equilibrium

The Grad-Schlüter-Shafranov equation. A very substantial simplification in
the system (1) is afforded by the assumption of axisymmetry. Under this assumption
the equation \( \nabla \cdot \mathbf{B} = 0 \) may be solved by choosing the representation,

\[
\mathbf{B} = F \nabla \phi + \nabla \psi \times \nabla \phi,
\]

where \( \phi \) is the ignorable angle in the cylindrical \((r, \phi, z)\) coordinate system, and \( F \) and
\( \psi \) are axisymmetric scalar functions. A similar representation is found for the current,

\[
\mu_0 \mathbf{J} = \nabla \times \mathbf{B} = -\Delta^* \psi \nabla \phi + \nabla F \times \nabla \phi.
\]

where the operator \( \Delta^* \) is given by

\[
\Delta^* \psi = r \frac{\partial}{\partial r} \left( r^{-1} \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2}.
\] (2)

From the force balance equation (1b) one may next derive \( \nabla \psi \times \nabla p = 0, \nabla F \times \nabla p = 0, \)
and \( \nabla \psi \times \nabla F = 0 \). It is taken to follow that \( \psi, F, \) and \( p \) are functionally related,
and one writes \( F = F(\psi) \) and \( p = p(\psi) \). These must be understood as local relations
in case a surface of constant \( \psi \) has disconnected parts. Finally, consideration of force balance along \( \nabla \psi \) leads to the Grad-Schlüter-Shafranov equation [7]–[9],

\[
\Delta \psi = -\mu_0 r^2 \frac{dp}{d\psi} - F \frac{dF}{d\psi} \quad (3)
\]

This equation is the basis for the study of axisymmetric ideal MHD equilibrium. For given profiles \( p(\psi) \) and \( F(\psi) \) it is an almost linear elliptic p.d.e., the nonlinearity occurring only in the right hand side. The various methods (not including multigrid) that have been used in the past to solve Eq. (3) have been reviewed in Refs. [10] and [11].

**Discretization scheme.** The conventional second order accurate discretization methods for the equilibrium equation (3), written as \( \Delta \psi = f(r, \psi) \), are of the five point molecule form.

\[
\begin{pmatrix}
* & * & * \\
* & * & \\
* & * & *
\end{pmatrix} \psi = f.
\]

Better methods are available for smooth \( f \), in particular a fourth order accurate 'compact' discretization of the shape.

\[
\begin{pmatrix}
* & * & * \\
* & * & * \\
* & * & *
\end{pmatrix} \psi = \begin{pmatrix}
* & * & * \\
* & * & \\
* & * & *
\end{pmatrix} f.
\]

Specifically: Consider a uniform rectangular mesh with spacing \( h r = h \) and \( \delta z = k \). Consider the natural splitting, \( \Delta \psi = \mathcal{L}_r + \mathcal{L}_z \), and define the second order accurate difference approximations \( \mathcal{L}_r^h \) and \( \mathcal{L}_z^k \) by,

\[
\mathcal{L}_r^h \psi (r, z) = \frac{1}{h^2} \left[ \left( \psi(r + h/2, z) - \psi(r - h/2, z) \right) - 2 \psi(r, z) \right]
\]

\[
\mathcal{L}_z^k \psi (r, z) = \frac{1}{k^2} \left[ \left( \psi(r, z + k) - \psi(r, z - k) \right) - 2 \psi(r, z) \right]
\]

Then a fourth order discretization of \( \Delta \psi = f \) is obtained from the identity.

\[
\left[ \mathcal{L}_r^h + \mathcal{L}_z^k + \frac{1}{12} (h^2 + k^2) \mathcal{L}_r^h \mathcal{L}_z^k + O(h^4 + k^4) \right] \psi
\]

\[
\quad = \left( 1 - \frac{1}{12} h^2 \mathcal{L}_r^h + \frac{1}{12} k^2 \mathcal{L}_z^k \right) f.
\]

A special treatment on the plasma-vacuum interface, on which \( f \) or its first order derivatives may be discontinuous, is required in order to gain full advantage of the higher accuracy obtained for the interior equations. (Such a treatment has not been implemented in our code).
Performance of multigrid. A demonstration code has been written, which employs multigrid relaxation to solve Eq. (3) on a rectangular region subject to Dirichlet boundary conditions. The mesh at level $i$ has size $(2^i + 1) \times (2^i + 1)$, where $1 \leq i$; the coarsest grid therefore contains only one single interior point. The cycling algorithm is of the full multigrid, full approximation storage variety, and employs adaptive switching. Full-weighting transfer is used for both the solution and the residuals, and bi-cubic interpolation is used for the corrections. Red-black point relaxation is employed on all grids except on the coarsest, where a nonlinear root-finding algorithm is employed. The special treatment on the coarsest grid is necessary because the equation generally admits multiple solutions; the algorithm that is employed on the coarsest grid is designed to find the interesting solution.

For an example calculation we assumed a right hand side in Eq. (3) of the form,

$$f(r, \psi) = \begin{cases} -r^2 g(\psi) - c, & \psi > 0 \\ 0, & \psi \leq 0 \end{cases}$$

in which $g$ is a second-degree polynomial and $c$ is a constant. The contour defined by $\psi = 0$ is the free boundary of the plasma. A calculation using 7 levels (the finest grid has size $129 \times 129$) required 120 msec on the Cray-1 to solve Eq. (3) to the level of the discretization error. The total number of passes over each of the levels 1-7 was 10, 19, 14, 8, 6, 4, and 2 respectively. The computing time was divided about evenly between evaluation of the r.h.s., evaluation of the residuals, bi-cubic interpolation, and all other chores.

Thus, full multigrid efficiency for the solution of the equation (3) has been achieved. For a linear problem this code is about 2-3 times slower than the well-optimized fast Buneman solver used at Garching [10], but for the more relevant nonlinear problems the codes based on a direct elliptic solver require Picard iteration, and multigrid relaxation is easily the fastest procedure available.

2.2. A Multigrid Code for Axisymmetric Equilibrium
2.3. Axisymmetric Equilibrium in Inverse Coordinates

For stability and transport calculations related to axisymmetric configurations it is required to have an explicit representation of the magnetic surfaces of the equilibrium (the contours of constant $\psi$), which are in this case assumed to form a nested set that converges to a single 'magnetic axis'. Such a representation may be found by a numerical mapping after having computed the equilibrium on a fixed spatial grid. In recent years, however, a class of procedures has become popular in which the equilibrium is computed in a formulation that immediately gives the spatial coordinates $r$ and $z$ as functions of $\psi$ and $\eta$, where $\eta$ is some angular variable. This 'inverse variables' method has been employed in Refs. [12]–[15]. There is considerable freedom in the choice of the angular variable $\eta$. It is defined via orthogonality in Ref. [12], via a specification of the Jacobian in Refs. [13] and [14], and in Ref. [15] it is suggested to choose $\eta$ such that contours of constant $\eta$ are straight rays.

The definition of the angular variable via orthogonality or via a Jacobian constraint leads to a system of differential–algebraic equations that is not easy to solve numerically. On the other hand, to choose the contours of constant $\eta$ to be straight rays is rather restrictive. A more general formulation for defining the angular variable is suggested by an analogy with the method of grid generation through elliptic equations [15], [16], [17]. This leads to a formulation of the inverse equilibrium problem in which the coordinate $\eta$ is defined as the solution of an elliptic equation, resulting in a quasilinear elliptic system of equations for $r(\psi, z)$ and $z(\psi, z)$. This formulation is discussed below, as it is the most suitable for multigrid treatment.

Grid generation through elliptic equations. We first consider by way of example the problem of constructing a boundary-fitted curvilinear coordinate system $(\xi^1, \xi^2)$ to cover the pseudo-rectangular region $G \in \mathbb{R}^2$. The Cartesian coordinates on $\mathbb{R}^2$ are $(x_1, x_2)$. $G$ is to be mapped to the unit square in the $(\xi^1, \xi^2)$ plane, and the points $A$, $B$, $C$, and $D$ on the boundary $\partial G$ are to be mapped to $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$. Using the method of grid generation through elliptic equations, the curvilinear coordinates $\xi^i$ are defined by Poisson equations,

$$\Delta \xi^i = F^i,$$

subject to Dirichlet conditions on $\partial G$: $\xi^1 = 0$ on $AB$, $\xi^1 = 1$ on $CD$, etc. In the simplest case one chooses $F^i = 0$, but a nonzero right hand side in Eq. (5) may be used to obtain more control over the resulting mesh.
The expression for the Laplacian on the curvilinear coordinates $\xi^i$ is,

$$\Delta u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^i} \left( \sqrt{g} g^{ij} \frac{\partial u}{\partial \xi^j} \right), \quad (6)$$

where summation over repeated indices is understood. Substituting $\xi^k$ for $u$ in the above equation and considering Eq. (5) one finds,

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^i} \left( \sqrt{g} g^{ij} \right) = F^j$$

and therefore,

$$\Delta u = g^{ij} \frac{\partial^2 u}{\partial \xi^i \partial \xi^j} + F^j \frac{\partial u}{\partial \xi^j} \quad (7)$$

To obtain numerically the transformation $x \rightarrow \xi$ it is convenient to solve in inverse coordinates, and obtain $x$ as a function of $\xi$ rather than $\xi$ as a function of $x$. (The whole point of the grid generation is that differential equations may be more easily solved on the transformed region). The components $x_i$ are obtained by solving

$$\Delta x_i = 0, \quad i = 1, 2 \quad (8)$$

on the unit square in the $(\xi^1, \xi^2)$ plane, again subject to Dirichlet boundary conditions. With $\Delta$ given by Eq. (7) this is a quasilinear elliptic system.

**Application to axisymmetric equilibrium.** The analogy between the above model problem of grid generation and the inverse coordinates approach to axisymmetric MHD equilibrium is quite obvious: one only has to replace $\Delta$ by $\Delta^*$ and change from a rectangular to a polar geometry, with the singularity at the magnetic axis. In a curvilinear coordinate system $(\xi, \eta)$ the operator $\Delta^*$ has the representation,

$$\Delta^* u = \left| \nabla \xi \right|^2 \frac{\partial^2 u}{\partial \xi^2} + 2(\nabla \xi \cdot \nabla \eta) \frac{\partial^2 u}{\partial \xi \partial \eta} + \left| \nabla \eta \right|^2 \frac{\partial^2 u}{\partial \eta^2} + \Delta^* \xi \frac{\partial u}{\partial \xi} + \Delta^* \eta \frac{\partial u}{\partial \eta} \quad (9)$$

which corresponds to Eq. (7). For the transformed coordinate $\xi$ one may choose $\xi = \psi$, or any function of $\psi$ alone, so that $\Delta^* \xi$ follows from the equilibrium equation (3), and

2.3. Axisymmetric Equilibrium in Inverse Coordinates
may be assumed known. The natural choice of the differential equation for \( \eta \) is \( \Delta^* \eta = 0 \). From Eq. (2) it is seen that

\[
\begin{align*}
\Delta^* r &= -r^{-1} \\
\Delta^* z &= 0
\end{align*}
\]  

(10)

Considering \( r \) and \( z \) as functions of \( \xi \) and \( \eta \), Eq. (10) naturally remains valid, but the operator \( \Delta^* \) is given by Eq. (9) instead of by Eq. (2). This quasi-linear elliptic system of equations (10) governs the equilibrium in inverse coordinates. The boundary conditions require periodicity in \( \eta \), specified \( r \) and \( z \) on the plasma boundary, and an appropriate expansion near the magnetic axis.

The system (10) appears suitable for multigrid treatment, although it is more complicated than the equilibrium equation in the form (3). The standard second order discretization has a symmetric nine-point stencil, and an incomplete Cholesky decomposition should probably be used for relaxation. Alternating direction line relaxation is an alternative, but point relaxation must not be employed on a polar grid. Higher order discretizations would be of interest, in particular a spectral method in the angular coordinate.

2.4. Prospects for Three-Dimensional Calculations

In this Section an analytical study of the use of multigrid for the difficult and (at present) very expensive area of 3-D MHD computations is initiated. The system of equations (1) is equivalent to the system that governs steady, inviscid, incompressible flow, as can be seen by making the substitutions,

\[
\frac{\mathbf{B}}{\sqrt{\mu_0}} \rightarrow \mathbf{v}, \quad p \rightarrow \frac{p}{\rho} - \frac{v^2}{2}.
\]

Progress in solving the corresponding hydrodynamic equations is therefore of immediate interest for magnetic confinement studies.
The Chodura and Schlüter approach. In attempting to develop a multigrid relaxation procedure for 3-D equilibrium I have found it convenient to take as the starting point the approach of Chodura and Schlüter [18], who employ a finite difference discretization on a fixed spatial mesh of the system (1) in primitive variables.

The solution procedure employed by Chodura and Schlüter is designed to find a constrained minimum of the potential energy, $W$, which is given by

$$W = \int_T \left( \frac{B^2}{2\mu_0} + \frac{p}{\gamma - 1} \right) dV,$$

where $T$ is the toroidal computational region, and $\gamma$ is the adiabatic index. Minimization of $W$ is performed through displacements of the form,

$$\begin{cases}
\delta B = \nabla \times (\xi \times B) \\
\delta \rho = -\nabla \cdot (\rho \xi)
\end{cases}$$

subject to $\xi = 0$ on the boundary $\partial T$. The relation $p = \rho^\gamma$ is assumed, and therefore $\rho$ corresponds to the mass density of the MHD fluid. Through these displacements, an arbitrary initial plasma and field configuration is transformed under the constraints of mass and flux conservation into a minimum energy state.

In leading order the change in energy due to the displacements (12) is given by

$$\delta W = -\int_T (F \cdot \xi) dV,$$

where $F = \frac{\mu_0}{2}(\nabla \times B) \times B - \nabla p$. Therefore, a state of minimum energy under the displacements (12) satisfies $F = 0$, and is a solution to the force balance equation (1b). The equation (1a) remains satisfied if it is satisfied initially. Eq. (13) also shows a possible route to the energy minimum, viz. to choose at each iteration $\xi = \alpha F$, where $\alpha > 0$ and $\alpha$ is sufficiently small to ensure stability, but this steepest descent algorithm is prohibitively slow. Chodura and Schlüter have employed both conjugate gradient acceleration and a second-order Richardson scheme, with good results, but some $10^3$ to $10^4$ iterations are still required for practical calculations.

2.4. Prospects for Three-Dimensional Calculations
A multigrid relaxation procedure. We now attempt to derive a relaxation procedure for the system (1) that effectively reduces the short wavelength error components, and that may therefore be suitable in the context of multigrid. A finite difference discretization of Eq. (1) on a staggered mesh is envisaged, but it turns out that the analysis of relaxation procedures can largely be carried out without reference to the discretized system of equations.

To satisfy flux conservation, \( \nabla \cdot \mathbf{B} = 0 \), is of course easy. At each relaxation sweep \( \mathbf{B} \) may be updated by a distributive scheme of the form \( \mathbf{B} \leftarrow \mathbf{B} + \delta \mathbf{B} \), where \( \delta \mathbf{B} = \nabla \chi \). To satisfy exactly \( \nabla \cdot \mathbf{B} = 0 \) after the iteration sweep one would have to find \( \chi \) as the solution to a Poisson equation, but here it suffices to approximate \( \chi \) locally by any kind of relaxation prescription that is suitable for Poisson equations. Notice that the replacement \( \mathbf{B} \leftarrow \mathbf{B} + \nabla \chi \) does not affect the force balance equation.

For the second equation, \( \mathbf{F} = 0 \), the work of Chodura and Schlüter suggests a relaxation scheme based on the coupled replacements \( \mathbf{B} \leftarrow \mathbf{B} + \delta \mathbf{B} \) and \( \rho \leftarrow \rho + \delta \rho \), where \( \delta \rho = \gamma \rho \delta \rho \), and where \( \delta \mathbf{B} \) and \( \delta \rho \) are given by Eq. (12). These replacements do not affect \( \nabla \cdot \mathbf{B} = 0 \). The question is how to choose the displacement vector \( \xi \) in Eq. (12) as a function of the current residual \( \mathbf{F} \).

To answer this question one must consider the principal terms of the change \( \delta \mathbf{F} \) under the displacements (12), viz. those terms in which \( \xi \) is twice differentiated:

\[
\delta \mathbf{F} \simeq \mu_0^{-1} (\mathbf{B} \cdot \nabla)(\mathbf{B} \cdot \nabla)\xi + (\mu_0^{-1} B^2 + \gamma \rho) \nabla (\nabla \cdot \xi)
\]

\[
- \mu_0^{-1} \mathbf{B}(\mathbf{B} \cdot \nabla)(\nabla \cdot \xi) - \mu_0^{-1} (\mathbf{B} \cdot \nabla) \nabla (\mathbf{B} \cdot \xi)
\]

(14)

Here, the operator \( \nabla \) acts on \( \xi \) only. A desirable relaxation scheme should lead to \( \delta \mathbf{F} \simeq -\mathbf{F} \), at least for the short wavelength components. Fourier analysis transforms \( \delta \mathbf{F} \) into \( \delta \tilde{\mathbf{F}} \) and \( \xi \) into \( \tilde{\xi} \), related by \( \delta \tilde{\mathbf{F}} \simeq \mathbf{A} \cdot \tilde{\xi} \), where

\[
\mathbf{A} = -\frac{B^2}{\mu_0} (k_\parallel^2 I + (1 + \beta) kk - k_\parallel (bk + kb))
\]

(15)

in which \( k \) is the wavevector, \( \beta = \mu_0 \gamma \rho / B^2 \), \( b = \mathbf{B} / \mathbf{B} \), and \( k_\parallel = k \cdot b \). \( \beta \) is a small parameter for magnetic confinement.

The operator \( \mathbf{A} \) may be inverted:

\[
\mathbf{A}^{-1} = -\frac{\mu_0}{B^2 k_\parallel^2} (I + \beta^{-1} bb - kk/k^2),
\]

(16)

34 2. MHD Equilibrium Calculations using Multigrid
and a desirable relaxation scheme should approximate $\ddot{\xi} = -A^{-1}\ddot{F}$ for short wavelength components. Of course $A^{-1}$ contains $k$, and is therefore not a local linear operator. Various things can be tried, for instance to drop the term $kk/k^2$ and to replace $k^2$ by $2\omega^{-1}(h_x^{-2} + h_y^{-2} + h_z^{-2})$, where $\omega$ is a constant of order unity, and $h_x$, $h_y$, and $h_z$ are the local mesh spacings. Via this prescription one obtains a relaxation procedure based on

$$\xi = R \cdot F,$$  \quad (17)

where $R$ is the operator,

$$R = \frac{\omega}{2} \frac{\mu_0}{B^2} (h_x^{-2} + h_y^{-2} + h_z^{-2})^{-1} (I + \beta^{-1}bb).$$  \quad (18)

The large coefficient on $bb$ in this relaxation prescription is worthy of note.

**Further analysis of the proposed procedure.** The relaxation scheme (17) must now be analyzed in order to see whether all short wavelength error modes are effectively reduced. Continuing with the linear analysis, and still considering principal terms only, one finds,

$$\delta F = A \cdot R \cdot F = -\frac{\omega}{2} (h_x^{-2} + h_y^{-2} + h_z^{-2})^{-1} (k_x^2 I + (1 + \beta)kk - k_y bb) \cdot \dot{F}.$$

It may be seen that the scheme is not satisfactory, as those modes for which

$$k \perp B \quad \text{and} \quad k \perp \dot{F}$$

(approximately) are not well eliminated. (In addition there may be problems related to the occurrence of different values of the grid spacing, but those difficulties are easily taken care of by line relaxation). The troublesome modes are the slow magnetosonic mode, for which $F$ and $B$ are nearly parallel, and the shear Alfvén mode, for which $B$, $F$, and $k$ form an orthogonal triad. These are the lowest frequency modes ($\nu \to 0$) in the MHD spectrum.

The reason that these modes are not well eliminated can also be understood on physical grounds. The perturbation related to the slow magnetosonic mode concerns the pressure only, and is characterized by a long wavelength along the magnetic field and a short wavelength across the field. As the restoring force for this perturbation acts along field lines, the relaxation procedure only becomes effective when the mesh
spacing corresponds to the length scale along the field, but on such a mesh (assuming equal coarsening in all directions) the perturbation will be invisible due to the rapid variation across the field. Similarly, the restoring force for the shear Alfvén mode is located in a plane in which the mode has a long wavelength, but perpendicular to this plane there is a rapid variation.

Both the form of the operator $A^{-1}$ in Eq. (16) and the physical picture outlined above point the way to a remedy. One needs line relaxation along the magnetic field (which allows to retain $k_\parallel$ in going from $A^{-1}$ to $R$) to eliminate effectively the slow magnetosonic mode, and either plane relaxation or semi-coarsening within flux surfaces to deal with the shear Alfvén mode. As the magnetic field configuration is unknown a priori this requires an adaptive grid, approximately tied to the field. Development of adaptive grid methods for 3-D MHD calculations is also important for reasons of numerical accuracy, but no satisfactory algorithm exists at present. Nevertheless, multigrid in conjunction with adaptive grid methods seems the most promising area of investigation towards efficient 3-D MHD equilibrium computations.

For time dependent three-dimensional calculations the scheme derived above may be more promising, as it would allow to follow accurately the evolution on the longest ideal MHD timescale, while eliminating efficiently the faster modes.

2.5. Conclusions

One objective in writing this paper has been to point out to both plasma physicists and multigrid experts that certain problems in computational MHD are of shared interest.

The axisymmetric equilibrium problem lends itself to a straightforward application of the multigrid procedure, and this has resulted in a code that is about 3 times faster than a code which uses a well optimized Buneman solver and Picard iteration. The main interest in very fast 2-D equilibrium calculations is for real-time data interpretation and control of an experiment, on a timescale of about 10 msec or less. Considering that in monitoring an experiment one is solving a chain of similar problems, and that a grid of modest size will suffice, our study has demonstrated at least the near-term feasibility of this application.

The problem of computing axisymmetric equilibrium in the inverse coordinates formulation is a more challenging (although hardly speculative) application of multigrid,
for which furthermore the relative gain over competing methods would be much larger, as rapid direct solvers are not available. Not all previous formulations of the inverse equilibrium problem are well suited for multigrid treatment, but the analogy with grid generation through elliptic equations shows the correct approach. In particular, any code for elliptic grid generation that can handle a polar geometry should almost immediately be applicable to the inverse coordinates MHD equilibrium problem.

The really difficult and expensive areas of work in computational MHD are the stability eigenvalue problem for axisymmetric equilibria (which has not been addressed in this paper), and the three-dimensional equilibrium and evolution problems. An impression of the complexity of the 2-D stability problem can be gained by noticing that it has required nearly a decade of work and the advent of the Cray-1 computer before the main result from the existing stability codes was obtained, viz. the Troyon scaling law [19]. For three-dimensional equilibrium and evolution problems a multigrid approach has been initiated here, but a fully satisfactory procedure has not yet been obtained. The main outstanding problem for these 3-D computations is to develop adaptive methods, in which the grid is adjusted to the (unknown) magnetic configuration.

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Acknowledgements
References

Abstract

The analytical theory and the computational methods that are available for the determination of MHD equilibrium characteristics from magnetic measurements on axisymmetric systems are reviewed and developed. The interpretation of these measurements relies to a large extent on two classes of integral relations that are due to L.E. Zakharov and V.D. Shafranov. Following and extending their work we provide an inventory of useful integral relations, including the contributions due to pressure anisotropy and plasma rotation. Effective methods to evaluate the required integrals from imperfect measurements are described. A full analysis of the magnetic diagnostics implies the determination of the MHD equilibrium, aiming at an optimal fit between the calculated magnetic field and the measured data. Published approaches to this problem are evaluated and a novel fast algorithm is proposed. Instead of the full equilibrium analysis several more limited problems are often considered, for which faster methods are available. The determination of only the plasma boundary requires the solution of a Cauchy problem for an elliptic equation. The published approaches are critically compared. Analytical theory provides approximations that are suitable for rapid estimation of characteristic parameters related to the plasma current, position, shape of the cross-section, pressure, and internal inductance. The most efficient algorithms are obtained by using the method of function parametrization. These algorithms are well suited to real-time control of the plasma. Appendices to the paper contain a discussion of the boundary conditions for the MHD equilibrium problem, a compendium of analytical solutions to the homogeneous equilibrium equation, and a re-examination of the possibility of determining the current distribution from knowledge of only the shape of the flux surfaces.
3.1. Introduction

The accurate and rapid determination of the magnetohydrodynamic (MHD) equilibrium configuration is of much importance for magnetic confinement experiments, be it for the purpose of feedback control of the plasma during machine operation, for the immediate on-line data analysis that takes place between successive discharges, or for the more extensive off-line analysis of an experiment. Measurements of the external magnetic field and flux provide the most basic information on the electromagnetic properties of the confined plasma, and are fundamental to the feedback control and to the further analysis of a discharge. A range of algorithms for the interpretation of these magnetic diagnostics is required, with different priorities as regards the speed vs. the accuracy or the scope of the computations. On the fastest timescale relevant to active control of a discharge (typically one to several milliseconds for tokamak operation, depending upon the skin-time of the vacuum vessel) one requires at least an estimate of the plasma position. Somewhat slower at present are the algorithms that determine accurately the location of the plasma boundary and compute estimates for such characteristic equilibrium parameters as the poloidal $\beta$ and the internal inductance. The complete determination of the equilibrium configuration and its time evolution is not yet a matter of routine. For all these tasks, significant advances in the sophistication and/or the speed of the analysis would be most welcome.

The subject matter of this paper is the analytical theory and the computational methods that are available for the determination of MHD equilibrium characteristics from magnetic measurements on axisymmetric systems, in particular on tokamaks. The paper provides a critical review of the literature, and also presents some new contributions to the analysis of magnetic measurements. In particular, recent work has shown that the accurate estimation of a wide set of characteristic equilibrium parameters can easily be done in the 1 millisecond timescale that is relevant to active control, and methods are outlined here that will allow even a full 2-D MHD equilibrium analysis of the plasma to be performed in only a few tens of milliseconds on present computing equipment. The paper should be of interest not only to those plasma physicists who are directly involved with magnetic diagnostics on a tokamak, but also to those involved with the interpretation of other basic plasma diagnostics, with machine control, or with MHD computations.

The paper is divided into two main parts. Sections 2-4 are of an analytical nature, and provide the fundamental equations that are required for the interpretation
of magnetic measurements in the context of MHD equilibrium theory. Sections 5–8 are oriented towards computation, and discuss the various numerical approaches to the analysis of these diagnostics. The remainder of this introduction provides a more detailed outline of the paper. Specific references to the literature can be found in the appropriate sections, and are therefore omitted here.

In Section 2 the equations and boundary conditions that govern axisymmetric confinement are summarized, both for the case of static, ideal MHD, and in the presence of pressure anisotropy and plasma rotation. Some useful properties of the equations are listed. The discussion of the boundary conditions is to some extent original.

Section 3 is concerned with one of the two classes of integral relations that were first discussed by L.E. Zakharov and V.D. Shafranov. This class of integral relations relies on Maxwell’s equations only, and relates measurements of the magnetic field and flux outside the plasma to moments of the current distribution in the interior. These moments involve solutions of the homogeneous equilibrium equation. Several families of analytical solutions are exhibited, and the issue of their completeness is discussed.

Section 4 is concerned with the other class of integral relations of Zakharov and Shafranov. This class relies on an equation for MHD equilibrium in addition to Maxwell’s equations, and relates the measured field and flux to moments of the energy density in the plasma. Following and extending the work of Cooper and Wootton, these relations are generalized to include the contributions due to pressure anisotropy and plasma rotation.

Section 5 contains an unconventional treatment of a rather elementary problem: the accurate approximation of the integrals of Sections 3 and 4 from imperfect measurements. The relevance of concepts from the numerical treatment of ill-conditioned equations and from statistical analysis is stressed.

Section 6 deals with methods for the difficult inverse problem of obtaining a complete solution to the equation for axisymmetric equilibrium, including a determination of free parameters that describe the current profile. We review the literature, and propose a novel algorithm to interleave the two iterative processes: the optimization of current profile parameters and the solution of the nonlinear equilibrium equation. In combination with a multigrid approach, this algorithm may yield a code that computes the complete equilibrium in real-time.

In Section 7 fast specialized methods for a more limited problem are discussed, viz. the determination of the plasma boundary contour and of the field on this contour from

3.1. Introduction
the external magnetic measurements. This involves a solution of one of the classical ill-posed problems of mathematical physics: the integration of an elliptic equation from Cauchy boundary data (or a similar problem). This problem, however, is well understood, and can easily be made well-posed in the sense of Tikhonov. The published approaches are compared.

Section 8 then deals with fast methods that seek to determine only a set of characteristic parameters of the equilibrium, related to the plasma current, position, shape, pressure, and internal inductance. Standard methods for this problem require a preliminary identification of the plasma boundary, and then employ analytical approximations that have been derived on the basis of a large aspect ratio expansion and a specific model for the plasma current distribution. Our recent work has shown that H. Wind's method of function parametrization can provide simple and accurate expressions that are suitable for real-time control of the experiment.

There are three Appendices. Appendix A contains a discussion of the free-field boundary conditions and their reduction to an integral equation over a finite boundary, and also deals with the accurate discretization of all possible boundary conditions. Appendix B provides several families of analytical solutions to the homogeneous equilibrium equation, appropriate to different coordinate systems. Appendix C re-examines the problem, first posed by Christiansen and Taylor, of the determination of the current profile from knowledge of only the shape of the flux surfaces.

3.2. Fundamental Relations for Axisymmetric Confinement

This Section serves to define some of the notation that will be employed throughout the paper, and to collect for later reference several important facts about axisymmetric magnetic fields and magnetohydrodynamic equilibrium. For more information one is referred to the original literature, notably Refs. [1], [8], and to standard texts and review papers [9]-[17].

Preliminaries. Throughout the paper, use is made of a righthanded cylindrical \((r, \phi, z)\) coordinate system, and where not noted otherwise, all occurring fields are assumed to be symmetric with respect to rotations about \(r = 0\). An axisymmetric toroidal region \(T\) serves as the domain for the discussions. \(T\) should completely enclose
the plasma, and may also contain a vacuum region and material regions. The cross-
section of \( T \) in the poloidal (half-)plane \((\phi = 0 \text{ and } r > 0)\) is denoted as \( \Omega \), and \( \partial T \)
and \( \partial \Omega \) are the boundaries of \( T \) and \( \Omega \). \( dV \) is the volume element on \( T \), \( dS \) the area
element on \( \Omega \), \( dA \) the area element on \( \partial T \), and \( ds \) the line element on \( \partial \Omega \). Therefore
\( dV = 2\pi r \, dS \) and \( dA = 2\pi r \, ds \). The positive orientation on \( \partial \Omega \) is such that \( \Omega \) lies
to the right. The normal and tangential derivatives on \( \partial \Omega \) are denoted as \( \partial / \partial n \) and
\( \partial / \partial s \). \( \Omega \) is assumed to be bounded, and \( \partial \Omega \) must be piecewise smooth. As a matter of
convenience it will also be assumed that \( \Omega \) is bounded away from \( r = 0 \), but in many
cases this restriction can be removed with little effort. \( \Omega \) need not be simply connected.

The general concern in this work is with how to derive information on the magnetic
field and the plasma in \( T \) (or \( \Omega \)) from knowledge of the magnetic field on \( \partial T \) (\( \partial \Omega \)).
In applications, \( \partial T \) is a surface on or near which the magnetic probes are located.
Often this is the inner or outer surface of the vacuum vessel, but it may happen that
the geometry is more complicated, for instance when passive conductors with magnetic
probes mounted on them are present inside the vacuum vessel.

In MHD confinement theory the magnetic permeability \( \mu \) is usually assumed to
be equal to the vacuum permeability \( \mu_0 \) throughout the domain of interest. This
is appropriate for the plasma and vacuum regions (all plasma currents being written
explicitly), but in the context of the interpretation of magnetic measurements one may
be forced to consider the presence of other material media as well, such as the vacuum
vessel and perhaps passive conductors located in \( \Omega \), and we therefore generally allow a
spatially varying permeability. Only linear magnetic material is considered in \( \Omega \); the
presence of nonlinear media causes substantial computational difficulties. Furthermore,
all nonconducting material has \( \mu = \mu_0 \) to sufficient accuracy for it to be treated here
as if it were part of the vacuum region. Accordingly we assume a decomposition of \( \Omega \)
in the form \( \Omega = \Omega_{\text{pl}} + \Omega_{\text{vac}} + \Omega_{\text{coil}} \); into a plasma region, a vacuum region, and a coil
region. In \( \Omega_{\text{pl}} + \Omega_{\text{vac}} \) it is assumed that \( \mu = \mu_0 \). The exterior region (the complement
of \( \Omega \) in the right half-plane) is denoted as \( T_{\text{ext}} \) (\( \Omega_{\text{ext}} \)). About this region we assume
only axisymmetry; it may carry any axisymmetric distribution of currents, and may
also contain nonlinear magnetic material (i.e. iron).

The magnetic field. In the case of axial symmetry the divergence-free magnetic
field \( \mathbf{B} \) may be represented as

\[
\mathbf{B} = F \nabla \phi + \nabla v \times \nabla \phi. \tag{1}
\]
in terms of two scalar functions, $F$ and $\psi$. $F$ is related to the toroidal field by $F = rB_t$, and $\psi$ is related to the toroidal component of the vector potential by $\psi = rA_t$. From Ampère's law, $\nabla \times \mathbf{H} = \mathbf{J}$, where $\mathbf{H} = \mu^{-1}\mathbf{B}$, a similar representation for the current is obtained,

$$\mathbf{J} = -\mu^{-1}\mathcal{L}^*\psi \nabla \phi + \nabla(\mu^{-1}F) \times \nabla \phi,$$

(2)

where the operator $\mathcal{L}^*$, in a coordinate-invariant representation, is the following,

$$\mathcal{L}^*\psi = r^2 \mu \nabla \cdot (r^{-2}\mu^{-1} \nabla \psi).$$

(3)

We consider $\mathcal{L}^*$ to be defined only for axisymmetric scalar fields. From Eq. (2) it follows that $\psi$ satisfies the elliptic equation,

$$\mathcal{L}^*\psi = -\mu \mathbf{j}_t,$$

(4)

where $\mathbf{j}_t$ is the toroidal current density.

For the case of uniform permeability, $\mu = \mu_0$, $\mathcal{L}^*$ reduces to the operator $\Delta^*$,

$$\Delta^*\psi = r^2 \nabla \cdot (r^{-2} \nabla \psi)$$

$$= r \frac{\partial}{\partial r} \left( r^{-1} \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2}.$$  

(5)

We also introduce the operator $\mathcal{L}$,

$$\mathcal{L}\psi = \mu^{-1} \nabla \cdot (\mu \nabla \psi),$$

(6)

which reduces to the Laplacian $\Delta$ for the case of uniform permeability. In a current-free region one may employ the representation $\mathbf{H} = \nabla g$, and $\nabla \cdot \mathbf{B} = 0$ is then equivalent to $\mathcal{L}g = 0$.

**Some useful identities.** Green's first identity for $\mathcal{L}^*$ is the following:

$$\int_{\Omega} r^{-1} \mu^{-1} \psi \mathcal{L}^* \chi \, dS = \oint_{\partial \Omega} r^{-1} \mu^{-1} \psi \frac{\partial \chi}{\partial n} \, ds - \int_{\Omega} r^{-1} \mu^{-1} \nabla \psi \cdot \nabla \chi \, dS,$$

(7)

and Green's second identity (Green's theorem) is:

$$\int_{\Omega} r^{-1} \mu^{-1} (\psi \mathcal{L}^* \chi - \chi \mathcal{L}^* \psi) \, dS = \oint_{\partial \Omega} r^{-1} \mu^{-1} \left( \psi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \psi}{\partial n} \right) \, ds.$$

(8)

44 3. The Interpretation of Tokamak Magnetic Diagnostics
Here, $\partial/\partial n$ is the outward normal derivative on $\partial \Omega$. These identities are easily obtained by application of the divergence theorem to the appropriate expressions on the torus $T$. Similar relations, with a factor $r\mu$ instead of $r^{-1}\mu^{-1}$, hold for $\mathcal{L}$.

We now turn to Green's third identity for the operator $\mathcal{L}^*$. Let the function $G(r, r')$ satisfy the equation $\mathcal{L}^*G = \mu \delta(r - r')$ in $\Omega$, where $G$ is considered as a function of $r$ at fixed $r'$. Boundary conditions on $G$ are not specified, so this function is determined up to an arbitrary solution to the homogeneous equation. Then $G$ is known as a Green function for the operator $\mathcal{L}^*$, and Green's third identity (Green's representation theorem) holds:

$$\psi'(r') = -\int_{\Omega\searrow} Gj_t \, dS + \int_{\partial\Omega} r^{-1}\mu^{-1} \left( \psi \frac{\partial G}{\partial n} - G \frac{\partial \psi}{\partial n} \right) ds. \quad (9)$$

Particularly useful specific Green functions are obtained by imposing homogeneous boundary conditions (either Dirichlet, Neumann, or mixed conditions, as appropriate for the problem at hand), which are furthermore independent of $r'$. Such Green functions satisfy the fundamental symmetry property, $G(r, r') = G(r', r)$.

If the external region $\Omega_{\text{ext}}$ also contains only linear magnetic material then Eq. (9) may be applied on the expanded region $\Omega + \Omega_{\text{ext}}$. If the specific Green function $G_0$ is then chosen to satisfy the free-field boundary conditions, $G_0(r, r') \to 0$ as $|r| \to \infty$ and as $r \to 0$, then the following simple representation is obtained:

$$\psi'(r') = -\int_{\Omega + \Omega_{\text{ext}}} G_0j_t \, dS. \quad (10)$$

$G_0(r, r')$ is therefore the influence function, equal to the flux at position $r'$ due to a negative unit current at position $r$. One is thereby led to define,

$$\psi_{\text{int}}(r') = -\int_{\Omega} G_0j_t \, dS,$$

$$\psi_{\text{ext}}(r') = \int_{\partial\Omega} r^{-1}\mu^{-1} \left( \psi \frac{\partial G_0}{\partial n} - G_0 \frac{\partial \psi}{\partial n} \right) ds,$$

so that $\psi = \psi_{\text{int}} + \psi_{\text{ext}}$ according to Eq. (9). The function $\psi_{\text{int}}$ may be understood as that part of the flux function that is due to the currents in $\Omega$, while $\psi_{\text{ext}}$ is associated with currents in the exterior region. $\psi_{\text{int}}$ is homogeneous in the exterior region $\Omega_{\text{ext}}$, and $\psi_{\text{ext}}$ is homogeneous in the interior region $\Omega$.

### 3.2. Fundamental Relations for Axisymmetric Confinement
An analytical expression for $G_0$ is available if $\mu = \mu_0$ everywhere:

$$G_0(r, r') = \frac{\mu_0}{k\pi} \sqrt{rr'} \left( E(k^2) - (1 - k^2/2)K(k^2) \right)$$

$$= \frac{\mu_0}{\pi} \sqrt{rr'} \left( \frac{1}{2} R_F(0, 1 - k^2, 1) - \frac{1}{3} R_D(0, 1 - k^2, 1) \right),$$

where

$$K(k^2) \quad \text{and} \quad E(k^2) \quad \text{are the complete elliptic integrals of the first and second kind, as defined in Refs. [18] and [19], whereas} \quad R_F \quad \text{and} \quad R_C \quad \text{are Carlson's forms of the elliptic integrals [20], which are more suitable for computation [21, ch. S21]. The function} \quad G_0 \quad \text{defined in (12) is called the free space Green function. Even if magnetic material is present one may employ this free space Green function in Eqs. (10) and (11), provided that all magnetization currents are treated as true currents.}$$

The following relations between $\Delta^*$ and $\Delta$ are sometimes useful [11]. For any sufficiently differentiable axisymmetric field $\xi$,

$$\Delta^* \left( r \frac{\partial \xi}{\partial r} \right) = r \frac{\partial}{\partial r} \Delta \xi,$$

and

$$r^{-1} \frac{\partial}{\partial r} \Delta^* \xi = \Delta \left( r^{-1} \frac{\partial \xi}{\partial r} \right).$$

Both $\Delta^*$ and $\Delta$ commute with $\partial/\partial z$.

**Ideal MHD equilibrium.** In ideal MHD equilibrium theory the magnetic field equations, $\nabla \cdot B = 0$ and $\nabla \times \mu^{-1} B = J$, are supplemented by a single equation of force balance, $\nabla p = J \times B$, where $p$ is the kinetic pressure of the plasma. It is furthermore assumed that $\mu = \mu_0$. Employing the representation (1) there are three unknown scalar fields: $\psi$, $F$, and $p$. Invoking axisymmetry one derives the relations, $\nabla \psi \times \nabla p = 0$ (from $B \cdot \nabla p = 0$ and Eq. (1)), $\nabla F \times \nabla p = 0$ (from $J \cdot \nabla p = 0$ and Eq. (2)), and, for good measure, $\nabla \psi \times \nabla F = 0$ (from toroidal force balance). It is taken to follow that, locally, $\psi$, $F$, and $p$ are functionally related, and one writes $F = F(\psi)$ and $p = p(\psi)$. (It bears saying that these relations need not hold globally in the case when a surface of constant $\psi$ has disconnected parts, as occurs in configurations having a divertor or
an internal separatrix). Consideration of force balance along $\nabla \psi$ then leads to the following expression for the toroidal current in the plasma,

$$j_t = r \frac{dp}{d\psi} + r^{-1} \mu_0^{-1} F \frac{dF}{d\psi},$$

and to what is commonly referred to as the Grad-Shafranov equation [1]-[3] for the flux function $\psi$,

$$\Delta^* \psi = -\mu_0 r^2 \frac{dp}{d\psi} - F \frac{dF}{d\psi}.\quad(16)$$

This 'almost linear' elliptic equation is the basis for the study of axisymmetric ideal MHD equilibrium. Notice that in the context of interpretation of experimental data the functions $p(\psi)$ and $F(\psi)$ must be regarded as unknown. The equation (16) restricts the right hand side of Eq. (4) to a form that involves just two arbitrary functions of a single variable.

The ideal MHD force balance equation can be written in conservation form [9],

$$\nabla \cdot T = 0,$$

where the stress tensor $T$ is given by

$$T = (p + B^2/2 \mu_0) I - \mu_0^{-1} BB.\quad(17)$$

A convenient related form in terms of the flux functions is the following,

$$\nabla p + \frac{1}{2 \mu_0} r^{-2} \nabla (F^2 - |\nabla \psi|^2) + \nabla \cdot \left( \frac{1}{\mu_0} r^{-2} \nabla \psi \nabla \psi \right) = 0,\quad(18)$$

as follows easily from Eq. (17).

**Pressure anisotropy and plasma rotation.** MHD equilibrium in the presence of anisotropy and flow is described in part by the following system of equations:

$$\nabla \cdot B = 0, \quad \nabla \cdot \rho v = 0,$$

$$\rho v \cdot \nabla v = (\nabla \times \mu_0^{-1} B) \times B - \nabla \cdot P,\quad(19)$$

$$v \times B = \nabla \Phi,$$

where $\rho$ is the mass density of the plasma, $v$ is the flow velocity, $P$ is the pressure tensor, and $\Phi$ is the electric potential. The general anisotropic pressure tensor has the form $P = p_\perp I + (p_\parallel - p_\perp) BB / B^2$, where $p_\perp$ and $p_\parallel$ are the perpendicular and parallel pressures. Two additional thermodynamic equations are required for $p_\perp$ and $p_\parallel$, but
the proper choice of these equations is open to dispute. One possibility is provided by the Chew-Goldberger-Low (CGL) equations [4], valid in the collisionless limit,

\[ \mathbf{v} \cdot \nabla \left( \frac{p_1 B^2}{\rho^3} \right) = 0, \quad \mathbf{v} \cdot \nabla \left( \frac{p_\parallel}{\rho B} \right) = 0. \]  

(20a)

Alternatively it is possible to assume isotropy, \( p_\perp = p_\parallel \) (= \( p \)), and then either ideal adiabatic flow,

\[ \mathbf{v} \cdot \nabla \left( p/\rho^\gamma \right) = 0, \]  

(20b)

where \( \gamma \) is the adiabatic index (\( \gamma = 5/3 \)), or constant temperature on flux surfaces,

\[ \mathbf{B} \cdot \nabla (p/\rho) = 0. \]  

(20c)

For the case of axisymmetry, the system of of equations (19) and (20b) governing ideal MHD flow equilibrium may be reduced to a system consisting of a single elliptic equation and a family of free flux functions [22]-[25], analogous to Eq. (16) for static equilibrium. A similar reduction in the case of axisymmetry can be given for the system (19) and (20c), essentially by setting \( \gamma = 1 \) and repeating the derivation appropriate to Eq. (20b). For the case of the CGL system (19) and (20c) such a reduction is not, as far as we know, available.

One important relation between \( \mathbf{v} \) and \( \mathbf{B} \), which is independent of the choice of equation (20), is obtained as follows. First, \( \nabla \cdot \mathbf{B} = 0 \) and \( \nabla \cdot \rho \mathbf{v} = 0 \) are solved by introducing the representations,

\[ \mathbf{B} = F \nabla \phi + \nabla \psi \times \nabla \phi, \]  

\[ \rho \mathbf{v} = G \nabla \phi + \nabla \omega \times \nabla \phi, \]  

(21)

so that there are eight unknown scalar fields: \( \psi, F, \omega, G, \rho, p_\perp, p_\parallel, \) and \( \Phi \). From \( \mathbf{v} \times \mathbf{B} = \nabla \Phi \), it follows, by taking the inner product with \( \mathbf{B}, \mathbf{v}, \) and \( \nabla \psi \), that \( \nabla \psi \times \nabla \Phi = 0, \nabla \omega \times \nabla \Phi = 0, \) and \( \nabla \psi \times \nabla \omega = 0 \), so that \( \omega \) and \( \Phi \) are functions of \( \psi \) alone, and the flow surfaces, equipotential surfaces, and flux surfaces coincide. Then from \( (\mathbf{v} \times \mathbf{B}) \cdot \nabla \psi = \nabla \Phi \cdot \nabla \psi \) it follows that

\[ r^{-2} \rho^{-1} (G - \omega' F') = \Phi', \]  

(22)

which is also a function of \( \psi \) alone. Thus, \( \rho \mathbf{v} = \omega' \mathbf{B} + r^2 \rho \Phi' \nabla \phi \). The mass flow can be decomposed on each individual flux surface into a divergence-free flow along \( \mathbf{B} \) and a rigid toroidal rotation.

3. The Interpretation of Tokamak Magnetic Diagnostics
The conservation form of the force balance equation, $\nabla \cdot \mathbf{T} = 0$, is obtained with

$$\mathbf{T} = (p_\perp + B^2/2\mu_0)\mathbf{l} + \rho \mathbf{v} \mathbf{v} - \sigma \mu_0^{-1} \mathbf{B} \mathbf{B},$$

(23)

where $\sigma = 1 + \mu_0(p_\perp - p_\parallel)/B^2$. The related form in terms of the scalar functions is,

$$\frac{1}{2} \nabla (p_\perp + p_\parallel + r^{-2} \rho^{-1} (G^2 + |\nabla \omega|^2))$$

$$+ \frac{1}{2} r^{-2} \nabla (\sigma \mu_0^{-1} (F^2 - |\nabla \psi|^2) - \rho^{-1} (G^2 - |\nabla \omega|^2))$$

$$+ \nabla \cdot (r^{-2} \sigma \mu_0^{-1} \nabla \psi \nabla \psi - r^{-2} \rho^{-1} \nabla \omega \nabla \omega) = 0.$$  

(24)

Eqs. (23) and (24) will be useful in Section 4.

**Boundary conditions and auxiliary equations.** The MHD equilibrium problem is supplied with both external and internal boundary conditions, and furthermore is usually posed as a parameter estimation problem, thus requiring additional constraint equations. The external boundary conditions may be local conditions, or they may have the form of a boundary integral equation.

The familiar local external boundary conditions prescribe the value of $\psi + \beta \partial \psi / \partial n$ on $\partial \Omega$, for given functions $\alpha$ and $\beta$. Dirichlet and Neumann conditions arise as special cases. Such boundary conditions may be obtained by fitting a curve through a sufficiently dense set of local field and flux measurements on $\partial \Omega$.

Alternatively the external boundary conditions may prescribe the behaviour of the solution on the axis ($r = 0$) and at infinity. Such conditions are obtained when the field $\psi_{ext}$ due to currents in external coils is known. On the finite computational domain $\Omega$, these free-field boundary conditions may be replaced by an integral equation relating $\psi$ and $\partial \psi / \partial n$ on $\partial \Omega$,

$$\frac{\varphi(r')}{2\pi} \psi(r') + \oint_{\partial \Omega} r^{-1} \mu^{-1} \psi \frac{\partial G}{\partial n} ds = \oint_{\partial \Omega} r^{-1} \mu^{-1} G \frac{\partial \psi}{\partial n} ds + \psi_{ext}(r'),$$

(25)

for $r' \in \partial \Omega$. Here, $\varphi(r')$ is the exterior angle subtended by $\partial \Omega$ at the point $r' \in \partial \Omega$, and $G(r, r')$ is the Green function for the problem, defined by the equation $\mathbf{L}^* G = \mu r' \delta (r - r')$, subject to the boundary condition $G \to 0$ as $r \to 0$ or as $|r| \to \infty$. Eq. (25) is a formulation of the free-field boundary conditions that has not been used before, although it is closely related to the formulation given by Von Hagenow and Lackner [26]. It is derived and discussed in Appendix A.

3.2. Fundamental Relations for Axisymmetric Confinement
Nonlocal boundary conditions involving integrals of $\psi$ and $\partial \psi / \partial n$ on $\partial \Omega$ can also arise from magnetic measurements using extended probes, or from measurements made not exactly on $\partial \Omega$. In the latter case one applies Green's theorem to represent the flux at points outside $\partial \Omega$ as an integral of $\psi$ and $\partial \psi / \partial n$ on $\partial \Omega$.

The internal boundary conditions (interface conditions) require the continuity of the normal component of $\mathbf{B}$ and of the tangential component of $\mathbf{H}$ on all interfaces (in the absence of skin currents). Furthermore, in the free boundary problem the plasma boundary contour $\partial \Omega_{pl}$ is unknown a priori, and must be determined as part of the solution. The proper characterization of $\partial \Omega_{pl}$ may vary somewhat from experiment to experiment, but a generally valid criterion is that $\Omega_{pl}$ shall be the largest simply connected region that is bounded by an isocontour of $\psi$ and that is wholly contained within a given limiting contour $L$. The plasma has a limiter geometry if $\partial \Omega_{pl}$ and $L$ have a point in common, otherwise it has a divertor geometry (and then $\partial \Omega_{pl}$ passes through a saddle point of $\psi$). In the vacuum region the pressure must vanish and the toroidal magnetic field must have the form $B_{t0} = r^{-1}F_0$, for some constant $F_0$. If no singular current density is allowed on $\partial \Omega_{pl}$ then $p$ and $F$ must satisfy the interface conditions $p = 0$ and $F = F_0$ on $\partial \Omega_{pl}$.

Additional constraints are required when the plasma current profile (or the contribution of other currents) is given in parametric form, or is treated as functionally unknown. In particular, a formulation is common in which the plasma current profile is given only up to an undetermined constant factor, and in which the value of the total current provides the necessary additional constraint. The value $\psi_b$ of the flux function on $\partial \Omega_{pl}$ is also in most cases an unknown parameter in the current profile, to be determined as part of the solution. In the context of the interpretation of diagnostics there should furthermore be some freedom in the shape of the current profile, and typically one to three additional free parameters are employed. A problem statement involving a functionally unknown current profile could be appropriate when a sufficiently sensitive set of diagnostics (more than just the external magnetic measurements) is to be interpreted. In a different context, the problem with a functionally unknown current profile arises when the $q$-profile is specified instead.
3.3. Moments of the Toroidal Current Density

This Section is concerned with a class of integral relations that express moments of the toroidal current density in \( \Omega \) as integrals of linear combinations of the poloidal magnetic field components on \( \partial \Omega \). In the context of magnetic confinement theory these integral relations were first given by Zakharov and Shafranov [27], but they appear in potential theory as an immediate corollary of Green's theorem. The moments involve solutions to the homogeneous equilibrium equation, and various families of solutions are provided here and in Appendix B.

An integral relation. Let \( \chi \) be an arbitrary function that satisfies the homogeneous equilibrium equation, \( \mathcal{L}^* \chi = 0 \) in \( \Omega \), and let \( \psi \) be the poloidal flux function, which satisfies \( \mathcal{L}^* \psi = -\mu r j_t \). Then by application of Green's second identity for the operator \( \mathcal{L}^* \), Eq. (8), to the pair \((\chi, \psi)\), one obtains the fundamental integral relation for the evaluation of moments of the toroidal current density,

\[
\int_{\Omega} \chi j_t \, dS = \int_{\partial \Omega} r^{-1} \mu^{-1} (\psi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \psi}{\partial n}) \, ds. \tag{26}
\]

The moments of the current density \( \int \chi j_t \, dS \) are thereby expressed in terms of the values of \( \psi \) and \( \partial \psi / \partial n \) on the boundary \( \partial \Omega \). Notice that from \( \mathcal{L}^* \chi = 0 \) it follows that \( \int r^{-1} \mu^{-1} (\partial \chi / \partial n) \, ds = 0 \), so in Eq. (26) there is no dependence upon the choice of the arbitrary additive constant in the potential \( \psi \). This can be made manifest by introducing together with \( \chi \) also a conjugate function \( \xi \) according to the equation,

\[
\nabla (r^{-1} \mu^{-1} \xi) = \mu^{-1} \nabla \chi \times \nabla \phi, \tag{27}
\]

which indeed admits a solution subject to \( \mathcal{L}^* \chi = 0 \). This definition implies the identity \( r^{-1} \mu^{-1} \partial \chi / \partial n = -\partial (r^{-1} \mu^{-1} \xi) / \partial s \), where \( \partial / \partial s \) is the derivative along \( \partial \Omega \) in the positive direction (clockwise on the outer boundary). By partial integration one may then eliminate \( \psi \) from Eq. (26) in favor of \( \partial \psi / \partial s \) to obtain

\[
\int_{\Omega} \chi j_t \, dS = \int_{\partial \Omega} r^{-1} \mu^{-1} (\xi \frac{\partial \psi}{\partial s} - \chi \frac{\partial \psi}{\partial n}) \, ds
= \int_{\partial \Omega} \mu^{-1} (\xi B_n + \chi B_s) \, ds \tag{28}
= \int_{\partial \Omega} (\xi H_n + \chi H_s) \, ds
\]

as an alternative form for Eq. (26).
In the customary cylindrical coordinates the relation between $\chi$ and $\xi$, Eq. (27), may be expressed as

\[ \begin{align*}
\frac{\partial}{\partial r} (r^{-1} \mu^{-1} \xi) &= -r^{-1} \mu^{-1} \frac{\partial \chi}{\partial z}, \\
\frac{\partial}{\partial z} (r^{-1} \mu^{-1} \xi) &= r^{-1} \mu^{-1} \frac{\partial \chi}{\partial r}.
\end{align*} \]

The function $\xi$ satisfies the equation $\mathcal{L}(r^{-1} \mu^{-1} \xi) = 0$, where $\mathcal{L}$ has been introduced in Eq. (6). Notice also that $\mathcal{L}(r^{-1} \mu^{-1} \chi) = \chi \mathcal{L}(r^{-1} \mu^{-1})$.

Zakharov and Shafranov [27] gave the following different derivation of the integral relation (26). Let the fields $q$ and $g$ satisfy $\mu^{-1} \nabla \times q = \nabla (\mu^{-1} g)$. Then

\[ \int_T q \cdot J \, dV = \int_T q \cdot (\nabla \times \mu^{-1} B) \, dV \]

\[ = \int_T (\nabla \cdot (\mu^{-1} B \times q) + \mu^{-1} B \cdot (\nabla \times q)) \, dV \]

\[ = \int_T (\nabla \cdot (\mu^{-1} B \times q) + B \cdot \nabla (\mu^{-1} g)) \, dV \]

\[ = \oint_{\partial T} ((\mu^{-1} B \times q) \cdot n + g \mu^{-1} B \cdot n) \, dA. \]

These identities do not require the assumption of axisymmetry. The previous result for the axisymmetric case is obtained when one sets $q = \chi \nabla \phi$ and $g = r^{-1} \xi$. In contrast to Ref. [27], the conjugate pair of functions $(\chi, \xi)$ has been defined in the present work in such a way that they have the same physical dimension. Our function $\chi$ corresponds to $f$ of Ref. [27], and our $\xi$ is $rg$ in their notation.

**Plasma current and position.** Specific analytical instances of these integral relations can only be given for the case of constant permeability. Let us therefore temporarily assume that $\Omega$ contains only the plasma and vacuum regions, $\Omega = \Omega_{pl} + \Omega_{vac}$, so that $\mu = \mu_0$ and $\mathcal{L}^* = \Delta^*$. Four simple independent pairs of conjugate solutions $(\chi, \xi)$ to $\Delta^* \chi = 0$ and $\Delta (r^{-1} \xi) = 0$ are: $\chi = 1, \xi = 0$, $\chi = 0, \xi = r$, $\chi = z, \xi = -r \log r$, and $\chi = r^2, \xi = 2rz$. These moments lead to the following specific integral relations,

\[ \int_\Omega j_\parallel \, dS = \oint_{\partial \Omega} \mu_0^{-1} B_s \, ds, \]

\[ 0 = \oint_{\partial \Omega} \mu_0^{-1} r B_n \, ds, \]

\[ \int_\Omega z j_\parallel \, dS = \oint_{\partial \Omega} \mu_0^{-1} (-r \log r B_n + z B_s) \, ds, \]

\[ \int_\Omega r^2 j_\parallel \, dS = \oint_{\partial \Omega} \mu_0^{-1} (2rz B_n + r^2 B_s) \, ds. \]
The first two will be recognized as the integral forms of $\nabla \times \mathbf{H} = \mathbf{J}$ and $\nabla \cdot \mathbf{B} = 0$, but the higher moments would not have been evident a priori. These equations suggest a characterization of the plasma position, $(r_c, z_c)$, according to the identities,

$$I_t = \int_\Omega j_t \, ds,$$

$$z_c I_t = \int_\Omega z j_t \, ds,$$

$$r_c^2 I_t = \int_\Omega r^2 j_t \, ds. \tag{36}$$

The current centre $(r_c, z_c)$ defined here is an intrinsic plasma property (not dependent upon the position of the contour $\partial \Omega$ as long as it completely encloses the plasma and encloses no other currents), which furthermore can be rigorously evaluated from external magnetic measurements. Alternative characterizations of the plasma position are: a geometric centre of plasma cross-section, or the position of the magnetic axis. These characterizations are more difficult to obtain from external measurements, and the plasma cross-section may not be a rigorously defined concept.

It is to be noted that Eqs. (35) and (36) are perfectly valid as definition of $z_c$ and $r_c$, but that the corresponding relations (32) and (33) are not immediately suitable for the computation of these quantities. In particular the uncritical use of Eq. (33) to compute $r_c$ is not to be recommended. Instead, after having obtained preliminary estimates $r_0$ and $z_0$ for the plasma position (e.g. as the centre of the vacuum vessel), $r_c$ and $z_c$ should be computed from the relations,

$$(z_c - z_0) I_t = \oint_{\partial \Omega} \mu_0^{-1} \left( -r \log \frac{r}{r_0} + (z - z_0) B_z \right) ds, \tag{37}$$

$$(r_c^2 - r_0^2) I_t = \oint_{\partial \Omega} \mu_0^{-1} \left( 2r(z - z_0) B_n + (r^2 - r_0^2) B_z \right) ds. \tag{38}$$

This computation may be iterated in order to obtain $(r_c, z_c)$ as that pair $(r_0, z_0)$ that causes the vanishing of some reasonable discrete approximation (in terms of the magnetic measurements) to the right hand sides of (37) and (38).

**Higher moments.** Specification of $\psi$ and $\partial \psi / \partial n$ on $\partial \Omega$ is equivalent to the specification of $\psi$ on $\partial \Omega$ together with the moments $q_i$ of $j_t$ with respect to a family of solutions $\{\chi_i\}_i$ to $\mathcal{L}^* \chi = 0$ that is complete on $\Omega$: $q_i = \int_\Omega \chi_i j_t \, ds$. This specification in terms of moments is very useful for the interpretation of magnetic measurements, as will be seen in Sections 6-8, and there is therefore an interest in (complete) families of higher moments of the toroidal current density.

### 3.3. Moments of the Toroidal Current Density
Analytical families of conjugate pairs of solutions to \( \Delta^* \chi = 0 \) and \( \Delta (r^{-1} \xi) = 0 \) can be generated in a number of ways. Zakharov and Shafranov [27, Eq. (61)] provide the first few even terms in a sequence of homogeneous polynomial solutions (but beware of two errors in that equation). Polynomial solutions to the homogeneous equilibrium equation have also been discussed in Refs. [28]–[30]. In our notation, and extended to all orders, these polynomial solutions are the family defined by,

\[
\begin{align*}
\chi_n &= \sum_{k=0}^{[n/2]-1} (-4)^{-k} \frac{(n-1)!/2}{k!(k+1)!(n-2k-2)!} r^{2k+2} z^{n-2k-2}, \\
\xi_n &= \sum_{k=0}^{[n/2]-1} (-4)^{-k} \frac{(n-1)!}{(k!)^2(n-2k-1)!} r^{2k+1} z^{n-2k-1},
\end{align*}
\]

for \( n > 0 \), together with the pair \((\chi_0 = 1, \xi_0 = 0)\). This family is not complete on any region that is of interest for tokamak studies, as is shown in Appendix B.

Elementary solutions to the homogeneous equilibrium equation can also be found by allowing a factor \( \ln r \) or a power of \( \sqrt{r^2 + z^2} \). Further analytical solutions may be obtained through separation of variables, in either cylindrical, spherical, or toroidal coordinates. All these forms are provided in Appendix B. Finally, a family of solutions to \( \mathcal{L}^* \chi = 0 \) may be generated by numerical solution of the elliptic equation for some family of boundary conditions for \( \partial \chi / \partial n \) on \( \partial \Omega \). This latter route is the only one available when the permeability in \( \Omega \) is not constant, and is also the most suitable procedure when the region \( \Omega \) does not have a regular shape.

### 3.4. Moments Involving a Generalized Pressure

This Section is concerned with a class of integral relations through which certain area (resp. volume) integrals of an energy density in \( \Omega \) (or \( T \)) may be expressed in terms of contour (surface) integrals of quadratic combinations of the magnetic field components on \( \partial \Omega \) (\( \partial T \)). In contrast to the relations of Section 3, which were derived using only the electromagnetic equations \( \nabla \cdot B = 0 \) and \( \nabla \times H = J \), the following relations are based also on an equation for the plasma equilibrium. In first instance the force balance equation for static, ideal MHD equilibrium will be assumed, viz. \( \nabla p = J \times B \). The resulting class of integral relations was first given in general form by Zakharov and Shafranov [27], although two important special cases had been given earlier [31]. In second instance we will consider the modifications due to pressure.
anisotropy and plasma rotation, thereby extending the work of Cooper and Wootton [32], who considered only the two cases of Ref. [31].

Throughout this Section it will be assumed that \( \Omega \) contains only a plasma and a vacuum region, \( \Omega = \Omega_{\text{pl}} + \Omega_{\text{vac}} \), as clearly an MHD force balance equation should not be assumed to hold in the coil region \( \Omega_{\text{coil}} \). In fact, application of the following integral relations is usually preceded by an identification of the plasma boundary and of the magnetic field on this boundary, using methods that are discussed in Section 7, in which case one may identify \( \Omega = \Omega_{\text{pl}} \). By a generalized pressure we understand any local expression in terms of \( p \) and \( B \) (and \( p_\parallel, p_\perp, \rho \) and \( v \) in the nonideal case) that has the physical dimension of a pressure.

**An integral relation for static, ideal MHD equilibrium.** Consider the equilibrium equation in the form of Eq. (18). Taking the scalar product with an arbitrary axisymmetric poloidal vector field \( Q \) gives,

\[
0 = Q \cdot \left[ \nabla p + \frac{1}{2 \mu_0} r^{-2} \nabla (F^2 - F_0^2 - |\nabla \psi|^2) + \nabla \cdot \left( \frac{1}{\mu_0} r^{-2} \nabla \psi \nabla \psi \right) \right].
\]

The constant \( F_0 \) is the value of \( F \) on the plasma boundary and in the vacuum region (related to the vacuum toroidal field by \( F_0 = r B_{t0} \)), which has been inserted here for convenience at a later stage. In order to arrive at a meaningful integral relation we rewrite this identity in a form that contains a total divergence,

\[
p \nabla \cdot Q + \frac{1}{2 \mu_0} (F^2 - F_0^2) \nabla \cdot (r^{-2} Q)
\]

\[
+ \frac{1}{\mu_0} \nabla \psi \cdot \left( r^{-2} \nabla Q - \frac{1}{2} \nabla \cdot (r^{-2} Q) I \right) \cdot \nabla \psi
\]

\[
= \nabla \cdot \left[ p Q + \frac{1}{2 \mu_0} r^{-2} (F^2 - F_0^2 - |\nabla \psi|^2) Q + \frac{1}{\mu_0} r^{-2} \nabla \psi \nabla \psi \cdot Q \right].
\]

This identity is next integrated over the volume of the torus, the divergence term being expressed as a surface integral, and this surface integral is simplified by employing that \( p = 0 \) and \( F^2 - F_0^2 = 0 \) on \( \partial T \). We choose to re-express the ensuing integrals over \( T \) and \( \partial T \) as integrals over \( \Omega \) and \( \partial \Omega \), and arrive at the following identity,

\[
\int_{\Omega} \left[ p \nabla \cdot Q + \frac{1}{2 \mu_0} (F^2 - F_0^2) \nabla \cdot (r^{-2} Q)
\right.
\]

\[
+ \frac{1}{\mu_0} \nabla \psi \cdot \left( r^{-2} \nabla Q - \frac{1}{2} \nabla \cdot (r^{-2} Q) I \right) \cdot \nabla \psi \left. \right] dS
\]

\[
= \frac{1}{\mu_0} \oint_{\partial \Omega} r^{-1} \left[ (Q \cdot \nabla \psi) (\nabla \psi \cdot n) - \frac{1}{2} |\nabla \psi|^2 (Q \cdot n) \right] ds.
\]

**3.4. Moments Involving a Generalized Pressure**

55
This is the desired integral relation. For any choice of the vector field \( Q \), the relation (40) expresses a certain volume integral of a generalized pressure in terms of a surface integral involving quadratic combinations of the magnetic field components on \( \partial \Omega \).

The integral in the left hand side of Eq. (40) is unfortunately not an invariant quantity, but depends on the choice of the region \( T \) (or \( \Omega \)), because although \( p \) and \( F^2 - F_0^2 \) vanish outside the plasma, \( \nabla \psi \) does not. In many applications of these relations \( \partial \Omega \) is identified with the plasma boundary in order to obtain intrinsic plasma properties, but in other cases \( \partial \Omega \) is taken as the contour on which the measurements are made.

Notice also the following related form,

\[
\int_{\Omega} r \left[ p \nabla \cdot Q + \frac{1}{2 \mu_0} (F^2 - F_0^2) \nabla \cdot (r^{-2} Q) \right] dS
= - \int_{\Omega} (Q \cdot \nabla \psi) j_4 \, dS.
\]

in which all integrals are invariant quantities, but in which the data on \( \partial \Omega \) do not enter.

**An alternative derivation.** A related but different derivation suggested by the work of Cooper and Wootton [32] is also worth noting. Let \( Q \) be an arbitrary vector field. Starting with the force balance equation in conservation form, \( \nabla \cdot T = 0 \), the following sequence of identities is derived:

\[
0 = \int_T Q \cdot (\nabla \cdot T) dV
= \int_T (\nabla \cdot (T \cdot Q) - T : \nabla Q) dV
= \int_{\partial \Omega} n \cdot T \cdot Q dA - \int_T T : \nabla Q dV.
\]

Inserting now \( T \) from Eq. (17) and using the fact that \( \mu = 0 \) on \( \partial T \), it follows that

\[
\int_T \left[ (p + \frac{1}{2 \mu_0} B^2) \nabla \cdot Q - \frac{1}{\mu_0} B \cdot \nabla Q \cdot B \right] dV
= \frac{1}{\mu_0} \int_{\partial \Omega} \left[ \frac{1}{2} B^2 (Q \cdot n) - (B \cdot Q) (B \cdot n) \right] dA.
\]

This relation is valid independent of axisymmetry. In the axisymmetric case Eq. (42) is equivalent to Eq. (40), as will be shown. First, without loss of generality \( Q \) may be required to be an axisymmetric vector field. Second, there is no reason to include a toroidal component in \( Q \). For suppose \( Q = \chi \nabla \phi \); then one obtains the integral relation,

\[
\int_T r B_1 B \cdot \nabla (r^{-2} \chi) dV = \int_{\partial T} r^{-1} B_1 B_n \chi dS.
\]
which is anyway trivial from $\mathbf{B} \cdot \nabla (rB_t) = 0$. Indeed, toroidal force balance was used earlier to prove that $rB_t$ is constant on flux surfaces. Next, restricting $\mathbf{Q}$ to be an axisymmetric vector field without toroidal component, one may separate in Eq. (42) the contributions of the toroidal and the poloidal fields, and subtract from both sides the contribution due to the vacuum toroidal field. This vacuum field is $\mathbf{B}_{00} = F_0 \nabla \phi$, with $F_0$ constant, and one uses the identities

$$\frac{1}{2\mu_0} B_{t0}^2 \nabla \cdot \mathbf{Q} - \frac{1}{\mu_0} \mathbf{B}_{t0} \cdot \nabla \mathbf{Q} \cdot \mathbf{B}_{t0} = \frac{1}{2\mu_0} B_{t0}^2 r^2 \nabla \cdot (r^{-2} \mathbf{Q}),$$

and $\mathbf{B}_t \cdot \nabla \mathbf{Q} \cdot \mathbf{B}_p = 0$, and $\mathbf{B}_p \cdot \nabla \mathbf{Q} \cdot \mathbf{B}_t = 0$. The result is the following form of the integral relation:

$$\int_T \left[ p \nabla \cdot \mathbf{Q} + \frac{1}{2\mu_0} (B_t^2 - B_{t0}^2) r^2 \nabla \cdot (r^{-2} \mathbf{Q}) \right] \nabla \cdot \mathbf{Q} = \frac{1}{\mu_0} \mathbf{B}_p \cdot \left( \nabla \mathbf{Q} - \frac{1}{2} (\nabla \cdot \mathbf{Q}) \mathbf{l} \right) \cdot \mathbf{B}_p \right] dV,$$

$$= \frac{1}{\mu_0} \oint_{\partial T} \left[ \frac{1}{2} B_p^2 \mathbf{Q} \cdot \mathbf{n} \cdot (\mathbf{Q} \cdot \mathbf{B}_p) (\mathbf{B}_p \cdot \mathbf{n}) \right] dA.$$

To simplify the right hand side it has been used that $B_t^2 - B_{t0}^2 = 0$ on $\partial T$. The final manipulations needed to demonstrate that this relation is equivalent to Eq. (40) may be left to the reader. We will henceforth work mainly with the last form, Eq. (43).

**Pressure anisotropy and plasma rotation.** It is a straightforward matter to repeat the derivation that lead to Eqs. (40) and (43), but starting from Eqs. (24) or (23). This leads to the identities,

$$\int_{\Omega} r \left[ \frac{1}{2} (p_\parallel + p_\perp + r^{-2} \rho^{-1} (|\nabla \chi|^2 + G^2)) \nabla \cdot \mathbf{Q} \right.$$

$$+ \left( \frac{\sigma F^2 - F_0^2}{2\mu_0} - \frac{1}{2} \rho^{-1} G^2 \right) \nabla \cdot (r^{-2} \mathbf{Q})$$

$$- \left( \frac{\sigma}{\mu_0} \nabla \psi \nabla \psi - \rho^{-1} \nabla \chi \nabla \chi \right) \cdot \left( r^{-2} \nabla \mathbf{Q} - \frac{1}{2} (\nabla \cdot (r^{-2} \mathbf{Q})) \mathbf{l} \right) \right] dS$$

$$= \frac{1}{\mu_0} \oint_{\partial \Omega} r^{-1} \left[ (\mathbf{Q} \cdot \nabla \psi)(\nabla \psi \cdot \mathbf{n}) - \frac{1}{2} |\nabla \psi|^2 (\mathbf{Q} \cdot \mathbf{n}) \right] ds,$$

3.4. **Moments Involving a Generalized Pressure**
and equivalently,

\[
\int T \left[ \frac{1}{2} (p_{||} + p_{\perp} + \rho \nu^2) \nabla \cdot Q + \left( \frac{\sigma B_{||}^2 - B_{10}^2}{2 \mu_0} - \frac{1}{2} \rho \nu^2 \right) r^2 \nabla \cdot (r^{-2} Q) - \left( \frac{\sigma}{\mu_0} B_p B_p - \rho \nu_p \nu_p \right) \cdot \left( \nabla Q - \frac{1}{2} (\nabla \cdot Q) I \right) \right] dV (45) \]

\[
= \frac{1}{\mu_0} \oint_{\partial T} \left[ \frac{1}{2} B_p^2 \nabla \cdot n - (B_p \cdot Q) (B_p \cdot n) \right] dA.
\]

This equation is related to the static, ideal MHD relation (43) through the substitutions,

\[
p \rightarrow \frac{1}{2} (p_{||} + p_{\perp} + \rho \nu^2),
\]

\[
B_{t}^2 - B_{10}^2 \rightarrow \sigma B_{t}^2 - B_{10}^2 - \mu_0 \rho \nu^2,
\]

\[
B_p B_p \rightarrow \sigma B_p B_p - \mu_0 \rho \nu_p \nu_p,
\]

and this indicates how pressure anisotropy and plasma rotation will affect the determination from magnetic measurements of the various terms describing the plasma energy content. Equation (45) has also been given by L.L. Lao [33], but in a form in which the relations (46) are not transparent.

**Definition of the parameters** \( \beta_i, \mu_i, \) and \( l_i \). A variety of definitions for these characteristic plasma parameters exists, and there would be good reason to avoid all of them and work directly with expressions for the energy content in the plasma: volume integrals of \( p \), \( \sigma (B_{t}^2 - B_{10}^2)/2 \mu_0 \), and of \( B_p^2/2 \mu_0 \) (dotless multiplication taking precedence over division). We therefore define,

\[
W_T = \int_{T_{pl}} p \, dV, \quad W_M = \int_{T_{pl}} \frac{B_{t}^2 - B_{10}^2}{2 \mu_0} \, dV, \quad W_L = \int_{T_{pl}} \frac{B_p^2}{2 \mu_0} \, dV, \quad (47)
\]

where \( T_{pl} \) is the plasma volume. As \( p \) and \( B_{t}^2 - B_{10}^2 \) vanish outside the plasma, the quantities \( W_T \) and \( W_M \) are completely unambiguous. \( W_L \) is dependent on which \( \psi \) contour is identified as the plasma boundary, and is therefore not such a good intrinsic plasma property.

The characteristic parameters \( \beta_i, \mu_i, \) and \( l_i \) are dimensionless quantities corresponding to \( W_T, W_M, \) and \( W_L \) respectively. In order to obtain the most suitable and unambiguous dimensionless characterizations of the energy content of the plasma we propose
the following definitions:

\[
\beta_I = \frac{4}{\mu_0 \tau_c I_t^2} \int_{T_{pl}} p \, dV,
\]

\[
\mu_I = -\frac{4}{\mu_0 \tau_c I_t^2} \int_{T_{pl}} \frac{B_t^2 - B_{t0}^2}{2\mu_0} \, dV,
\]

\[
l_t = \frac{4}{\mu_0 \tau_c I_t^2} \int_{T_{pl}} \frac{B_p^2}{2\mu_0} \, dV,
\]

where \( I_t \) is the toroidal plasma current and \( \tau_c \) is defined by Eq. (36). The particular scaling factor \( \frac{1}{4}\mu_0 \tau_c I_t^2 \) has been chosen because it gives the customary cylindrical limit, is an intrinsic plasma property, and can be rigorously determined from the external measurements.

Instead of the present normalizing energy (a) \( \frac{1}{4}\mu_0 \tau_c I_t^2 \) one also sees (b) \( \frac{1}{4}\mu_0 R_0 I_t^2 \), (c) \( V(B_{p0}^2/2\mu_0) \), (d) \( V B_{p0}^2/2\mu_0 \), (e) \( \mu_0 V I_t^2/8\pi S \), or (f) \( \mu_0 V I_t^2/2s^2 \), where \( R_0 \) is the major radius of the confinement vessel, \( V \) is the plasma volume, \( S \) is the area of the poloidal cross-section of the plasma, \( s \) is the circumference of the poloidal cross-section, and

\[
\langle B_p^2 \rangle = \frac{\int_{\Omega_{pl}} B_p \, ds}{\int_{\Omega_{pl}} B_p^{-1} \, ds},
\]

\[
B_p^2 = \frac{2}{1 + \kappa^2} \left( \frac{\mu_0 I_t}{2\pi a} \right)^2,
\]

in which \( a \) is the plasma minor radius and \( \kappa = b/a \) the elongation. Occasionally one also sees definitions in which the integrals of \( p \), \( (B_t^2 - B_{t0}^2)/2\mu_0 \), and \( B_{p0}^2/2\mu_0 \) are taken over the poloidal cross-section instead of over the volume, and the normalizing constant is reduced by a factor \( 2\pi \tau_c \) or something equivalent.

In our opinion the choice (a) is preferable to any of these alternatives. (b) is not an intrinsic plasma property. (c) is singular for a plasma bounded by a magnetic separatrix, and is therefore suspect in all cases. (d) contains the two geometric quantities \( a \) and \( \kappa \) of which the definition is ambiguous, in particular for unsymmetric configurations. (e) and (f) are closest to our definition, and preferable to (b)–(d), but still have to make reference to the plasma boundary, thereby introducing an unnecessary ambiguity in the definitions of \( \beta_I \) and \( \mu_I \). Finally, those definitions in which the integrals are taken over the plasma cross-section instead of over the plasma volume lose the rigorous connection with the energy content in the plasma.

3.4. Moments Involving a Generalized Pressure
The parameter $\mu_I$ is closely related to the change in the toroidal flux due to the plasma. In the limit of small toroidicity and small $|B_t - B_{t0}|/B_{t0}$,

$$\mu_I \approx -\frac{8\pi F_0}{\mu_0 r_c T_1} \int_{\Omega} (B_t - B_{t0}) dS. \quad (50)$$

The integral on the right hand side can be measured using a diamagnetic flux loop.

**Some specific multipole moments.** Three important instances from the general class of integral relations given by Eq. (43) follow. The integrals $s_1$, $s_2$, and $s_3$ can all be rigorously determined from knowledge of the magnetic field on $\partial\Omega$.

1. Selecting $Q = re_r + ze_z$:

$$s_1 = \int_T \left( 3p + \frac{B_t^2 - B_{t0}^2}{2\mu_0} + \frac{B_\phi^2}{2\mu_0} \right) dV \quad (51)$$

In the case when $T$ is the plasma volume, $s_1$ is related to $3\beta_I - \mu_I + l_i$.

2. Selecting $Q = e_r$:

$$s_2 = \int_T r^{-1} \left( p - \frac{B_t^2 - B_{t0}^2}{2\mu_0} + \frac{B_\phi^2}{2\mu_0} \right) dV \quad (52)$$

When $T$ is the plasma volume, $s_2$ is related to $\beta_I + \mu_I + l_i$.

3. Selecting $Q = re_r$:

$$s_3 = \int_T \left( 2p + \frac{B_\phi^2}{\mu_0} \right) dV \quad (53)$$

When $T$ is the plasma volume, $s_3$ is related to $2\beta_I + l_i$.

The integrals $s_1$ and $s_2$ were given in Ref. [31]. They can be combined to eliminate one of the three quantities $\beta_I$, $\mu_I$, and $l_i$; in particular, after eliminating $\mu_I$ they provide an estimate for $\beta_I + l_i/2$. This calculation does involve a large aspect ratio approximation, because the expression for $s_2$ has a factor $r^{-1}$ in the integration, and that for $s_1$ does not. If an independent measurement of $\mu_I$ is available (a diamagnetic flux loop), then separate estimates can also be obtained for $\beta_I$ and $l_i$. L.L. Lao [33] employs $s_3$ together with $s_1$ and $s_2$ in order to obtain a separate identification of $\beta_I$ and $l_i$ without the use of a diamagnetic measurement. This approach relies on the volume average $\langle B_\phi^2/\mu_0 \rangle$ being different from $\langle B_t^2/2\mu_0 \rangle$. It provides analytical underpinning for the empirical observation (discussed further in Section 6) that full MHD equilibrium calculations can provide a separate identification of $\beta_I$ and $l_i$ for sufficiently large deviations from circularity.
**Systematic sets of moments.** The class of all axisymmetric poloidal vector fields is still too large to deal with in a systematic manner. In order to see how much further \( \mathbf{Q} \) may be restricted without loss of information it helps to consider the case where \( \partial T \) coincides with the plasma boundary. (No measurements further out can provide more information about the plasma). In this case the right hand side of Eq. (43) reduces to

\[
\oint_{\partial T} \frac{1}{2 \mu_0} B^2_p \mathbf{Q} \cdot \mathbf{n} dA
\]

and it is seen that \( \mathbf{Q} \) may be restricted to any class of axisymmetric, poloidal fields for which \((\mathbf{Q} \cdot \mathbf{n})\) generates a complete set of functions on \( \partial \Omega \). Using this freedom, Eq. (43) may be simplified in several ways:

a) Let \( \mathbf{Q} = \nabla \chi \), where \( \Delta \chi = 0 \); this eliminates the toroidal field term. A family of solutions to \( \Delta \chi = 0 \) can be obtained in a variety of ways, the most generally useful analytical approach being separation in toroidal coordinates. See Appendix B.

b) Let \( \mathbf{Q} = \nabla \xi \), where \( \Delta \xi = 0 \); this eliminates the pressure term. Separation in toroidal coordinates is again indicated.

c) Let \( \mathbf{Q}_r + i \mathbf{Q}_z = f(r + iz) \) for analytic \( f \) (and \( i = \sqrt{-1} \)); this simplifies the poloidal field term to a form that involves only \( B^2_p \). Analytic function theory provides many different families of solutions, of which the set of monomials, \( f = w^m \) and \( f = -iw^m \), for \( w = (r - r_0) + i(z - z_0) \) and \( (r_0, z_0) \) an interior point of \( \Omega \), is the simplest.

The choices (a) and (b) do not quite provide a complete set of solutions for \( \mathbf{Q} \), as the differential equation imposes a certain consistency constraint on the boundary values of \((\nabla \chi \cdot \mathbf{n})\); all functions \( \mathbf{Q} \) constructed by method (a) satisfy \( \oint r^{-1}(\mathbf{Q} \cdot \mathbf{n}) ds = 0 \), and those constructed by method (b) satisfy \( \oint r(\mathbf{Q} \cdot \mathbf{n}) ds = 0 \). Therefore, to a complete set of solutions obtained from either \( \Delta \chi = 0 \) or from \( \Delta \xi = 0 \) one additional function \( \mathbf{Q} \) must be added for which \((\mathbf{Q} \cdot \mathbf{n})\) does not satisfy the associated constraint. A function that is suitable for either case (a) or case (b) is \( \mathbf{Q} = (r - r_0) \mathbf{e}_r + (z - z_0) \mathbf{e}_z \), where \((r_0, z_0)\) is an interior point of \( \Omega \).

The first few integrals obtained by method (c) will now be shown explicitly. For that purpose we define the generalized pressures \( P_0 \) and \( P_1 \) according to,

\[
P_0 = p + \frac{1}{2 \mu_0} (B^2_t - B^2_{t(0)}),
\]

\[
P_1 = p - \frac{1}{2 \mu_0} (B^2_t - B^2_{t(0)}) + \frac{1}{2 \mu_0} B^2_p.
\]
Notice that $P_0$ vanishes outside the plasma, and therefore the contribution of $P_0$ to the integrals is an invariant quantity in the sense introduced earlier, viz. independent of the precise location of the contour $\partial\Omega$. The contribution of $P_1$ does not have that pleasant property.

Then, selecting $f = 1$, $Q = e_r$:

$$\int_T r^{-1} P_1 \, dV.$$ 

Selecting $f = -i$, $Q = -e_z$:

$$\int_T 0 \, dV.$$ 

Selecting $f = w$, $Q = xe_r + ye_z$:

$$\int_T (2P_0 + r^{-1}xP_1) \, dV.$$ 

Selecting $f = iw$, $Q = ye_r - xe_z$:

$$\int_T r^{-1} yP_1 \, dV.$$ 

Selecting $f = w^2$, $Q = (x^2 - y^2)e_r + 2xye_z$:

$$\int_T (4xP_0 + r^{-1}(x^2 - y^2)P_1) \, dV.$$ 

Selecting $f = -iw^2$, $Q = 2xye_r + (y^2 - x^2)e_z$:

$$\int_T (4yP_0 + 2r^{-1}xyP_1) \, dV.$$ 

All the above integrals can be rigorously evaluated from the external magnetic measurements. Notice that whenever both $P_0$ and $P_1$ appear in an integral, the contribution of $P_1$ is less by a factor involving the inverse aspect ratio. In the cylindrical limit only the moments of $P_0$ survive.

62 3. The Interpretation of Tokamak Magnetic Diagnostics
3.5. Evaluation of the Current Moments from Measured Data

In this section we are concerned with the problem of constructing numerical approximations in terms of the magnetic measurements for the boundary integrals that occurred in Sections 3 and 4. It is shown how methods that are familiar from the treatment of ill-posed linear equations and from multivariate statistical analysis can be used to obtain accurate discrete approximations, and an approach to the robust treatment of failing or wildly erroneous signals is initiated. The methods that are discussed in this section are applicable in a wide variety of circumstances, and for many readers this may be the most valuable part of the paper. It should also serve as an introduction, in a linear context, to the method of function parametrization [34]-[36], to which we return in Section 8. A general discussion of the physical characteristics of the various kinds of magnetic diagnostics may be found in Refs. [37] and [38], while Ref. [39] provides a detailed discussion of the engineering issues related to the implementation of these diagnostics on one particular machine (TFTR).

General considerations. In practice there may arise several complications when an integral such as \( \int_{\partial \Omega} \mu^{-1}(\xi B_n + \chi B_s) \, ds \), for given functions \( \xi \) and \( \chi \), is to be approximated from the magnetic measurements: (a) Only a finite set of measurements of \( B_n \) and \( B_s \) is made, and the precise nature of these measurements is dictated more by engineering considerations than by considerations from numerical analysis. (b) Actually, instead of a local \( B_n \) or \( B_s \) one often measures integrals of type \( \int_{\partial \Omega} w_i(s) B_n(s) \, ds \) or \( \int_{\partial \Omega} w_i(s) B_s(s) \, ds \). (c) \( B_n \) and \( B_s \) may be measured on different contours, or not even on smooth contours at all. (d) The measurements involve a random error. (e) Sometimes an individual signal may be completely wrong.

Complication (e) will be ignored initially, but we return to it in the final subsection. The discussion here will be restricted to the problem of the approximation of the integrals given in Section 3, which are linear in the magnetic field components, but the analogous treatment for the integrals of Section 4 can easily be developed by the reader.

Let numerical approximations in terms of \( m \) measurements, \( \{q_i\}_{1 \leq i \leq m} \), be wanted for a collection of \( n \) moments, \( \{p_j\}_{1 \leq j \leq n} \). Each of the \( q_i \) and \( p_j \) is defined by a linear expression in terms of the poloidal components of the magnetic field on \( \partial \Omega \). For each moment one may therefore postulate a numerical approximation that is linear in the measurements: \( p_j \simeq \sum_i c_{ij} q_i \), or \( \mathbf{p} \simeq \mathbf{C}^T \mathbf{q} \). (\( \mathbf{C} \) has size \( m \times n \)). The problem is to determine an optimal coefficient matrix \( \mathbf{C} \). Presumably \( \mathbf{C} \) will be used many times, and it is therefore assumed that efficiency in obtaining this matrix is not an issue.

3.5. Evaluation of the Current Moments from Measured Data 63
A linear equation. By numerical simulation and/or in the course of calibrating
the diagnostic system, one may obtain for a large number of current distributions
(indexed by $\alpha$, $1 \leq \alpha \leq N$) the values of the current moments $p_\alpha$ and of the associated
measurements $q_\alpha$. One may then attempt to determine the matrix $C$ by straightforward
least squares optimization:

$$C \text{ to minimize } \sum_{\alpha=1}^{N} w_\alpha \| p_\alpha - C^T q_\alpha \|^2 / \sum_{\alpha=1}^{N} w_\alpha. \quad (55)$$

for given nonnegative weights $w_\alpha$. The averaging over $\alpha$ that is employed in Eq. (55)
will henceforth be denoted by the bracket pair $\langle \cdot \rangle$, so the objective function above
becomes $\langle \| p - C^T q \|^2 \rangle$. The minimum is attained for $C$ determined by

$$\langle qq^T \rangle \cdot C = \langle qp^T \rangle. \quad (56)$$

which has a unique solution provided that $\langle qq^T \rangle$ is not singular.

The obvious difficulty with this line of approach is that $\langle qq^T \rangle$ is likely to be very
ill-conditioned. Routine methods are however available for dealing with such near-
singularity in a system of linear equations in order to obtain an approximate solution
that is stable with respect to small changes in the data. Numerical analysts refer to
Refs. [40]-[42] and employ some form of quasi-inversion: either by selecting a least-
squares solution in a subspace on which $\langle qq^T \rangle$ is well-conditioned (‘truncation’), or
through the introduction of a stabilizing functional (‘damping’). Statisticians employ
the same methods, but call these principal components regression and ridge regression
respectively; see for instance Refs. [43, ch. 8], [44, ch. 8], and [45, ch. 6]. A brief review
of these methods is given in the next subsection.

First, however, it is useful to provide a slight generalization of (55) and (56), namely
to allow also a constant term in the linear relation between $p$ and $q$. Thus we seek
to determine coefficients $p_0$ and $C$ so as to obtain in a stable manner an approximate
minimum of the objective function,

$$I = \langle \| p - p_0 - C^T q \|^2 \rangle. \quad (57)$$

The minimum of this function is attained for $C$ given by

$$\langle (q - \bar{q})(q - \bar{q})^T \rangle \cdot C = \langle (q - \bar{q})(p - \bar{p})^T \rangle, \quad (58)$$
and for $p_0 = \bar{p} - C^T \bar{q}$, in which $\bar{q} = \langle q \rangle$ and $\bar{p} = \langle p \rangle$. The matrix that occurs on the left hand side of Eq. (58) is known as the (sample) dispersion matrix associated with the data $\langle q_a \rangle_a$.

$$S = \langle (q - \bar{q})(q - \bar{q})^T \rangle.$$ (59)

The generalization to Eqs. (57) and (58) is needed for instance when all the magnetic measurements and moments are scaled to correspond to unit plasma current, as there is then no longer a natural origin at $p = 0$ and $q = 0$.

**Stable solution methods.** The dispersion matrix $S$ is symmetric and positive semi-definite, and therefore has $m$ eigenvalues, $\lambda_1^2 \geq \ldots \geq \lambda_m^2 \geq 0$, with corresponding orthonormal eigenvectors, $a_1, \ldots, a_m$. Then,

$$S = \sum_{i=1}^{m} \lambda_i^2 a_i a_i^T, \quad S^{-1} = \sum_{i=1}^{m} \lambda_i^{-2} a_i a_i^T,$$ (60)

where the inverse exists only if all $\lambda_i^2 > 0$. $S$ is ill-conditioned if $\lambda_m^2 / \lambda_1^2 \ll 1$, and in order to make Eq. (58) well-posed it is then necessary to reduce in some sense the influence of the smaller eigenvalues.

The first popular procedure for obtaining a stable approximate solution to the linear equation (58) is variously known as truncation, selection, quasi-inversion, or principal components regression. The method is simply to truncate the expansion for $S^{-1}$ given in Eq. (60) at some index $m_0 \leq m$ (possibly $m_0 \ll m$), and thus to set

$$C = \sum_{i=1}^{m_0} \lambda_i^{-2} a_i a_i^T \langle (q - \bar{q})(p - \bar{p})^T \rangle.$$ (61)

The choice of the value of $m_0$ must depend on the accuracy with which the measurements are made. In particular, if the measurements $q$ are assumed to suffer independent random errors coming from a normal distribution with mean 0 and width $\sigma$, then a value of $m_0$ should be chosen such that $\lambda_{m_0}^2 \geq \sigma^2$. A preliminary transformation of the measurements in order to make the expected distribution of their errors equal and independent is therefore advisable.

The other popular procedure for obtaining a stable approximate solution to Eq. (58) is to employ damping (equivalently, to employ a stabilizing functional, ridge regression). Using that approach, $S$ is replaced in Eq. (58) by $S + \sigma^2 I$, with $\sigma^2$ a small parameter. Then

$$C = \sum_{i=1}^{m} \frac{1}{\lambda_i^2 + \sigma^2} a_i a_i^T \langle (q - \bar{q})(p - \bar{p})^T \rangle.$$ (62)

3.5. Evaluation of the Current Moments from Measured Data
More generally one can use $S + E$, for any well-conditioned positive definite $E$, and determine $C$ from the equation

$$(S + E) \cdot C = \langle(q - \bar{q})(p - \bar{p})^T \rangle. \quad (63)$$

Increasing $\sigma^2$ or $E$ makes the matrix equation better conditioned, but also increases the bias in the resulting coefficients $C$. In the present context a good case can be made for the damping method, when $E$ is chosen to correspond to an estimate of the dispersion matrix for the random errors in the measurements. Assuming $S$ has been computed from idealized data, the coefficient matrix $C$ will be optimal in a least squares sense for the actual measurements.

**Further discussion.** An important aspect of the above procedure is that it can work well with measurements that would be less suitable if analytical procedures were to be used to derive the approximations for $p$ in terms of $q$. For instance, returning to our original concern of obtaining the value of the integral $\int_{\Omega} \mu^{-1} (\xi B_n + \gamma B_s) \, ds$ from measured data, the use of analytical approximations would favor equidistant point measurements of the magnetic field, whereas more accurate data are obtained with somewhat extended coils, and the distribution of the coils over the vacuum vessel will anyway be restricted by engineering considerations. Local measurements of the poloidal field in particular suffer from alignment errors and from perturbations due to nonaxisymmetric nearby eddy currents, and these measurements should almost certainly be abandoned in favor of measurements made using saddle loops and partial or variable-winding Rogowski coils.

The procedure outlined above can easily be employed so as to make good use of redundant information, such as a mixture of point $B_p$ measurements, partial or variable-winding Rogowski coils, full flux loops, saddle coils, and data on the currents in the external poloidal field generating coils. It is furthermore possible to combine this procedure with any standard method that yields a numerical approximation for the moments in terms of the measurements, namely by applying the presently described methods in a defect correction manner [46]. A statistician would consider this to be a Bayesian procedure. Such an approach has the advantage that a stronger stabilizing term can be employed to achieve comparable overall accuracy.
Treatment of erroneous measurements. As the magnetic signals are used both for machine control and for routine data analysis, it is particularly important to have an algorithm that will deal effectively and efficiently with failing or erroneous signals. About the convenient assumption of normally distributed errors it has been said that "everybody trusts it [...] for experimenters believe it is a mathematical theorem, whereas the mathematicians see it as an experimental fact" (see [47, p. 2]). In fact diagnostics do fail on occasion, and we discuss now how to deal with outlying data.

Let us assume that the measurements \( \{ q_i \}_{1 \leq i \leq m} \) are independent, and that all components are expected to suffer random errors coming from a distribution having mean 0 and width \( \sigma \) if the outlying data are disregarded. A preliminary transformation of the data may be performed in order to achieve this condition. Next let \( \mathbf{A} \) be the matrix that has as its columns the eigenvectors \( \mathbf{a}_i \) \((1 \leq i \leq m)\) of the dispersion matrix \( \mathbf{S} \). Define for any measurement vector \( q \) the transformed measurement vector \( x \) according to \( x = \mathbf{A}^T (q - q) \), and define the functional \( J(q; \sigma) \) by

\[
J(q; \sigma) = \sum_{i=1}^{m} \frac{x_i^2}{\lambda_i^2 + \sigma^2}.
\]

Then \( (J) \simeq m \) if outliers are discarded, and those actual measurements \( q^{\text{exp}} \) for which \( J(q^{\text{exp}}; \sigma) > m \) are suspect.

Consider now an actual measurement \( q^{\text{exp}} \). If it is known that one or more specific components \( q_i^{\text{exp}} \) are in error, then these components can be restored to that set of values by which the quadratic form \( J \) is minimized. This is a simple and valuable procedure, immediately available as a by-product of an eigenanalysis on \( \mathbf{S} \), and requires only the solution of a system of linear equations having dimension equal to the number of failing signals.

It is however far from easy to design a procedure that will decide effectively and efficiently whether one (or more!) signals really are wrong. The preferred approach for related problems in statistical analysis is to use a robust method [47, 48], viz. a method that is not overly sensitive to outlying data without requiring their explicit identification. (The simplest example is the use of the median rather than the mean for estimating a location). Such a robust procedure to deal with possibly corrupt data \( q^{\text{exp}} \) is the following: routinely perform the data analysis in terms of a vector \( q = q^{\text{exp}} + h \), where \( h \) is such that

\[
\begin{align*}
J(q^{\text{exp}} + h; \sigma) &\leq m \\
\|h\|_1 \text{ is minimal}
\end{align*}
\]

3.5. Evaluation of the Current Moments from Measured Data
where $\| \cdot \|_1$ is the $l_1$ norm. In fact there is a choice of other norms that are preferable to $l_1$ from a computational point of view and that also lead to a robust procedure: see [47, ch. 6]. An algorithm as outlined here must necessarily be nonlinear, and some further investigation will be required in order to develop a fully satisfactory procedure. Nevertheless, the simple structure of the objective function $J$ provides confidence that such a procedure can be developed.

### 3.6. Full Equilibrium Determination From Magnetic Measurements

The problem that is considered in this Section is to determine the profiles $\mu_0 p'(\psi)$ and $FF'(\psi)$ such that the corresponding solution to the equation for axisymmetric ideal MHD equilibrium provides an optimal fit to the external magnetic measurements. We review the published studies in this area, initially concentrating on the physical content of the work, and then comparing the various numerical methods that have been employed. A novel fast algorithm for the current profile determination is proposed.

The discussion is restricted to isotropic, static equilibria, for which the representation $j_i = r \frac{dp}{d\psi} + r^{-1} \mu_0^{-1} FdF/d\psi$ holds in $\Omega_{pl}$, with a parametrization $\mu_0 p' = g_1(\psi, \alpha)$ and $FF' = g_2(\psi, \alpha)$. The problem is then to determine the parameter vector $\alpha$, the value $\psi_b$ of the flux function at the plasma boundary, the plasma boundary contour $\partial \Omega_{pl}$, and the solution $\psi(r, z)$ throughout $\Omega$.

**General considerations.** In order to make possible a determination of the profiles $\mu_0 p'$ and $FF'$ the available measurements must provide sufficient redundancy beyond what is required to solve the elliptic boundary value problem for $\psi$ with a known current profile, Eq. (16). For instance one may have data for both $\psi$ and $\partial \psi/\partial n$ on $\partial \Omega$, or alternatively one may know the contribution to the field due to the currents in external coils and in addition have some local measurements of the total magnetic field. In either case the determination of $\mu_0 p'$ and $FF'$ from the external field measurements is a typical difficult ‘inverse’ problem, and is certainly ill-posed [40]-[42] if no further restrictions on these profiles are given. The fundamental task for the numerical analyst is thus to find a suitable method of quasi-inversion or stabilization for this problem.

There is, however, remarkably little mathematical understanding. Even in straight geometry (the limit of infinite aspect ratio), in which the equilibrium is governed by...
a Poisson equation with only one unknown profile function, \( \Delta \psi = -f(\psi) \), the basic questions of the existence and uniqueness of solutions to the inverse problem are unanswered. A special case that is understood arises in straight geometry, when in addition the measurements are consistent with a solution that has concentric circular flux surfaces; in that case the measurements are automatically consistent with any profile function that gives the correct total current, and there is therefore an infinite degeneracy. In toroidal geometry there exists also a family of equilibria, all having the same set of flux surfaces, that is degenerate with respect to the interpretation of the magnetic measurements [49]-[52]. This is discussed further in Appendix C. It is not known whether any degeneracy remains in case the measurements do not correspond to one of these special cases, but clearly the problem remains ill-posed. The fundamental difficulty is that the magnetic field measurements are sensitive only to poloidal variations in the current profile, whereas \( f(\psi) \) in straight geometry, or \( \mu_{0} p'(\psi) \) and \( Fp'(\psi) \) in toroidal geometry, primarily influence the radial distribution of the current.

The studies of Luxon and Brown. The first published extensive numerical studies that involved a current profile optimization aiming to fit a set of magnetic measurements were done for the Doublet IIa and Doublet III experiments, and were presented in Ref. [53]. This paper showed clearly the possibilities and limitations of the magnetics analysis, and it is reviewed here in some detail.

The Doublet III studies reported in Ref. [53] are based on a system of magnetic diagnostics consisting of 24 one-turn loops, measuring the poloidal flux near each of the 24 poloidal field shaping coils, and 12 partial Rogowski coils, measuring the average poloidal field over a segment spanning two field-shaping coils in the poloidal direction. A plasma current measurement obtained from a full Rogowski coil is used for comparison purposes only, and there is no diamagnetic measurement. The arrangement of these diagnostics and of the poloidal field shaping coils on Doublet III is illustrated in Fig. 1. This Figure also shows the location of 11 point magnetic field probes, which have been used in different work.

Luxon and Brown employ a variety of current profiles, of which the following (with unimportant change in notation) is illustrative,

\[
\mathbf{j}_{l} = \begin{cases} 
\alpha \left( \beta \frac{r}{R_0} + (1 - \beta) \frac{R_0}{r} \right) g(\psi; \gamma) & \text{in } \Omega_{pl} \\
0 & \text{in } \Omega_{vac}
\end{cases}
\]  

(66)
Fig. 1. Schematic view of the magnetic diagnostics and the field shaping coils on Doublet III.

- flux loop
- magnetic probe
- partial Rogowski coil
- field shaping coil
where

\[ g(\tilde{\psi}; \gamma) = \exp(-\gamma^2(1 - \tilde{\psi})^2). \]  

(67)

The normalized flux function \( \tilde{\psi} \) is defined by \( \tilde{\psi} = (\psi - \psi_b)/(\psi_a - \psi_b) \), where \( \psi_a \) is the value of \( \psi \) on the magnetic axis and \( \psi_b \) is the value on the plasma boundary. \( \psi_a \) and \( \psi_b \) are not known a priori. The quantity \( R_0 \) is a characteristic major radius of the machine. The free parameters, \( \alpha, \beta, \) and \( \gamma \), are selected to minimize the (chi-squared) cost function,

\[ J = \sum_i \frac{(B_i - \hat{B}_i)^2}{\sigma_i^2} \]  

(68)

where \( B_i \) and \( \hat{B}_i \) are the measured and the calculated values of some component of the poloidal field at position \( i \), and \( \sigma_i \) is the standard error of the measurement. The \( \hat{B}_i \) are calculated as functions of \( (\alpha, \beta, \gamma) \) by solving the equilibrium equation subject to boundary conditions obtained from the measurements of the poloidal flux \( \psi \). These boundary conditions are imposed in an indirect manner. The 'infinite domain' Green function equilibrium solver GAQ [54] is employed, and the currents in the external coils are adjusted in order to let the computed solution match the boundary data for \( \psi \). Minimization of \( J \) as a function of \( (\alpha, \beta, \gamma) \) is carried out with the aid of a standard library routine.

By comparison of Eq. (66) with the equilibrium relation, Eq. (16), one sees that the term containing \( r/R_0 \) corresponds to the contribution of \( \mu_0 p' \), and the term containing \( R_0/r \) corresponds to \( FF' \). The parameters \( \alpha, \beta \) and \( \gamma \) may be seen to be related roughly to the toroidal current, the poloidal \( \beta \), and the internal inductance (or the peakedness of the current profile). Notice that the parametrization in Eq. (66) assigns the same shape, \( g(\tilde{\psi}; \gamma) \), but independent weighting factors to the contributions from \( \mu_0 p' \) and from \( FF' \) in the current density.

Luxon and Brown give contour plots of \( J \) as a function of \( \beta \) and \( \gamma \), for fixed, optimal \( \alpha \). A well-defined minimum generally exists for non-circular equilibria, but for near-circular equilibria the contours become very elongated ellipses, and a separate identification of \( \beta \) and \( \gamma \) is no longer possible. This is understood to correspond to the impossibility of separately determining \( \beta_f \) and \( l_i/2 \) for circular cross-section. They proceed to study different expressions for the current profile, including some with four free parameters instead of three, but conclude that three parameters, equivalent to \( I_t, \beta_I, \) and \( l_i \), are adequate to fit the magnetic measurements. Determination of a fourth parameter becomes marginally possible only at the most highly shaped equilibria.

3.6. Full Equilibrium Determination From Magnetic Measurements
Approximately the same minimum value of \( J \), and near-identical values for \( I_t \), \( \beta_t \), and \( l_i \) at the optimum, are obtained for a variety of mathematical parametrizations. This shows that, for noncircular cross-section, these three physical parameters are indeed well-determined by the external magnetic measurements. For circular cross-section one is only able to determine the two parameters \( I_t \) and \( \beta_t + l_i/2 \). Other properties of the plasma that are reported to be accurately determined by the magnetic analysis are the location of the plasma boundary (and as a consequence also the value of the safety factor at the boundary), and the position of the magnetic axis. For the plasma boundary this comes as no surprise; we will see in Section 7 that the plasma boundary is well determined even without the need for a determination of the current profile in the plasma. As regards the location of the magnetic axis the result is less transparent, as analytical approximations for the magnitude of the shift of the magnetic axis with respect to the geometric centre of the cross-section have to rely on a specific model for the current distribution in the interior of the plasma. Apparently this shift depends principally on \( \beta_t + l_i/2 \), and is not too sensitive to further details of the current distribution.

As the determination of a fourth parameter in the current profile is at best marginally possible, it follows that the magnetic measurements alone do not provide sufficient information to determine separately the shape of the \( \mu_{0p}' \) and \( FF' \) terms. Additional information, in particular a measurement of the location of the \( q = 1 \) surface, is employed in the analysis of D-III data in order to obtain a more accurate current profile.

**MHD equilibrium determination on JET.** Descriptions of the experimental system and an overview of the various codes employed for magnetic data analysis on the JET tokamak have been given in Refs. [55] and [56]. Here we are concerned with the methods for full MHD equilibrium analysis, developed by J. Blum and co-workers, and described in Refs. [57]–[60]. The physics content of these studies is similar to that of Ref. [53], but the numerical methods employed are entirely different.

On JET the poloidal flux function is measured at 14 locations on the outside surface of the vacuum vessel, using 8 full flux loops and 14 saddle coils. The component of the poloidal field tangential to the vacuum vessel is measured by a system of 18 local magnetic probes, mounted on the inside of the vessel. An independent measurement of the plasma current is available, but not used in the work described here, and no mention is made of a diamagnetic flux measurement. Fig. 2 shows the layout of the JET magnetic diagnostics.
Fig. 2. Schematic view of the magnetic diagnostics on JET.
- flux loop
- magnetic probe
For the JET studies the current profile is parametrized as,

\[
\mathbf{j}_t = \begin{cases} 
\alpha \left( \beta \frac{r}{R_0} g(\tilde{\psi}; \gamma_1) + (1 - \beta) \frac{R_0}{r} g(\tilde{\psi}; \gamma_2) \right) & \text{in } \Omega_{\text{pl}} \\
0 & \text{in } \Omega_{\text{vac}}
\end{cases}
\] (69)

where, as in Ref. [53], \( \tilde{\psi} = (\psi - \psi_b)/(\psi_a - \psi_b) \) (in which \( \psi_a \) and \( \psi_b \) are the value of \( \psi \) on the magnetic axis and on the boundary), and \( R_0 \) is a characteristic major radius of the device. For the profile function \( g(\tilde{\psi}; \gamma) \) either a polynomial

\[
g(\tilde{\psi}; \gamma) = \tilde{\psi} + \gamma \tilde{\psi}^2
\] (70a)

or a power function

\[
g(\tilde{\psi}; \gamma) = \tilde{\psi}^\gamma
\] (70b)

is selected. There are at most four free parameters, \( \alpha, \beta, \gamma_1 \) and \( \gamma_2 \), but one or two of these may be fixed in advance, or it may be required that \( \gamma_1 = \gamma_2 \).

As in Ref. [53], the optimization criterion is minimization of a cost function \( J \) defined in terms of the poloidal field measurements, Eq. (68), whereas the measured flux values provide the boundary conditions for the equilibrium solver. Information on the currents in external coils is not needed as input, nor is it obtained from the analysis. The equilibrium solvers used at JET are the IDENTB and IDENTC codes, which are related to the SCED code of J. Blum [58]. These codes employ a finite element discretization together with a Newton iteration scheme, as described in detail in Ref. [60].

The JET studies show that from the magnetic measurements alone, two or three parameters can be determined: \( I_t \) and \( \beta_1 + l_i/2 \) for low-\( \beta \), near-circular plasmas, and \( I_t, \beta_1 \), and \( l_i \) for non-circular plasma or at high \( \beta \). This experience is consistent with the results obtained in the D-III modelling. The minimum elongation for which the parameters \( \beta_1 \) and \( l_i \) can be separated at low \( \beta \) is reported to lie around \( b/a \approx 1.25 \). An indication of the minimum \( \beta_1 + l_i/2 \) for which \( \beta_1 \) and \( l_i \) can be separated at circular cross-section is not available.

The IDENTC code allows specification of the value of the pressure on axis, or of the radial profile of the pressure, in addition to the magnetic data. With this additional information one more parameter in the current profile can be determined, separating \( \gamma_1 \) and \( l_i \) in the circular case, and providing both the coefficients \( \gamma_1 \) and \( \gamma_2 \) for elongated cross-sections.
A study for ASDEX. Winter and Albert present in Ref. [61] a method for determining the separatrix location from magnetic measurements, with application to the ASDEX tokamak. The next Section will discuss specialized methods for plasma boundary identification, but as Winter and Albert rely on a solution of the MHD equilibrium equation, their work is discussed here.

Only a limited number of local magnetic measurements is available on ASDEX: the toroidal plasma current, and two flux- and two field measurements (one of each on the outer and on the inner side of the cross-section), and these are reduced further to three signals by considering only $I_t$ and the difference signals $\delta \psi$ and $\delta B_p$. However, in Ref. [61] the field due to the external currents is also assumed to be known. The current profile parametrization has the form,

$$j_t = \alpha \left( \beta \frac{r}{R_0} + (1 - \beta) \frac{R_0}{r} \right) \left( \psi + \gamma \psi^2 \right)$$  \hspace{1cm} (71)

in the plasma region $\Omega_{pl}$, and $j_t = 0$ in $\Omega_{vac}$. In the present case, $R_0$ is the major radius of the geometric centre of the plasma boundary, and $\tilde{\psi} = \psi - \psi_b$.

As ASDEX has a near-circular cross-section, a separate identification of the parameter $\beta$ is not expected to be feasible. This expectation is confirmed by the analysis. Winter and Albert therefore supply an estimate of $\beta$ based on other diagnostics, and perform the minimization of the cost function over the parameters $\alpha$ and $\gamma$ only. The Garching free boundary equilibrium code [26] is employed. This code solves the equilibrium equation for given external field. The three signals mentioned before are therefore all available for the profile determination.

The procedure has been tested on numerically generated magnetic field data, which may correspond to a different functional form for the current profile than the one used in the reconstruction, Eq. (71). Good agreement between the reconstructed magnetic measurements and the input data is obtained at the optimum $(\alpha, \gamma)$ for a range of estimated $\beta$, and over this range the computed separatrix location remains relatively immobile. This confirms that $\beta$ is not well determined by the magnetic analysis, and shows that the separatrix location is well determined. Winter and Albert report that $\beta + l_t/2$ is also accurately determined.

An interesting limitation on the performance of the reconstruction algorithm is noted in Ref. [61]. A current distribution according to Eq. (71) has $l_t$ values in the range $0.8 < l_t < 2.2$ (they claim). If the guessed input value of $\beta$ is such that the correct value of $\beta + l_t/2$ cannot be reproduced using this current distribution, then it turns

3.6. Full Equilibrium Determination From Magnetic Measurements
out that there may also be a large error in the computed separatrix location. This is to be taken as an injunction to employ a current profile parametrization that admits a sufficiently large range of \( i_l \) values.

**Recent work for Doublet III.** L.L. Lao et al. describe in Ref. [62] a new code for MHD equilibrium determination, EFIT. This code is significantly faster than the one used by Luxon [53], and is presently employed for routine analysis of the D-III magnetic measurements. The diagnostics considered in EFIT include the 24 flux loops and 12 partial Rogowski coils that were used also in Ref. [53], and furthermore include 11 local magnetic field probes, one full Rogowski loop, and optionally a diamagnetic flux loop. The layout of these diagnostics on D-III has been shown in Fig. 1. In addition the value of the safety factor on axis, \( q_a \), may be specified as input to EFIT.

The two terms in the plasma current profile, corresponding to \( \mu_0 \psi' \) and to \( FF' \), are parametrized using a polynomial model, constrained by \( \psi = 0 \) on the plasma boundary. Good results are obtained by using a third degree polynomial for \( \mu_0 \psi' \) and a linear function for \( FF' \):

\[
\tilde{j}_l = (\alpha_1 \tilde{\psi} + \alpha_2 \tilde{\psi}^2 + \alpha_3 \tilde{\psi}^3) r + \beta_1 \tilde{\psi} r^{-1}
\]  

(72)

in \( \Omega_{pl} \), and \( j_l = 0 \) in \( \Omega_{vac} \), where again \( \tilde{\psi} = (\psi - \psi_b)/(\psi_a - \psi_b) \). As an alternative a second degree polynomial model for both \( \mu_0 \psi' \) and \( FF' \) has been used.

An 'infinite domain' equilibrium solver is employed in EFIT, and the vector of unknowns contains the profile parameters \((\alpha_1, \alpha_2, \alpha_3, \beta_1)\) as well as the values of the currents in the external coils. The optimization criterion is a cost function defined in terms of all the measurements and any constraints between the profile parameters:

\[
J = \sum_{i=1}^{N_m} \frac{(M_i - \hat{M}_i)^2}{\sigma_i^2} + \sum_{i=1}^{N_c} \frac{(H_i - \hat{H}_i)^2}{\zeta_i^2}
\]

(73)

where \( N_m \) and \( N_c \) denote the number of measurements and the number of constraints. \( M_i, \hat{M}_i, \) and \( \sigma_i \) denote the measured value, the computed value, and the error associated with the \( i \)-th measurement, and \( H_i, \hat{H}_i, \) and \( \zeta_i \) denote the given value, the computed value, and the uncertainty associated with the \( i \)-th constraint.

The results described in Ref. [62] are consistent with those obtained by Luxon and Brown [53], and in the JET studies [56]. Without a specification of \( q_a \) or a diamagnetic measurement, two independent parameters can be determined for approximately

3. The Interpretation of Tokamak Magnetic Diagnostics
circular plasmas, and three for elongated configurations. Assuming isotropic pressure, a diamagnetic measurement provides the information needed to separate $\beta_I$ and $l_i$ in the circular case. When furthermore $q_a$ is specified a fourth parameter can be determined. Good agreement is found between the diamagnetic $\beta_I$ and the $\beta_I$ obtained from poloidal field and flux measurements for elongated plasma ($b/a > 1.15$) "covering a wide range of plasma operating conditions". However, there is no quantitative discussion in Ref. [62] of the influence of high power neutral injection (which will cause the pressure to become anisotropic) on the difference between these two methods for determining $\beta_I$.

Recent work for Tuman-3. A code for full MHD equilibrium determination from magnetic measurements on the Tuman-3 tokamak has been described in Ref. [63]. The diagnostics used in this work include measurements of the poloidal flux function along two different contours $L_1$ and $L_2$ encircling the plasma column, an independent measurement of the toroidal plasma current, and a diamagnetic measurement. The parameterization for the plasma current is

$$j_t = \alpha_1 (\alpha_2 \tilde{\psi} + (1 - \alpha_2) \tilde{\psi}^2) \frac{r}{R_0} + \beta_1 (\beta_2 \tilde{\psi} + (1 - \beta_2) \tilde{\psi}^2) \frac{R_0}{r} \quad (74)$$

in $\Omega_{pl}$, at $j_t = 0$ in $\Omega_{vac}$, where again $\tilde{\psi} = (\psi \cdot \psi_n)/(\psi_n \cdot \psi_n)$.

A Green function method is employed to solve the equilibrium equation (16) for given parameters ($\alpha_1, \alpha_2, \beta_1, \beta_2$), subject to Dirichlet boundary conditions obtained from the flux measurements on the contour $L_1$ (presumably $L_1$ is located outside $L_2$). The parameters are determined in a two stage procedure. For given values $\alpha_2$ and $\beta_2$ the parameters $\alpha_1$ and $\beta_1$ are determined from the condition of reproducing exactly the measured toroidal current and diamagnetic flux, while $\alpha_2$ and $\beta_2$ themselves are determined from the condition of chi-squared minimization of the error in the measurements made on $L_2$, viz. minimization of the cost function

$$J = \sum_{i=1}^{N} \frac{(\psi_i - \hat{\psi}_i)^2}{\sigma_i^2} \quad (75)$$

Here, $N$ is the number of measurements made on $L_2$, $\psi_i$ is the measured value and $\hat{\psi}_i$ the computed value of the flux function at location $i$, and $\sigma_i$ is the standard error of the $i$-th measurement.

Consistent with all the studies described previously, the Tuman-3 studies show that only three independent parameters can be determined using this set of diagnostics;
the parameters $\alpha_2$ and $\beta_2$ cannot be separated. Along each line of constant $J$ in the $(\alpha_2, \beta_2)$ plane, the resulting current profile shows very little change. Ref. [63] suggests that beyond the plasma current and the diamagnetic flux also the internal inductance ($l_i$), the poloidal beta ($\beta_p$), the safety factor on the plasma boundary ($q_b$), and the safety factor on axis ($q_a$) are well determined by the magnetic analysis. As regards $q_a$ a more specific investigation seems desirable, as all other work suggests that the Tuman-3 diagnostics would not suffice for accurate determination of this parameter.

Possibilities and limitations of magnetic analysis. All the studies described above show that the external magnetic measurements, even if these include a diamagnetic flux loop, provide only limited information on the interior structure of the plasma current profile. One obtains the plasma current $I_t$, the parameters $\beta_p$ and $l_i$, the plasma boundary $\partial\Omega_{pl}$, the current centre $(r_c, z_c)$, the position of the magnetic axis $(r_a, z_a)$, and the safety factor $q_b$ on the boundary. The field due to the external currents is accurately obtained even if these currents are not measured directly, and with that information included only three of the characteristic plasma parameters (namely $I_t$, $\beta_p$, and $l_i$) can be considered to be independent within the accuracy of the measurements.

In Sections 7 and 8 we will discuss fast specialized methods for the identification of the plasma boundary and for the determination of characteristic parameters of the equilibrium, including all the parameters listed above. It will be seen that these methods can be both very efficient and accurate, and the full equilibrium analysis as described in this Section is therefore not required for routine data analysis, but only to provide a standard of comparison for the more rapid and specialized methods. In fact it must be recognized that the full equilibrium analysis over-fits the data, by producing a complete solution to the equilibrium equation in the interior of the plasma, although the measurements only warrant a specification of some integral characteristics. One must be careful to avoid assigning too much meaning to those details of the solution that are strongly dependent upon the specific mathematical parametrization that is employed.

An important virtue of the studies described above is that they have demonstrated the limitations to the analysis of magnetic measurements alone. Clearly it remains a desirable objective to be able to determine on a routine basis the complete MHD equilibrium configuration and its time evolution throughout a discharge, and the methods used in the above studies will remain relevant for equilibrium determination from an appropriate extended set of diagnostics. The possible role of information on the safety factor on axis, $q_a$, or the location of the $q = 1$ surface (if it exists) has already been
demonstrated. Significant additional information on the current profile in the interior of the plasma could come from Faraday rotation measurements [64]–[67], where it would be important to analyze these measurements in conjunction with the magnetic diagnostics, and not in isolation. Further diagnostic input could be in the form of purely geometric information on the shape of the flux surfaces, as available from electron temperature measurements in particular. The use of such geometric information for current profile determination was proposed by Christiansen and Taylor [68], and is developed further in Appendix C.

It should be pointed out that some codes exist for equilibrium determination and transport analysis based on a consistent interpretation of a wide range of diagnostic systems, notably the ZORNOC code [69], [70], developed for the analysis of ISX-B data, and the TRANSP code [71], developed at Princeton. Neither of these codes appears suitable for routine analysis of many time slices for a single discharge, and development of procedures for efficient MHD equilibrium analysis based on a range of diagnostic systems remains an open challenge.

Comparison of numerical methods. So far in this Section the review of published work has concentrated on the physics content of the studies, without more than a brief mention of the numerical procedures employed. We now turn to that issue.

First it must be pointed out that each of the groups whose work was discussed above has used a different equilibrium solver; we have seen finite difference methods involving a Buneman rapid solver, a finite element method employing Newton iteration, a ‘moments’ method, and even the Green function method (which must compete with unaccelerated relaxation and Gaussian elimination for being the worst possible procedure). Another, more fundamental, distinction is that between the use of a ‘finite domain’ or an ‘infinite domain’ equilibrium solver, corresponding respectively to fitting only to local measurements of the field and flux, and to using also information on the currents in external coils. These distinctions are not the subject of this paper, but they should be noticed as a warning against facile comparisons between the the work of the different groups.

The optimization problem, determining the current profile that fits best to the magnetic measurements, contains two different sources of nonlinearity. First there is the complicated dependence of the cost function $J$ on the unknown parameters, while for fixed parameter values the solution of the p.d.e., Eq. (16), is also in general a nonlinear problem. One would like to have a reasonably efficient numerical scheme for the combined problem.

3.6. Full Equilibrium Determination From Magnetic Measurements
In the work of Luxon and Brown [53] the iterative procedures for these two nonlinearities are nested: an outer iteration varies the estimated values of the parameters, and for each parameter set the cost function is evaluated from a fully converged solution to the differential equation. In fact, the Jacobian of $J$ with respect to variations in the parameters is also evaluated repeatedly, and this has to be done numerically, so that the p.d.e. must be solved even more often. This treatment of the outer nonlinearity, together with the use of an inefficient basic equilibrium solver, explains the extremely long running times of the algorithm of Ref. [53].

In Blum's work [57]-[60] the equilibrium equation is discretized by a finite element method, which is solved using Newton iteration [58]. For the combined parameter estimation and equilibrium problem again a Newton iteration scheme is employed, as described in detail in Ref. [60]. This leads to an efficient algorithm, with running time quoted as several seconds on a Cray-1.

In the work of Winter and Albert [61] (following a suggestion of K. Lackner), and also in work of Lao et al. [62], a functional form for the current density is employed that is linear in the unknown parameters: $\mu_0 r j_i = \sum_k a_k g_i(r, \psi)$. This makes it possible to interleave the two iterative procedures in a relatively straightforward manner. At stage $n$ of the procedure one has the approximate parameter vector $\vec{a}^{(n)}$ and the approximate solution $\psi^{(n)}$. Then the following linear problems are solved by a suitable direct method: for each $i$ the inhomogeneous equation, $\Delta \psi_i^{(n-1)} = -g_i(r, \psi^{(n)})$ in $\Omega$, subject to homogeneous boundary conditions ($\psi_i^{(n+1)} = 0$ on $\partial\Omega$), and in addition the homogeneous equation, $\Delta \chi^{(n+1)} = 0$, with the correct inhomogeneous boundary conditions. (In the standard case the boundary conditions are linear, so that $\chi$ has to be computed only once). Next $\vec{a}^{(n+1)}$ and $\psi^{(n+1)}$ are computed from the solution to a linear least squares problem; with

$$\vec{a}^{(n+1)} = \sum_i a_i \psi_i^{(n+1)},$$

(76)

$\vec{a}^{(n+1)}$ is that parameter vector $\vec{a}$ by which the cost function $J(\psi)$ is minimized, and $\psi^{(n+1)}$ is the corresponding minimizing function $\psi$.

A widely used method to solve the equilibrium problem with known current profile is to employ Picard iteration, solving at each stage $\mathcal{L}^* \psi^{(n+1)} = -\mu r j_i(r, \psi^{(n)})$ by the use of a rapid direct solver [26]. The work involved in the procedure for equilibrium determination that is employed in Refs. [61] and [62] is seen to be a small multiple (corresponding to the number of free parameters in the current profile) of the work of

80 3. The Interpretation of Tokamak Magnetic Diagnostics
involved in the solution of the problem with known current profile via Picard iteration. A limitation is that the form of the profile is restricted to be linear in the unknown parameters.

Fast optimization of parameters. It appears possible to construct a considerably faster algorithm for the optimization (with respect to the measured data) of the parameters describing the current profile, and this without requiring that the parameters enter linearly in the profile: it may be assumed that \( r_{ji} = f(r, \psi, \alpha) \) in the plasma, without specific restrictions on the functional form of \( f \). The proposal is to determine the parameters \( \alpha \) not from the requirement of obtaining a best fit to the external magnetic measurements directly, but rather from the requirement of obtaining a best fit to a set of moments of the current, as obtained from these measurements according to the theory discussed in Section 3.

We take as the starting point any efficient algorithm for solution of the equilibrium equation with known current profile (nonlinear in \( \psi \)); our favorite method, and the fastest available, is multigrid relaxation [72,75], but some form of Picard iteration based on a rapid direct solver [26] or a Newton iteration scheme as used by Blum in the SCED code [58] is also suitable. Next, in between the iterations of this basic equilibrium solver we interleave the parameter optimization procedure, correcting the parameters \( \alpha \) in order to improve the fit between the moments derived from the measured data and those computed from \( r_{ji}(\psi, \alpha) \). In this way the correction to the parameters is found by purely algebraic methods, and does not require the solution of any auxiliary p.d.e. as does the method of Refs. [61] and [62].

Notice that this procedure is closely related to the usual way in which a total current constraint is imposed on the equilibrium problem. In that case one is asked to determine \( \psi \) and \( \lambda \) such that \( \Delta \psi = \lambda \mu_0 r f(r, \psi) \) subject to the integral constraint \( \lambda \int_{\Omega} f dS = I_t \), for given current shape function \( f(r, \psi) \) and total current \( I_t \). No one writing a standard equilibrium code would seek to impose this constraint in the manner of Ref. [53]. Instead, after each iteration of the equilibrium solver, \( \lambda \) is adjusted in order to obtain the desired total current. This standard procedure may obviously be extended to more than one constraint, and also to an overdetermined system, to be solved in a least-squares sense.

The author's present equilibrium code [74], [75], was written purely to provide a 'proof of principle' for multigrid as a method for computing MHD equilibrium. The code solves equation (16) subject to Dirichlet boundary conditions on a rectangle, which is of no

3.6. Full Equilibrium Determination From Magnetic Measurements
practical interest. However, as a demonstration code it has been entirely successful, achieving full multigrid efficiency, and solving the equilibrium equation for nonlinear right hand side on a 128 x 128 grid in \( \approx 120 \) msec on the Cray-1. In the context of the interpretation of experimental data a much coarser grid will be adequate, and it becomes realistic to strive for full MHD equilibrium determination in about 20 msec on a Cray-1. This is in fact close to the timescale that is relevant for active control of an experiment.

3.7. Fast Identification of the Plasma Boundary

In this Section we consider fast specialized methods for the determination of the plasma boundary contour \( \partial \Omega_{pl} \) and of the magnetic field on \( \partial \Omega_{pl} \) from the external magnetic measurements. Knowledge of the location of the plasma boundary is important for control of the experiment, and if in addition the field on the plasma boundary is known, then the theory presented in Section 4 can be used to obtain estimates for a number of internal plasma parameters. We take the contour \( \partial \Omega_{pl} \) to be the largest closed flux surface inside a given limiting contour \( L \). The plasma has a limiter geometry if \( \partial \Omega_{pl} \) and \( L \) have a point in common, otherwise it has a divertor geometry.

General considerations. As for the more general profile determination problem that was discussed in the previous Section, the measurements may provide data about both \( \psi \) and \( \partial \psi / \partial n \) on \( \partial \Omega \), or they may provide a specification of the field due to the external currents, together with some local field measurements. The basis for all fast specialized methods for plasma boundary identification is that in the vacuum region \( \Omega_{vac} \), bounded by \( \partial \Omega_{pl} \) and \( \partial \Omega \), the flux function \( \psi \) satisfies the homogeneous equation \( \mathcal{L} \psi = 0 \). A solution to this equation that agrees with the given boundary conditions, and that is valid throughout a region including \( \Omega_{vac} \), therefore suffices to determine the plasma boundary. Thus, if both \( \psi \) and \( \partial \psi / \partial n \) are given on \( \partial \Omega \), the plasma boundary identification problem involves one of the classical ill-posed problems of mathematical physics (in the sense of Hadamard): the integration of an elliptic equation from Cauchy boundary data. A similar ill-posed type of problem arises in case the boundary conditions include a specification of the field produced by the external currents.
Fortunately, the Cauchy problem for elliptic equations is well-understood [40]–[42]. Stable solution methods may be obtained either by restricting the class of allowed solutions \( \psi \) on \( \Omega_{\text{vac}} \) to an appropriate finite-dimensional space (truncation, quasi-inversion), or through the introduction of a stabilizing functional (damping), or through a combination of these two methods. Most of the successful procedures for fast plasma boundary identification therefore rely on an approximation of the flux function \( \psi \) by a finite series in terms of solutions to the homogeneous equilibrium equation,

\[
\hat{\psi} = \psi^0 + \sum_{j=1}^{N} \hat{\epsilon}_j \chi_j. \tag{77}
\]

Here, \( \psi^0 \) represents any known contribution to the flux function (this term may be absent), and the basis functions \( \chi_j \) all satisfy the homogeneous equation, \( \mathcal{L}^* \chi_j = 0 \), on some (annular) region \( \Omega_0 \) that is known to include the vacuum region \( \Omega_{\text{vac}} \). The coefficients \( \hat{\epsilon}_j \) are to be determined from the measured data.

Let us denote the relevant actual measurements by \( y_i \), where \( 1 \leq i \leq M \). The expectation values of the measurements depend linearly on the magnetic field and flux, and there exists therefore a response matrix \( Q \) such that

\[
\hat{y}_i = \hat{y}_i^0 + \sum_{j=1}^{N} Q_{ij} \hat{\epsilon}_j, \quad 1 \leq i \leq M \tag{78}
\]

where \( \hat{y}_i \) is the expectation value of the \( i \)-th measurement when the flux function \( \psi \) is given by Eq. (77). \( \hat{y}_i^0 \) is associated with \( \psi^0 \) and the matrix element \( Q_{ij} \) is associated with \( \chi_j \). The response matrix \( Q \) and the vector \( \hat{y}^0 \) will be assumed to be known exactly.

The usual least squares approach to determining the coefficients \( \hat{\epsilon}_j \) is to minimize the (chi-squared) cost function,

\[
J = \sum_{i=1}^{M} \frac{(y_i - \hat{y}_i)^2}{\sigma_i^2}, \tag{79}
\]

where \( \sigma_i \) is the standard error of the \( i \)-th measurement, and the relation (78) is assumed. This minimization criterion gives rise to a linear representation for the coefficients \( (\hat{\epsilon}_j)_j \) in terms of the measurements \( (y_i)_i \). Whether this procedure is stable depends on the choice of the set of basis functions \( (\chi_j)_j \). A more general approach is to replace the cost function \( J \) defined in Eq. (79) by a function \( J_e \) of the form,

\[
J_e = \sum_{i=1}^{M} \frac{(y_i - \hat{y}_i)^2}{\sigma_i^2} + \epsilon \sum_{j=1}^{N} \frac{(\epsilon_j - \hat{\epsilon}_j)^2}{\eta_j^2}, \tag{80}
\]

3.7. Fast Identification of the Plasma Boundary
where \( \epsilon, (c_j)_{ij}, \) and \( (\eta_j)_{ij} \) are constants. The second term in \( J_\epsilon \) is intended to provide numerical stabilization, and reflects a priori knowledge about the range of values for the coefficients \( \hat{c}_j \). One may set \( c_j \) to some 'average' value of \( \hat{c}_j \) over all possible states of the system, and set \( \eta_j \) to a measure for the dispersion of the values of \( \hat{c}_j \). Then \( \epsilon \) is a tuning constant of order unity.

Using the minimization criterion (80) the coefficients \( (\hat{c}_j)_{ij} \) are still linearly related to the measurements \( (y_i)_{ij} \), and the matrix elements in this linear relation can be pre-computed. Specifically, these matrix elements are obtained by inversion of the relation,

\[
(Q^T \sigma^{-1} Q + \epsilon \eta^{-1}) \hat{c} = Q^T \sigma^{-1} (y - y^0) + \epsilon \eta^{-1} c,
\]

where \( D_\sigma = \text{diag}(\sigma^2) \) and \( D_\eta = \text{diag}(\eta^2) \). In this way a stable approximation \( \psi \) to the true flux function \( \psi \) in the vacuum region \( \Omega_{vac} \) is rapidly computed, after which the plasma boundary \( \partial \Omega_{pl} \) is obtained by finding the largest closed flux surface inside the given limiting contour. A stable approximation to the magnetic field on \( \partial \Omega_{pl} \) is obtained at the same time. Inside the plasma region \( \Omega_{pl} \), however, the function \( \psi \) must not be considered an approximation to the true flux function \( \psi \).

In many cases the series (77) splits naturally into two parts: a solution \( \psi_{\text{ext}} \) that is due to the current distribution on the exterior region \( \Omega_{\text{ext}} \) and that is regular throughout \( \Omega \), and a solution \( \psi_{\text{int}} \) that is associated with the plasma current. The treatment of the contribution due to the plasma current is critical for the plasma boundary identification problem, as this is the part that causes the problem to be ill-posed. The solution due to the external currents may be known from direct measurements of these currents, or it may be computed directly from the magnetic measurements via Green's representation theorem in the form of Eq. (11), or it may be found together with the solution \( \psi_{\text{int}} \) through optimization of Eq. (79) or (80). In any case, the determination on the interior region of the solution due to external currents is a stable process.

Several different representations for the field due to the plasma current have been used: the most important ones being an expansion of the field in toroidal eigenfunctions \([76]-[79]\), a discrete current filament model \([80]-[83], [33], [62]\), and a representation via a single layer potential on a control surface \([57], [60], [84]\). The field due to the external currents is sometimes assumed known \([80]-[83]\). It has also been represented by an expansion having undetermined coefficients, either based on a filament model \([33], [62]\), or on toroidal eigenfunctions \([76]-[79]\). A Green function or single layer potential representation for the field due to external currents has been used in Refs. \([57], [60]\),
and [84]. The following three subsections review in more detail the studies based on an expansion in toroidal harmonics, on a filamentary current model, and on a single layer potential.

**Expansion in toroidal harmonics.** One well established stable method to integrate the equation $\Delta^* \psi = 0$ inwards from the boundary data on $\partial \Omega$ is to employ a truncated series expansion in toroidal eigenfunctions of the homogeneous equation. The toroidal $(\zeta, \eta)$ coordinate system about the point $(r = r_0, z = z_0)$, where $r_0 > 0$, is defined by

$$
\begin{align*}
\zeta &= r_0 \sinh \zeta / (\cosh \zeta - \cos \eta) \\
\eta &= r_0 \sin \eta / (\cosh \zeta - \cos \eta)
\end{align*}
$$

where $\zeta > 0$, $0 \leq \eta \leq 2\pi$, and the ignorable coordinate $\phi$ is ignored. The contour defined by $\zeta = \zeta_0$ is a circle in the right half-plane, having radius $r_0 \sinh \zeta_0$ and centre at $(r, z) = (r_0 \coth \zeta_0, z_0)$. The corresponding torus has aspect ratio $\cosh \zeta_0$. The circle degenerates to the singular point $(r_0, z_0)$ for $\zeta_0 \rightarrow \infty$, and to the axis of symmetry, $r = 0$, for $\zeta_0 \rightarrow 0$. The metric coefficients of the coordinate system are given by $h_\zeta = h_\eta = r_0/(\cosh \zeta - \cos \eta)$ and $h_\phi = r_0 \sinh \zeta / (\cosh \zeta - \cos \eta)$.

The most general solution to $\Delta^* \psi = 0$ outside the singular point is given by

$$
\psi = \psi_{\text{ext}} + \psi_{\text{int}},
$$

where

$$
\psi_{\text{ext}} = \frac{r_0 \sinh \zeta}{\sqrt{\cosh \zeta - \cos \eta}} \left[ \sum_{n=0}^{\infty} \alpha_n P_{n-\frac{1}{2}}^1(\cosh \zeta) \cos(n\eta) \right. \\
\left. - \sum_{n=1}^{\infty} \beta_n Q_{n-\frac{1}{2}}^1(\cosh \zeta) \sin(n\eta) \right],
$$

and

$$
\psi_{\text{int}} = \frac{r_0 \sinh \zeta}{\sqrt{\cosh \zeta - \cos \eta}} \left[ \sum_{n=0}^{\infty} \alpha_n P_{n-\frac{1}{2}}^1(\cosh \zeta) \cos(n\eta) \right. \\
\left. + \sum_{n=1}^{\infty} \beta_n P_{n-\frac{1}{2}}^1(\cosh \zeta) \sin(n\eta) \right].
$$

$P_n^m$ and $Q_n^m$ are Legendre functions, for which we follow the notation of Refs. [18] and [19] (See also Appendix B). The solution $\psi_{\text{ext}}$ is finite throughout the right half plane and corresponds to a field due to currents located on the axis of symmetry, $r = 0$, while the solution $\psi_{\text{int}}$ has a singularity at $(r_0, z_0)$ and corresponds to the field of a multipole current distribution at the singular point.

**3.7. Fast Identification of the Plasma Boundary**
Similar series representations may be derived for the magnetic field components,

\[ B_\eta = -\frac{1}{r h_\zeta} \frac{\partial \psi}{\partial \zeta}, \quad B_\zeta = \frac{1}{r h_\eta} \frac{\partial \psi}{\partial \eta}. \]  

(85)

It will be noticed that (having to deal with orthogonal coordinate systems only) we do not use tensor notation here or elsewhere in the paper, but have defined \( B_\zeta = \mathbf{B} \cdot \nabla \zeta / |\nabla \zeta| \) and \( B_\eta = \mathbf{B} \cdot \nabla \eta / |\nabla \eta| \). The explicit representations for the field components follow from the relations,

\[
\begin{align*}
-\frac{1}{r h_\zeta} \frac{\partial}{\partial \zeta} \left( \frac{r_0 \sinh \zeta}{\sqrt{\cosh \zeta - \cos \eta}} Q^{1}_{n-1/2} (\cosh \zeta) \right) & \quad \text{(86)} \\
& = \frac{\sqrt{\cosh \zeta - \cos \eta}}{r_0} \left( n + \frac{1}{2} \right) \left( (n - \frac{1}{2}) \cos \eta Q^0_{n-1/2} - n \cosh \zeta Q^0_{n-1/2} + \frac{1}{2} Q^0_{n+1/2} \right) \\
& = \frac{\sqrt{\cosh \zeta - \cos \eta}}{r_0} \left( n - \frac{1}{2} \right) \left( (n + \frac{1}{2}) \cos \eta Q^0_{n-1/2} - n \cosh \zeta Q^0_{n-1/2} - \frac{1}{2} Q^0_{n-3/2} \right)
\end{align*}
\]

and

\[
\begin{align*}
-\frac{1}{r h_\eta} \frac{\partial}{\partial \eta} \left( \frac{r_0 \sinh \zeta}{\sqrt{\cosh \zeta - \cos \eta}} P^1_{n-1/2} (\cosh \zeta) \right) & \quad \text{(87)} \\
& = \frac{\sqrt{\cosh \zeta - \cos \eta}}{r_0} \left( n + \frac{1}{2} \right) \left( (n - \frac{1}{2}) \cos \eta P^0_{n-1/2} - n \cosh \zeta P^0_{n-1/2} + \frac{1}{2} P^0_{n+1/2} \right) \\
& = \frac{\sqrt{\cosh \zeta - \cos \eta}}{r_0} \left( n - \frac{1}{2} \right) \left( (n + \frac{1}{2}) \cos \eta P^0_{n-1/2} - n \cosh \zeta P^0_{n-1/2} - \frac{1}{2} P^0_{n-3/2} \right)
\end{align*}
\]

and

\[
\frac{1}{r h_\eta} \frac{\partial}{\partial \eta} \left( \frac{r_0 \sinh \zeta}{\sqrt{\cosh \zeta - \cos \eta}} \cos(n\eta) \right) \quad \text{(88)}
\]

\[
= \frac{\sqrt{\cosh \zeta - \cos \eta}}{r_0} \left( \frac{n + 1}{2} \sin((n - 1)\eta) - n \cosh \zeta \sin(n\eta) + \frac{n - 1}{2} \sin((n + 1)\eta) \right)
\]

and

\[
\frac{1}{r h_\eta} \frac{\partial}{\partial \eta} \left( \frac{r_0 \sinh \zeta}{\sqrt{\cosh \zeta - \cos \eta}} \sin(n\eta) \right) \quad \text{(89)}
\]

\[
= -\frac{\sqrt{\cosh \zeta - \cos \eta}}{r_0} \left( \frac{n + 1}{2} \cos((n - 1)\eta) - n \cosh \zeta \cos(n\eta) + \frac{n - 1}{2} \cos((n + 1)\eta) \right).
\]

By truncating the series for \( \psi \) and for the magnetic field components, and demanding that the corresponding field and flux on \( \partial \Omega \) provide a least squares fit to the measured
data, a linear algebraic system for the coefficients $a_n^i$ and $b_n^i$ of an approximate solution $\hat{v}$ is obtained. The condition number of this system will depend on the number of terms retained in the series, and a damping term may be added in order to provide further stabilization.

After having obtained in this way a stable approximate solution $\hat{v} = \hat{v}_{\text{ext}} + \hat{v}_{\text{int}}$ to the ill-posed boundary value problem for $v$, the plasma boundary $\partial \Omega_{\text{pl}}$ is identified as the largest closed flux surface inside the given limiter contour. For this procedure to provide a meaningful result it is necessary that the point $(r_0, z_0)$ shall have been chosen to lie inside the plasma region (which is unknown a priori), and preferably not too close to the boundary. In the vacuum region $\Omega_{\text{vac}}$, the solutions $\hat{v}_{\text{ext}}$ and $\hat{v}_{\text{int}}$ may be understood to correspond to the fields due to the external and to the internal currents respectively. For this interpretation, however, all magnetization currents must be considered as true currents.

The method of expansion in toroidal harmonics has been implemented by a number of authors.

Lee and Peng [76] describe a numerical study of this method for use on an idealization of the ISX-B tokamak. The vacuum chamber of ISX-B has a rectangular cross-section, and a system of magnetic probes provides point measurements of both $B_r$ and $B_z$ at 18 points along the circumference. The idealized device of Ref. [76] has the same rectangular cross-section, on which a numerical grid of 34 × 56 points is imposed, and measurements of the poloidal flux function are made at each of the 344 points in the outermost two layers of the grid. No reason is given for the replacement of the real ISX-B diagnostic system by this construction.

The singular point $(r_0, z_0)$ is positioned at the centre of the cross-section, and the following truncated series expansion is employed:

$$\hat{v} = \hat{a}_0 + \frac{\sinh \zeta}{\sqrt{2} (\cosh \zeta - \cos \eta)} \left[ N_\zeta \sum_{n=0}^{N_r} \hat{a}_n^1 (\cosh \zeta) + \hat{a}_n^2 Q_{n - \frac{1}{2}}^1 (\cosh \zeta) \cos(n \eta) \right]$$

$$+ \sum_{n=1}^{N_r} \left[ \hat{b}_n^1 P_{n - \frac{1}{2}}^1 (\cosh \zeta) + \hat{b}_n^2 Q_{n - \frac{1}{2}}^1 (\cosh \zeta) \sin(n \eta) \right].$$

The difference of a factor $\sqrt{2} r_0$ compared to the series in Eqs. (83) and (84) is of course unimportant, but their inclusion of the constant term $\hat{a}_0$ is somewhat surprising, as it

3.7. Fast Identification of the Plasma Boundary
is not independent of the infinite series in Eq. (83). In fact

\[ 1 = \frac{\sinh \zeta}{\sqrt{\cosh \zeta - \cos \eta}} \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \frac{Q_{n-\frac{1}{2}}(\cosh \zeta) \cos(n\eta)}{n^2 - \frac{1}{4}} \]  

(91)

where the prime on the summation signifies that the term for \( n = 0 \) should be taken with a factor \( \frac{1}{2} \). It is true that the truncated system (90) remains non-singular even with the term \( a_0 \) included, but it will tend to be badly conditioned.

Two test cases are studied in Ref. [76], one elongated D-shape and one circular plasma, both symmetric about the midplane. These shapes are well recovered by Eq. (90) for \( N_c \) in the range 1–3 and for \( N_s = 1 \), when the simulated \( \psi \) measurements are perturbed by a random error of not more than \( \sim 5\% \). The maximum relative error in \( B_p \) is usually 25–35 times larger (this factor must be the ratio of the gradient scale length to their grid spacing) and Lee and Peng are led to conclude that the method of expansion in toroidal harmonics would be useful in general if the maximum relative random error of the poloidal \( B \) field data does not exceed 100%. This author is very sceptical. Indeed, although these simulated \( B_p \) measurements have a maximum relative error in excess of 100%, the correlation structure of the errors is such that integrals of \( B_p \) can be built with much higher accuracy than would be the case for actual \( B_p \) measurements at a similar error level.

Simultaneously with the work of Lee and Peng [76], the use of an expansion in toroidal harmonics for plasma boundary identification was also proposed, but not actually implemented, by Kuznetsov and Naboka [77]. Their proposal was further developed and studied numerically in Ref. [78], and actual results from an implementation for the Tuman-3 tokamak are reported in Ref. [79]. This last reference will be the basis for our discussion.

As described in Ref. [79], the system of magnetic diagnostics on Tuman-3 consists of 24 probes distributed uniformly along a circular contour encircling the vacuum vessel, with 12 probes measuring the tangential component and 12 measuring the normal component of the magnetic field. In addition there is an independent measurement of the plasma current, which is used to obtain an overall multiplicative correction to the tangential field measurements after which a discretized Ampère's law holds exactly. In the same spirit the measurements of the normal component of the field are adjusted in order to satisfy the integral form of \( \nabla \cdot \mathbf{B} = 0 \).
In the Tuman-3 study the expansions (83) and (84) are employed, all four series being truncated after the same index $N$. Thus there are $4N + 2$ unknown coefficients, which are determined from the condition of minimization of the cost function,

$$J_\epsilon = J + \epsilon \sum_{n=0}^{\mathcal{N}} \left( (a_n^\epsilon)^2 + (b_n^\epsilon)^2 + (a_n^i)^2 + (b_n^i)^2 \right).$$  \hfill (92)

Common sense dictates that $J$ in the above expression should have the form (79), but whether this is what is intended by Eq. (11) of Ref. [79] is not clear. It is reported that $N \geq 3$ provides a good reconstruction of the plasma boundary.

The parameter $\epsilon$ is not a constant in Ref. [79], but is determined from the condition of minimization of the functional

$$\Phi(\epsilon) = \left\{ \epsilon^2 \sum_n \left[ \left( \frac{da_n^\epsilon}{d\epsilon} \right)^2 + \left( \frac{db_n^\epsilon}{d\epsilon} \right)^2 + \left( \frac{da_n^i}{d\epsilon} \right)^2 + \left( \frac{db_n^i}{d\epsilon} \right)^2 \right] \right\}^{\frac{1}{2}}. \hfill (93)$$

Although this is an interesting and correct mathematical procedure, it does have the drawback of requiring an iterative algorithm, for which furthermore the matrix elements of the inverse of Eq. (81) cannot be precomputed.

**Filamentary current model.** Another widely used expansion method for plasma boundary identification is based on a model of the plasma current as a finite set of current filaments,

$$j_{\text{f}} = \sum_{j=1}^{\mathcal{N}} I_j \delta(\mathbf{r} - \mathbf{r}_j) \hfill (94)$$

in $\Omega_{\text{pl}}$, where the coefficients $I_j$ are to be determined. The fixed points $\mathbf{r}_j$ must lie inside the (unknown) plasma region. The field in $\Omega$ due to the external currents may be known from measurements of these currents, or it may be obtained by an application of Green's representation theorem employing the measured field and flux on $\partial\Omega$, or it may also be modelled in terms of current filaments of unknown strength. The filamentary current method has been in routine use on a number of experiments.

A.J. Wootton [80] describes a study, for the TOSCA tokamak, in which the discrete filamentary current model is employed to determine the shape of the plasma boundary. This is in fact the first published study on fast methods for plasma boundary identification. The currents in the external windings are known in Ref. [80], and plasma-induced currents external to the measuring contour are assumed to be negligible. Wootton has
a measurement of the total current, and furthermore uses three variable winding Rogowski coils and three saddle coils to measure the first three harmonics of the tangential and normal components of the magnetic field on the vacuum vessel, viz. the coefficients $a_j$ and $b_j$ ($1 \leq j \leq 3$) in the expansions,

\[
B_r \simeq \frac{\mu_0 I}{2\pi d} \left( 1 + \sum_{j=1}^{3} a_j \cos j\omega \right)
\]

\[
B_n \simeq \frac{\mu_0 I}{2\pi d} \sum_{j=1}^{3} b_j \sin j\omega
\]

where $\omega$ is the angular variable in a local polar coordinate system, and $d$ is the minor radius of the measuring contour. This expansion and the further analysis in Ref. [80] assumes symmetry about the horizontal midplane. It is also clearly oriented towards large aspect ratio, as for finite aspect ratio the chosen representation of $B_n$ is inconsistent with $\nabla \cdot \mathbf{B} = 0$.

Only three current filaments (apparently in carefully optimized locations) are employed to represent the plasma current. Within this framework the most natural computational procedure would seem to be to determine the 3 unknown coefficients from the requirement of obtaining a least squares fit to the magnetic measurements. Wootton instead uses the measurements to evaluate large aspect ratio approximations to 3 moments of the current density, and then determines the coefficients to obtain an exact fit to these moments. The results in Ref. [80] show that even strongly distorted shapes of the plasma boundary, as computed using an MHD equilibrium code, may be accurately retrieved by this analysis.

A filamentary current model was also employed by D.W. Swain and G.H. Neilson for plasma boundary identification on ISX-B [81], [82]. They have point measurements of the $r$ and $z$ components of the magnetic field in 18 locations distributed around the vacuum vessel, and also know the currents through each of the four groups of poloidal field generating coils (all coils within one group are connected in series). This diagnostic system and the field coils are illustrated in Fig. 3.

Due to the presence of an iron core in the main transformer coil of ISX-B, the free space Green function cannot be used to compute the field produced by a given filamentary current. Instead an analytically tractable model for the main transformer is employed in Refs. [81] and [82], in which it is represented as an infinitely permeable centre post of the correct radius and infinite axial extent, with an annular iron cylinder
Fig. 3. Schematic view of the magnetic diagnostics and the field shaping coils on ISX-B.

- $B_r$ magnetic probe
- $B_z$ magnetic probe
- field shaping coil
modelling the return leg. Two free parameters in their model are the radius of the outer annulus, and the iron permeability. The thickness of the annulus is specified by requiring that the cross-sectional area of the core and of the annulus be equal. These two parameters are adjusted so as to give a good fit to the experimental data.

Using this model for the iron core, the response matrices $Q^c$ and $Q^p$ are calculated, which linearly relate the expectation values of the magnetic measurements to the currents in the external coils and in the model plasma filaments:

$$\hat{y}_i = \sum_{j=1}^{N_c} Q^c_{ij} I^c_j + \sum_{j=1}^{N_p} Q^p_{ij} I^p_j, \quad 1 \leq i \leq M. \tag{96}$$

Here, $N_c$, $N_p$, and $M$ are the number of groups of poloidal field generating coils, the number of plasma current filaments, and the number of measurements. $I^c_j$ is the current through the $j$-th group of poloidal field coils, $I^p_j$ is the $j$-th filamentary current, and $\hat{y}_i$ is the expectation value of the $i$-th magnetic field measurement corresponding to this set of currents. $N_c = 4$, $N_p = 6$ typically, and $M = 36$.

Given actual magnetic field measurements $y_i$ and coil currents $I^c_j$, the filamentary currents $I^p_j$ are determined from the requirement of obtaining a least squares fit in Eq. (96), viz. from minimization of a cost function as given in Eq. (79). The resulting expansion is employed to trace out the plasma boundary surface and to determine the field on the plasma boundary. The lowest order moments of the plasma current distribution are also evaluated. Comparison tests using a free boundary equilibrium code show that the plasma boundary and the dipole and quadrupole moments of the current distribution are accurately recovered.

Lao et al. [33], [62] have implemented the filamentary current model in a code (MFIT) for the determination of plasma shape on Doublet III. The diagnostics used in these calculations consist of 24 flux loops, 12 partial Rogowski coils, 11 local magnetic probes, and one full Rogowski coil, as illustrated in Fig. 1. The plasma current is modelled using 6 filaments in fixed positions, while the field due to the external currents is also considered unknown, and is modelled using 24 filaments in the positions of the field shaping coils. Doublet III does not have an iron core transformer, and the response matrix in Eq. (96) is calculated by using the free space Green function. The currents $I^c_j$ and $I^p_j$ are determined from the requirement of minimization of the following cost function,

$$J_e = \sum_{i=1}^{M} \frac{(y_i - \hat{y}_i)^2}{\sigma_i^2} + \epsilon \left( \sum_{j=1}^{N_c} (I^c_j)^2 + \sum_{j=1}^{N_p} (I^p_j)^2 \right). \tag{97}$$
where \( \epsilon \) is a constant.

It is mentioned in Ref. [62] that for large sized plasmas, which simultaneously touch the top, the inner, and the outer limiters of D-III, the filamentary current method becomes inaccurate, also in comparison with the results obtained using the full equilibrium determination code EFIT. This is quite interesting, as at first sight there is no reason to suspect a deterioration of the filamentary current method for large sized plasmas. No explanation of the phenomenon is offered in Ref. [62], but two possible explanations come to mind: (1) For large plasma the boundary is located close to the many magnetic diagnostics of D-III, hence it is in principle very accurately determined, and 6 current filaments are too few to obtain full accuracy. (2) For large plasma the boundary is located much closer to the external current filaments than to the internal filaments, and it becomes improper to use the same value of \( \epsilon \) in Eq. (97) to dampen both the term related to the external currents and the one related to the internal currents. The internal currents should be damped less strongly.

**Single layer potential methods.** The two approaches to be described in this subsection both employ a control surface \( C_0 \), which is a fixed closed contour in the poloidal plane, located in a position where it is guaranteed to lie inside the plasma region. The function \( \psi \) is determined as a solution to the homogeneous equilibrium equation on the region \( \Omega_0 \) which is bounded by \( C_0 \) and \( \partial \Omega \). The methods that rely on a control surface are closely related to the single layer approach in potential theory.

Feneberg, Lackner and Martin [84] have developed a fast boundary identification code (FASTB) for JET using such an approach. We recall the system of magnetic diagnostics on JET (Fig. 2), which consists of 14 flux loops on the outside of the vacuum vessel and 18 tangential field probes on the inside.

In FASTB the unknown plasma current is represented by an expansion in \( N \) Fourier modes of a singular current density on the control surface \( C_0 \). The external currents are similarly represented by \( N' \) Fourier modes on the surface \( \partial \Omega \), which is coincident with the contour on which the poloidal flux function is measured, and therefore lies just outside the contour on which measurements of the poloidal field are made. The free space Green function is employed to compute the field and flux due to the basic current distributions on \( \partial \Omega \), while the field and flux due to the currents on \( C_0 \) are computed subject to the homogeneous Dirichlet condition on \( \partial \Omega \). This leads to an approximate flux function of the form

\[
\psi = \sum_{n=1}^{N} \xi_n \lambda_n^i + \sum_{n=1}^{N'} \xi_n^e \lambda_n^e, \tag{98}
\]

3.7. Fast Identification of the Plasma Boundary
where the basis functions \( \chi_n^i \) vanish on \( \partial \Omega \). In Ref. [84], first the coefficients \( (c_n^i)_n \) are computed from the requirement of obtaining a least squares fit to the flux measurements on \( \partial \Omega \), and afterwards the coefficients \( \hat{c}_n \) follow from the requirement of a least squares fit to the poloidal field measurements. Typically, \( N = 6 \) and \( N' = 14 \). As there are also 14 flux loops, the flux measurements are fitted exactly.

FASTB is used routinely for the analysis of JET discharges. After determining the plasma boundary and the field on the boundary, integral relations are employed to calculate characteristic parameters of the configuration.

Another single layer potential approach was followed by J. Blum in an early study for plasma boundary identification on JET [57], [60]. In his work the problem \( \mathcal{L}^* \hat{\psi} = 0 \) is solved in the region \( \Omega_0 \), bounded by \( C_0 \) and \( \partial \Omega \), subject to the boundary conditions \( \hat{\psi} = f \) on \( \partial \Omega \) and \( \hat{\psi} = \psi \) on \( C_0 \), where \( f \) is obtained from the poloidal flux measurements and \( \psi \) is unknown. The function \( \psi \) has to be determined from the condition of minimization of (a discrete approximation to) the cost function.

\[
J_\epsilon(v) = \sum_j \left( \frac{1}{r_j} \frac{\partial \hat{\psi}}{\partial n} (r_j) - g_j \right)^2 - \epsilon \int_{C_0} \left( \frac{1}{r} \frac{\partial \hat{\psi}}{\partial n} \right)^2 ds,
\]

where the \( g_j \) are the poloidal field measurements, and the second term is included in order to stabilize the problem. Using a finite element discretization for \( \psi \) and \( \hat{\psi} \), the minimization criterion (99) is rewritten as a linear equation that relates \( \psi \) to the flux and field measurements. The matrix elements of this equation can be pre-computed.

**Irregular methods.** The various methods for plasma boundary identification that have been discussed above, whether based on an expansion in toroidal eigenfunctions [76]-[79], on a filamentary current model [80]-[83], [33], [62], or on an expansion employing a single layer potential on a control surface [84], [57], [60], are rather alike in their basic structure. In each case the unknown flux function \( \psi \) in the vacuum region \( \Omega_{\text{vac}} \) is expanded in a (small) series of mutually independent solutions to the homogeneous equation. This gives rise to a system of linear equations, which may be stabilized if necessary by the use of a regularizing functional. This is correct mathematical procedure. Several nonstandard treatments of the plasma boundary identification problem have also appeared in the literature.

One such method was described by Lee and Peng [76] under the name of “local fitting”. (The method of expansion in toroidal harmonics is proposed in the same paper, and referred to as global fitting). In this method of local fitting, a large number
of overlapping circular regions is placed on the domain $\Omega$, and in each of these regions a local Taylor series solution to the homogeneous equilibrium equation is found. These local solutions are computed so as to obtain a fit to the measured boundary values that fall inside the local region, and to any interior values that have already been computed by a fit in a different region (and that are not inside the plasma). The value of the flux function at any point is then computed as the average of the values obtained from all local Taylor series representations that cover that point. We refer to Ref. [76] for the details. The method of local fitting appears to be without virtue.

Another nonstandard procedure was presented in Ref. [78] as the “integral method”. (These authors too discuss the method of expansion in toroidal harmonics, which they call the differential method). In this method the vector of discrete unknowns represents the values of the current at all the points of a two-dimensional mesh. The associated matrix equation obtained from the requirement of chi-squared minimization with respect to the measurements is then large and extremely ill-conditioned. The equation is successfully stabilized by the use of a regularizing functional, defined as the sum of squares of the mesh currents. The principal objection to this integral method is that the resulting matrix equation is much larger than the one associated with any of the standard methods, without any real advantage. Although the calculated current distribution is smooth, it does not satisfy the equilibrium equation, and therefore still has no meaning inside the plasma.

The third of the nonstandard procedures that we wish to mention is a method employed by Iida and Toyama [85] for boundary identification on TNT-A (Tokyo Noncircular Tokamak). Their model flux function does not satisfy the homogeneous equilibrium equation, but satisfies the two-dimensional Laplace equation instead. Some local fitting procedure is employed to integrate the Laplace equation inwards from the boundary data. The statement that this procedure is 50 times as fast as a filamentary current method cannot be based on a reasonable implementation of the latter method.

3.8. Fast Determination of Characteristic Parameters

In this Section we consider fast specialized methods for the approximation of characteristic parameters of the MHD equilibrium configuration. After providing an overview of the available analytical approaches, this Section will be devoted mainly to a recent study that has demonstrated the method of function parametrization [34; 36] as an
extremely effective procedure for obtaining a wide variety of characteristic parameters from the magnetic measurements [86], [87].

**Some important parameters.** The following is a (not exhaustive) list of parameters that one may wish to determine from magnetic measurements. They are not all independent, even if it is generally impossible to provide explicit connecting formulae.

- \( I_t \) toroidal plasma current
- \( (r_c, z_c) \) position of the current centre, Eqs. (35) and (36)
- \( (r_a, z_a) \) position of the magnetic axis
- \( (r_g, z_g) \) position of a geometric centre of \( \Omega_{pl} \)
- \( a, b \) horizontal and vertical minor radius of \( \Omega_{pl} \)
- \( \vartheta, \lambda, \ldots \) triangularity, indentation, \ldots, of \( \Omega_{pl} \)
- \( S \) area of \( \Omega_{pl} \)
- \( V \) plasma volume
- \( A \) plasma surface area
- \( (r_x, z_x) \) position of some saddle point
- \( \psi_a - \psi_b \) flux difference between the magnetic axis and \( \partial \Omega_{pl} \)
- \( \psi_b - \psi_r \) flux difference between \( \partial \Omega_{pl} \) and a reference location
- \( q_a \) safety factor on axis
- \( q_b \) safety factor on \( \partial \Omega_{pl} \)
- \( \beta_l \) poloidal \( \beta \)
- \( l_i \) internal inductance
- \( \mu_l \) toroidal diamagnetism parameter
- \( \beta_l + l_i/2 \) Shafranov parameter
- \( \ldots \) etc.

(Recall that \( \Omega_{pl} \) denotes the poloidal cross-section of the plasma).

An important question is to what extent these parameters may be determined (semi-)directly from the magnetic measurements, without requiring a complete solution of the equilibrium equation. In this context a method will be called **direct** when it provides a physical parameter as an explicit function of the measured data, and **semi-direct** when it relies on a preliminary identification of the plasma boundary.
Review of analytical methods. A direct and rigorous determination of $I_t$, $r_c$, and $z_c$ is possible using the theory of Section 3. After identification of the plasma boundary, with the aid of one of the methods described in Section 7, all the geometric quantities $(r_g, z_g, a, b, \theta, \delta, S, V, A, r_x, z_x)$, as well as $\psi_b - \psi_r$ and $q_b$, are rigorously determined. All the fast procedures for plasma boundary identification can also provide the magnetic field on the boundary. The theory of Section 4 then provides approximations for $\beta_I - \mu_I$ and for $\beta_I + \mu_I/2$, and if a diamagnetic measurement is available or if the plasma is sufficiently shaped, also for $\beta_I$, $l_i$, and $\mu_I$ separately. These expressions are exact only in the limit of large aspect ratio, but a good approximation to $\beta_I + \mu_I/2$ is generally obtained. The approximations derived by Cordey, Lazzaro et al. [88], [56] and by Lao et al. [62] in order to separate $\beta_I$ and $l_i$ without the use of a diamagnetic measurement rely on a specific model of the equilibrium current distribution, but appear to work well in the application to JET and D-III data.

An analytical formula for the radial displacement of the magnetic axis with respect to the geometric centre of the plasma, $r_a - r_g$, was given by Shafranov [9, Eq. (6.33)],

$$r_a - r_g = \frac{a^2}{2r_0}(\beta_I + \mu_I/2), \quad (100)$$

where $a$ is the plasma minor radius and $r_0$ is a characteristic major radius. The derivation of this relation relies on the assumptions of up-down symmetry, circular cross-section and on a specific model for the current distribution, but in fact equilibrium calculations for ASDEX have shown it to be accurate over a wide range of current distributions. No satisfactory approximations in terms of the magnetic measurements alone exist for the parameters $\psi_a - \psi_b$ and $q_a$, and the studies described in Section 6 indicate that the determination of these parameters requires additional information.

Throughout this paper we have emphasized those methods that make as few assumptions as possible about the shape of the plasma cross-section and about the current distribution in the plasma. In this case, for almost all of the parameters listed at the beginning of this Section it is necessary to rely on one of the methods for fast plasma boundary equation, followed by application of the integral relations of Section 4. In fact, it is only with the advent of the recent generation of devices with shaped cross-section, and in particular D-III, JET, ISX-B and Tuman-3, that application of these fairly general methods has become routine. For configurations having approximately circular cross-section, direct approximations are available for many of the characteristic parameters, and these are used in practice.

3.8. Fast Determination of Characteristic Parameters
The most widely used of these direct approximations are again due to Shafranov [8], [9, Section 6], [10, Section 4.7], and provide the radial plasma position and the parameter $\beta_1 + l_i/2$ in terms of the magnetic measurements. A local polar coordinate system $(\rho, \omega)$ is employed, centred at a fixed position $(r_0, 0)$, so that $r = r_0 + \rho \cos \omega$ and $z = \rho \sin \omega$. Then Shafranov's approximations for the flux function $\psi$ and the poloidal field components $B_\omega$ and $B_\rho$ outside the plasma, but inside any other conducting surface, are

$$
\psi(\rho, \omega) \approx -\frac{\mu_0 r_0 I_i}{2\pi} \left( \ln \frac{\rho}{8r_0} + 2 \right)
- \frac{\mu_0 I_i}{4\pi r_0} \left[ \left( 1 - \frac{a^2}{\rho^2} \right) \left( \beta_1 + \frac{l_i - 1}{2} \right) + \ln \frac{\rho}{a} \frac{r_0}{\rho^2} - \frac{2r_0}{\rho^2} \right] \cos \omega
$$

(101)

$$
B_\omega(\rho, \omega) \approx -\frac{\mu_0 I_i}{2\pi \rho} \left[ \left( 1 - \frac{a^2}{\rho^2} \right) \left( \beta_1 + \frac{l_i - 1}{2} \right) + \ln \frac{\rho}{a} - 1 + \frac{2r_0}{\rho^2} \right] \cos \omega
$$

(102)

$$
B_\rho(\rho, \omega) \approx -\frac{\mu_0 I_i}{4\pi r_0} \left[ \left( 1 - \frac{a^2}{\rho^2} \right) \left( \beta_1 + \frac{l_i - 1}{2} \right) + \ln \frac{\rho}{a} \frac{r_0}{\rho^2} - \frac{2r_0}{\rho^2} \right] \sin \omega
$$

(103)

where $\Delta = r_0 - r_0$. Assume that the plasma current $I_i$ is measured and that an estimate of the minor radius $a$ is available. Then two flux measurements and two measurements of $B_\omega$, preferable made on a contour of constant $\rho$, suffice to determine the coefficients of $\cos \omega$ in Eqs. (101) and (102), and thereby to determine $\beta_1 + l_i/2$ and $\Delta$. As a straightforward variation on this procedure, the flux measurements may be replaced by measurements of the other component of the poloidal magnetic field, $B_\rho$, and if more measurements are available, then a least squares fit can be employed to determine $\beta_1 + l_i/2$ and $\Delta$.

**Function parametrization.** We turn now to a discussion of a radically different method for estimating the physical parameters from magnetic measurements. Function parametrization was originally developed by H. Wind (CERN) for fast momentum determination from spark chamber data in high energy physics experiments [34], [35], but it has a much wider range of applicability, and can be considered whenever many measurements are to be made with the same diagnostic setup. In a recent paper [36] we have presented function parametrization in the context of controlled fusion research, in particular with a view towards the interpretation of magnetic diagnostics.

3. The Interpretation of Tokamak Magnetic Diagnostics
An application to the magnetic data analysis on the ASDEX experiment has been described in Refs. [86] and [87], and is summarized below.

The method relies on an analysis of a large data set of simulated experiments, aiming to obtain an optimal representation of some simple functional form for intrinsic physical parameters of a system in terms of the values of the measurements. Statistical techniques for dimension reduction and multiple regression are used in the analysis. The resulting function may be chosen to involve only low-order polynomials in only a few linear combinations of the original measurements; this function can therefore be evaluated very rapidly, and needs only minimal hardware facilities.

Three steps have to be made for experimental data evaluation based on function parametrization. (1) A numerical model of the experiment and of the relevant diagnostics is used to generate a data base of simulated states of the physical system, in which each state is represented by the values of the relevant physical parameters and of the associated measurements. (2) This data base is made the object of a statistical analysis, with the aim to provide a relatively simple function that expresses the physical parameters in terms of the measurements. (3) The resulting function is then employed for the fast interpretation of real measurements.

As an initial study we applied function parametrization to the determination of a limited set of characteristic equilibrium parameters for the ASDEX experiment. The relevant measurements consist of three differential flux measurements, four field measurements, the current through the multipole shaping coils, and the plasma current. However, the plasma current can be scaled out of the problem, so that 8 independent measurements remain. The physical parameters to be determined include the position of the magnetic axis, the geometric centre of the cross-section, the current centre, the horizontal and vertical minor radius, \( \beta_I + l_1/2 \), a normalized \( q \)-value at the separatrix, the flux difference between the separatrix and a reference position, the position of the lower and upper saddle points, and the point of intersection of the separatrix with each of the four divertor plates.

The details of this study are given in Refs. [86] and [87], and are highly encouraging. A second order polynomial fit involving 4 principal components is found to provide good accuracy also in the presence of realistic measurement errors. For instance the fits for the plasma position are accurate to about 5 mm, while \( \beta_I + l_1/2 \) is fitted with an accuracy of about 5%. This application has shown function parametrization to be a straightforward and effective way in which to obtain numerical approximations for
a variety of characteristic equilibrium parameters in terms of the magnetic measurements. These approximations are not only extremely easy to evaluate, but are also more accurate than the analytic approximations that are now in common use. The procedure does not require very specific assumptions about the MHD equilibrium, and is also well suited to a consistent analysis of a system consisting of several different diagnostics.

It can be expected that in the future function parametrization will have an important role both for on-line data analysis and for real-time plasma control. To the latter point, it is estimated in Ref. [87] that a small processor, having a few kilowords of 32-bit store and a speed of about 5 Mflops/sec for modest size matrix-vector multiplication, suffices to evaluate the approximations to a set of about 25 interesting physical parameters within 1 msec, which is the timescale that is relevant for active control.

3.9. Conclusions

We have reviewed the analytical theory that is available for MHD equilibrium determination from magnetic measurements on axisymmetric systems, emphasizing the utility of the two classes of integral relations due to Zakharov and Shafranov. These relations have been extended to take full account of pressure anisotropy and plasma rotation, and an inventory of analytical forms has been provided. The main emphasis in this work, however, has been placed on a reconsideration and development of the relevant numerical methods.

For the interpretation of magnetic signals we indicated a target code performance less than 1 msec for the estimation of a set of global parameters characterizing at least the plasma position, shape, pressure and internal inductance, and of about 20 msec for a full equilibrium determination. The 1 msec timescale is below the skin time of the vacuum vessel in present experiments (e.g. 8 msec for Asdex, 3 msec for JET), and is therefore relevant to active feedback control of the plasma. As most tokamak experiments in operation strive to operate near the beta limit, or instability threshold, the need for accurate plasma control is particularly acute.

The ~ 20 msec timescale which we believe is needed for a full 2-D equilibrium determination is still above what is required for active position control, but may be relevant to the programming of auxiliary heating, and of course to rapid inter-shot analysis.
However, a full equilibrium analysis can only provide genuine additional information over that delivered by the fast specialized methods if additional diagnostics are taken into account, and the codes for rapid analysis of a spectrum of diagnostic systems remain to be developed. The immediate aim of such an effort should be to achieve the routine determination after every shot of the complete time-evolution of the 2-D equilibrium configuration throughout the discharge, and this aim appears realistic using presently available laboratory computers.

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Appendix 3.A. Transformation of the Free-Field Boundary Conditions

In the case when the field due to the currents in external coils is specified, the proper boundary conditions for the equilibrium problem demand regularity as \( r \to 0 \) and as \( |r| \to \infty \) for that part of the flux function that is due to currents in the plasma. On a finite computational domain \( \Omega \), which must completely enclose the plasma, these conditions may be replaced by an integral equation relating \( \psi \) and \( \partial \psi / \partial n \) on the boundary \( \partial \Omega \). A method of this nature has been developed by Von Hagenow.
and Lackner [26], but in their formulation it is required to solve an auxiliary problem having homogeneous boundary conditions. This Appendix shows a somewhat different approach.

A boundary integral equation. Let $\psi_{\text{ext}}$ be the (known) flux function due to the currents in the external coils, and define $\psi_{\text{int}} = \psi - \psi_{\text{ext}}$. Let $G(r, r')$ be the Green function for the problem in the infinite domain: $L^*G = \mu r' \delta(r - r')$ for each fixed $r'$, and $G$ vanishes as $r \to 0$ and as $|r| \to \infty$. We recall Green's second identity for $L^*$ on $\Omega$, Eq. (8), and note that a similar relation, with only a change of sign for the normal derivative on $\partial\Omega$, holds for $L^*$ on the exterior region $\Omega_{\text{ext}}$. By application of Green's second identity to the functions $\psi_{\text{int}}$ and $G(r, r')$ on $\Omega_{\text{ext}}$, on which $L^*\psi_{\text{int}} = 0$, one obtains,

$$c(r')\psi_{\text{int}}(r') + \int_{\partial\Omega} r^{-1} \mu^{-1} \psi_{\text{int}} \frac{\partial G}{\partial n} \, ds = \int_{\partial\Omega} r^{-1} \mu^{-1} G \frac{\partial \psi_{\text{int}}}{\partial n} \, ds, \quad (A.1)$$

where

$$c(r') = \begin{cases} 
0, & \text{if } r' \text{ is an interior point of } \Omega, \\
\frac{\varphi(r')}{2\pi}, & \text{if } r' \in \partial\Omega, \\
1, & \text{if } r' \text{ is an interior point of } \Omega_{\text{ext}}.
\end{cases} \quad (A.2)$$

in which $\varphi(r')$ is the exterior angle subtended by $\partial\Omega$ at the point $r' \in \partial\Omega$. Similarly, by application of Green's second identity to the functions $\psi_{\text{ext}}$ and $G(r, r')$ on $\Omega$, one finds,

$$-(1 - c(r'))\psi_{\text{ext}}(r') + \int_{\partial\Omega} r^{-1} \mu^{-1} \psi_{\text{ext}} \frac{\partial G}{\partial n} \, ds = \int_{\partial\Omega} r^{-1} \mu^{-1} G \frac{\partial \psi_{\text{ext}}}{\partial n} \, ds, \quad (A.3)$$

and combining these two relations,

$$c(r')\psi(r') + \int_{\partial\Omega} r^{-1} \mu^{-1} \psi \frac{\partial G}{\partial n} \, ds = \int_{\partial\Omega} r^{-1} \mu^{-1} G \frac{\partial \psi}{\partial n} \, ds + \psi_{\text{ext}}(r'), \quad (A.4)$$

which, when specialized to the case $r' \in \partial\Omega$, is the desired integral equation to connect $\psi$ and $\partial \psi / \partial n$ on $\partial\Omega$. Alternatively, if $\psi_{\text{int}}$ instead of $\psi$ is selected as the unknown field, then Eq. (A.1) specialized to $r' \in \partial\Omega$ may be used as the boundary condition.

102 3. The Interpretation of Tokamak Magnetic Diagnostics
**Discretization.** A natural discretization of the boundary condition in Eq. (A.1) or (A.4) has the form of a matrix equation connecting the values of $\psi$ on the boundary points of the mesh to the values of $\psi$ on those interior points that border on the boundary. Let $\vec{\psi}_0$ denote the vector of values of $\psi$ on the boundary mesh points, taken in some definite order, and let $\vec{\psi}_1$ denote the vector of values of $\psi$ on the neighbouring interior points. Then the discretization of the boundary conditions, Eq. (A.4), should obtain the form,

$$\vec{\psi}_0 = A\vec{\psi}_1 + b. \quad (A.5)$$

In the case of Eq. (A.1) the inhomogeneous term, $b$, will be absent. That this form of a discrete boundary condition is reasonable follows also from the well-posedness of the exterior problem for the equation $A'\psi = -\mu r j_r$ on $\Omega_{ext}$.

In order to obtain an accurate discretization of the above form, the methods discussed in Section 4 are again applicable. These methods are in this case best employed in the defect correction mode, after first obtaining a low order discretization by straightforward analytical procedures.

The boundary condition in the form (A.5) leads to an iterative procedure, and is therefore useful if the equilibrium equation itself is also solved iteratively (as in [74], [75]; classical iterative methods are not to be recommended for this equation). In case the equilibrium equation is solved by a rapid direct method, the treatment of the boundary conditions as given in [26] remains indicated.

**Appendix 3.B. The Homogeneous Equilibrium Equation**

In Section 2 it has been shown that certain moments of the toroidal current density in the region $\Omega$ can be expressed as integrals of linear combinations of the poloidal magnetic field components on the boundary $\partial \Omega$. This theory involves conjugate pairs of solutions to the homogeneous equations, $L^* \chi = 0$ and $L(r^{-1} \mu^{-1} \xi) = 0$, where functions $\chi$ and $\xi$ are called conjugates if Eq. (27) holds. Furthermore, in Section 6 it has been shown how the location of the plasma boundary can be found by fitting a solution of $L^* \chi = 0$ to the external measurements. These two topics demonstrate the need for solutions to the homogeneous equilibrium equation, and in particular for one or more families of solutions that are complete on some appropriate domain. Whereas the evaluation of moments of the current density requires only solutions valid on the
interior region $\Omega$, for the identification of the plasma boundary also solutions valid on $\Omega_{ext}$, and corresponding to a current density inside $\Omega$, are required.

In the important practical case when the magnetic permeability is constant throughout $\Omega$, the equations reduce to $\Delta^* \chi = 0$ and $\Delta (r^{-1} \xi) = 0$, and analytical solutions can be found. This Appendix provides solutions in the form of polynomials and other elementary functions in $r$ and $z$, and solutions obtained by separation of variables in cylindrical, spherical, and toroidal coordinates. In each case pairs of conjugate functions $\chi$ and $\xi$ are given. As $\Delta^* \chi = r \Delta (r^{-1} \chi) - r^{-2} \chi$, it is seen that each solution to $\Delta^* \chi = 0$ corresponds to an \( \"m = \pm 1\" \) solution, $r^{-1} \chi$, to the three-dimensional Laplace equation, where $m$ is the toroidal mode number. Of course, $r^{-1} \xi$ is an \( \"m = 0\" \) solution to the Laplace equation.

Solutions involving elementary functions. The operators $\Delta^*$ and $\Delta$ both have the property that they will transform a homogeneous polynomial of degree $n$ in $r$ and $z$ into a homogeneous polynomial of degree $n - 2$. It follows that, if $\Delta^* \chi = 0$ and if $\chi$ admits a power series expansion about the origin, then by collecting terms of like order, a family of homogeneous polynomial solutions to the homogeneous equilibrium equation will be generated. Such a family may therefore be expected to exist, and indeed by elementary analysis it is found that the following pairs $(\chi_n, \xi_n)$ are all solutions to $\Delta^* \chi = 0$ and to $\Delta (r^{-1} \xi) = 0$, and satisfy the conjugacy relation, Eq. (27) or (29).

\[
\begin{align*}
\chi_0 &= 1, \quad \xi_0 = 0 \\
\chi_1 &= 0, \quad \xi_1 = r \\
\chi_2 &= \frac{1}{2} r^2, \quad \xi_2 = rz \\
\chi_3 &= r^2 z, \quad \xi_3 = r z^2 - \frac{1}{2} r^3 \\
\chi_4 &= \frac{3}{2} r^2 z^2 - \frac{3}{8} r^4, \quad \xi_4 = r z^3 - \frac{3}{2} r^3 z \\
\chi_5 &= 2 r^2 z^3 - \frac{3}{8} r^4 z, \quad \xi_5 = r z^4 - 3 r^3 z^2 + \frac{3}{8} r^5 \\
\chi_6 &= \frac{5}{2} r^2 z^4 - \frac{15}{4} r^4 z^2 + \frac{5}{16} r^6, \quad \xi_6 = r z^5 - 5 r^3 z^3 + \frac{15}{8} r^5 z
\end{align*}
\]

etc. The general formula, for $n > 0$ is,
The solution for $n = 0$ does not fit well into this scheme. The moments for $n = 2$, $n = 4$, and $n = 6$ were given by Zakharov and Shafranov [27], who employ a different normalization of the functions. The motivation for the present choice of normalization will become clear when we discuss the moments obtained by separation of variables in spherical coordinates. Notice that the relevant Eq. (61) of [27] contains two errors, only one of which is obvious.

A related family of elementary solutions can be found by allowing a factor $\ln r$. There results the sequence:

$$
\chi_1^f = r, \quad \xi_1^f = r \ln r,
$$

$$
\chi_2^f = \chi_2 \ln r - \frac{1}{2} z^2 - \frac{1}{4} r^2, \quad \xi_2^f = \xi_2 \ln r,
$$

$$
\chi_3^f = \chi_3 \ln r - \frac{1}{3} z^3 - \frac{1}{2} r^2 z, \quad \xi_3^f = \xi_3 \ln r + \frac{1}{2} r^3.
$$

$$
\chi_4^f = \chi_4 \ln r - \frac{1}{4} z^4 - \frac{3}{4} r^2 z^2 + \frac{15}{32} r^4, \quad \xi_4^f = \xi_4 \ln r + \frac{3}{2} r^3 z,
$$

etc. The general formula, for $n \neq 0$ is,

$$
\chi_n^f = \sum_{k=0}^{[n/2]-1} (-4)^{-k} \frac{(n - 1)!/2}{k! (k + 1)! (n - 2k - 2)!} \times r^{2k-2} \cdot z^n \cdot 2^{k-2} \left( \ln r - \sum_{j=1}^{k} \frac{1}{2(k + 1)} \right) \frac{1}{n^2}
$$

$$
\xi_n^f = \sum_{k=0}^{[n/2]-1} (-4)^{-k} \frac{(n - 1)!}{(k!)^2 (n - 2k - 1)!} \times r^{2k+1} \cdot z^{n-2k-1} \left( \ln r - \sum_{j=1}^{k} \frac{1}{j} \right)
$$

Other elementary solutions can be found by allowing a power of $\sqrt{r^2 + z^2}$. For $n > 1$ the pair

$$
\begin{align*}
\chi &= \frac{n}{1-n} (r^2 + z^2)^{1/2-n} \chi_n \\
\xi &= (r^2 + z^2)^{1/2-n} \xi_n
\end{align*}
$$

Appendix 3.B.
also provides a solution.

The polynomial solutions can be combined in a form that yields approximations to the lowest order multipole moments about an expansion point \((R_0, 0)\), for arbitrary \(R_0 > 0\):

\[
f_n^{(\text{even})} = (-R_0)^n 2^{-n} \left[\chi_0 - \sum_{m=1}^{n} \left(\frac{2}{R_0}\right)^{2m} \frac{n! m!}{(2m)! (n-m)!} \chi_{2m}\right]
\]

\[
f_n^{(\text{odd})} = (-R_0)^{n-1} 2^{-n} \sum_{m=0}^{n} \left(\frac{2}{R_0}\right)^{2m} \frac{n! m!}{(2m)! (n-m)!} \chi_{2m+1}
\]

The functions \(g_n^{(\text{even})}\) and \(g_n^{(\text{odd})}\) are similarly defined with replacement of \(\chi\) by \(\xi\) throughout. In leading order, \(f_n^{(\text{even})} + if_n^{(\text{odd})} \simeq (x + iz)^n\) and \(g_n^{(\text{even})} + ig_n^{(\text{odd})} \simeq -i(x + iz)^n\), where \(x = r - R_0\) and \(i = \sqrt{-1}\). On the symmetry plane, \(z = 0\), the exact formulae are

\[
f_n^{(\text{even})} = (1 - x/2R_0)^n x^n,
\]

\[
g_n^{(\text{odd})} = -(1 + x/R_0)(1 + x/2R_0)^n x^n.
\]

while \(g_n^{(\text{even})}\) and \(f_n^{(\text{odd})}\) vanish for \(z = 0\). We do not have a simple formula for these polynomials that is valid also for \(z \neq 0\).

The corresponding approximations to the multipole moments about an arbitrary expansion point \((R_0, Z_0)\) are obtained by substituting \(\chi(r, z - Z_0)\) for \(\chi\) in Eqs. (B.6) and (B.7), together with a similar substitution for \(\xi\), and then employing the representations,

\[
\chi_n(r, z - Z_0) = \sum_{k=1}^{n} (-Z_0)^{n-k} \frac{(n-1)!}{(n-k)! (k-1)!} \chi_k(r, z).
\]

\[
\xi_n(r, z - Z_0) = \sum_{k=1}^{n} (-Z_0)^{n-k} \frac{(n-1)!}{(n-k)! (k-1)!} \xi_k(r, z).
\]

As the multipoles \((x + iy)^n\), where \(x = r - R_0\) and \(y = z - Z_0\), form a complete family of solutions to the Laplace equation on any disk centered at \((R_0, Z_0)\), one may be led to assume that the polynomial solutions given in Eq. (B.2) must also form a complete family on such a disk. This is not the case, however, and the fallacy in the reasoning is that the approximation of Eqs. (B.6) and (B.7) to the real and imaginary parts of \((x + iy)^n\) is not uniform in \(n\). We will see later that the polynomial solutions correspond to a certain family of solutions obtained by separation of variables in spherical coordinates. These polynomial solutions form a complete family on any
circular region centered at the origin, which may be of interest for compact toroids or for spheromaks, but hardly for tokamak studies. It has been suggested [30] that the polynomial solutions may be combined with solutions involving negative powers of $p = \sqrt{r^2 + z^2}$, as in Eq. (B.5), to form a complete family. Such a family, however, will only be complete on a region of the form $\rho_a < p < \rho_b$, which is not of interest. Useful complete families of analytical solutions are only obtained by separation of variables in toroidal or in cylindrical coordinates.

**Solutions from separation in cylindrical coordinates.** Separated variables solutions to the Laplace equation in cylindrical coordinates involve Bessel functions and modified Bessel functions of $r$ together with hyperbolic and trigonometric functions of $z$. To obtain separated solutions to $\Delta \chi = 0$ or to $\Delta (r^{-1} \xi) = 0$ the indicated ansatz is $\chi = rf(r)g(z)$ or $\xi = rf(r)g(z)$, which leads to the pair of ordinary differential equations:

\[
\begin{align*}
  r^2 f'' + rf' + (\kappa r^2 - m^2)f &= 0 \\
  g'' - \kappa g &= 0
\end{align*}
\]

where $m^2 = 1$ for the solution $\chi$, and $m^2 = 0$ for $\xi$. There are three classes of solutions, depending upon the choice of sign of the separation constant $\kappa$.

(a) $\kappa = \lambda^2$, with $\lambda < 0$. Then,

\[
\begin{align*}
  \chi &= r(c_1 J_1(\lambda r) + c_2 Y_1(\lambda r))(c_3 \cosh(\lambda z) + c_4 \sinh(\lambda z)) \\
  \xi &= r(c_1 J_0(\lambda r) + c_2 Y_0(\lambda r))(c_3 \sinh(\lambda z) + c_4 \cosh(\lambda z))
\end{align*}
\]

$J$ and $Y$ are Bessel functions, in the notation of [18], [19]. Here and elsewhere $c_1$, $c_2$, $c_3$, and $c_4$ are arbitrary real constants, and the functions $\chi$ and $\xi$ given as a pair are conjugate functions in the sense of Eq. (27).

(b) $\kappa = -\lambda^2$, with $\lambda > 0$. Then,

\[
\begin{align*}
  \chi &= r(c_1 I_1(\lambda r) - c_2 K_1(\lambda r))(c_3 \cos(\lambda z) - c_4 \sin(\lambda z)) \\
  \xi &= r(c_1 I_0(\lambda r) + c_2 K_0(\lambda r))(c_3 \sin(\lambda z) + c_4 \cos(\lambda z))
\end{align*}
\]

$I$ and $K$ are modified Bessel functions. The families (a) and (b), restricted to a discrete set of values of the separation constant, may be used to provide a system that is complete on any desired rectangular region ($r_a < r < r_b$ and $z_a < z < z_b$) in the right half plane.

*Appendix 3.B.*
(c) $\kappa = 0$. This yields again the simplest polynomial solutions.

$$\chi = (c_1 + c_2 r^2)(c_3 + c_4 z)$$

$$\xi = (d_1 r + d_2 r \ln r)(d_3 + d_4 z)$$

(B.13)

These are not directly conjugate to each other, but Eq. (B.1) and Eq. (B.3) may be consulted to find the conjugate functions.

All these solutions may be expanded in terms of the solutions $(\chi_n, \xi_n)$ and $(\chi^f_n, \xi^f_n)$ given earlier, by the use of the following formulae.

$$r J_0(\lambda r) \exp(\lambda z) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \chi_n$$

$$r J_1(\lambda r) \exp(\lambda z) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \chi_n$$

$$r Y_0(\lambda r) \exp(\lambda z) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \left(\frac{\xi_n'}{\lambda_n} + (\gamma + \ln(\lambda/2))\xi_n\right) - \frac{2}{\pi} \lambda^{-1} \xi_0$$

$$r Y_1(\lambda r) \exp(\lambda z) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \left(\frac{\chi_n'}{\lambda_n} + (\gamma + \ln(\lambda/2))\chi_n\right) - \frac{2}{\pi} \lambda^{-1} \chi_0$$

$$r I_0(\lambda r) \exp(i\lambda z) = \sum_{n=1}^{\infty} \frac{(i\lambda)^{n-1}}{(n-1)!} \xi_n$$

$$r I_1(\lambda r) \exp(i\lambda z) = -i \sum_{n=1}^{\infty} \frac{(i\lambda)^{n-1}}{(n-1)!} \chi_n$$

$$r K_0(\lambda r) \exp(i\lambda z) = -\sum_{n=1}^{\infty} \frac{(i\lambda)^{n-1}}{(n-1)!} \left(\frac{\xi_n'}{\lambda_n} + (\gamma - \ln(\lambda/2))\xi_n\right)$$

$$r K_1(\lambda r) \exp(i\lambda z) = -i \sum_{n=1}^{\infty} \frac{(i\lambda)^{n-1}}{(n-1)!} \left(\frac{\chi_n'}{\lambda_n} + (\gamma - \ln(\lambda/2))\chi_n\right)$$

The solutions obtained by separation in cylindrical coordinates are appropriate for very elongated (belt-shaped) configurations and for devices with rectangular cross-section, and may also be convenient for certain modelling studies involving a rectangular grid.

108 3. The Interpretation of Tokamak Magnetic Diagnostics
Solutions from separation in spherical coordinates. In spherical coordinates, 
\( r = \rho \sin \vartheta, \quad z = \rho \cos \vartheta \), the representation for \( \Delta' \) is,

\[
\Delta' \chi = \frac{\partial^2 \chi}{\partial \rho^2} + \frac{1}{\rho^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \frac{1}{\sin \vartheta} \frac{\partial \chi}{\partial \vartheta} \right),
\]  

(B.14)

and the relation between the conjugate functions \( \chi \) and \( \xi \) is,

\[
\begin{align*}
\frac{\partial}{\partial \rho} (r^{-1} \xi) &= r^{-1} \rho^{-1} \frac{\partial \chi}{\partial \vartheta} \\
\frac{\partial}{\partial \vartheta} (r^{-1} \xi) &= -r^{-1} \rho \frac{\partial \chi}{\partial \rho}
\end{align*}
\]  

(B.15)

Starting from the ansatz \( \chi = \rho \sin \vartheta f(\rho)g(\vartheta) \), and a similar ansatz for \( \xi \), one obtains for \( \Delta' \chi = 0 \) and \( \Delta (r^{-1} \xi) = 0 \) the systems,

\[
\begin{align*}
\rho^2 f'' + 2\rho f' - \kappa f &= 0 \\
g''(\vartheta) + \frac{\cos \vartheta}{\sin \vartheta} g'(\vartheta) + \left( \kappa - \frac{m^2}{(\sin \vartheta)^2} \right) g(\vartheta) &= 0
\end{align*}
\]  

(B.16)

with separation constant \( \kappa \in \mathbb{R} \). \( m^2 = 1 \) corresponds to the equation for \( \chi \), and \( m^2 = 0 \) corresponds to \( \xi \). Three cases must be distinguished: \( \kappa = \frac{1}{4} \), \( \kappa = -\frac{1}{4} \), and \( \kappa = -\frac{1}{4} \).

In the case \( \kappa = \frac{1}{4} \), there exists a real \( \alpha = \frac{1}{2} \) such that \( \kappa = \alpha(\alpha - 1) \). The first equation then admits the two independent solutions \( f = \rho^{\alpha-1} \) and \( f = \rho^{-\alpha} \). The second equation admits for almost every value of \( \alpha \) (namely \( \alpha \) not an integer in the range \( 1 - m < \alpha < m \)) the following two independent solutions.

\[
g(\vartheta) = P_{\alpha-1}^m(\cos \vartheta) \quad \text{(notation of [18], [19])}
\]

\[
= (-1)^m \frac{\Gamma(m + \alpha)}{m! \Gamma(\alpha - 1)} (\tan \frac{1}{2} \vartheta)^m \left( \frac{1}{2} \right)^m F \left( 1 - \alpha, \alpha; 1 + m; (\sin \frac{1}{2} \vartheta)^2 \right)
\]

\[
= (-1)^m \frac{\Gamma(m + \alpha)}{2^m m! \Gamma(\alpha - 1)} (\sin \vartheta)^m \left( \frac{1 - \alpha + m}{2}, \frac{\alpha - m}{2}; 1 + m; (\sin \vartheta)^2 \right)
\]

and

\[
g(\vartheta) = Q_{\alpha-1}^m(\cos \vartheta) \quad \text{(notation of [18], [19])}
\]

\[
= (-1)^m Q_{\alpha-1}^m(\cos \vartheta) \quad \text{(notation of [89])}
\]  

(B.18)

Appendix 3.B.
\( F \) denotes the \( {}_2F_1 \) hypergeometric function. In the sequel we follow the notation of [18] and [19] for the Legendre functions \( P \) and \( Q \). The solutions given in Eqs. (B.17) and (B.18) are invariant under the replacement of \( \alpha \) by 1 - \( \alpha \). One therefore obtains the following two families of conjugate solutions to the equations \( \Delta \chi = 0 \) and \( \Delta (r^{-1}\xi) = 0 \):

\[
\begin{align*}
\chi(\rho, \vartheta) &= -\frac{1}{\alpha} \rho^\alpha \sin \vartheta P_{\alpha-1}^1(\cos \vartheta) \\
\xi(\rho, \vartheta) &= \rho^\alpha \sin \vartheta P_{\alpha-1}^0(\cos \vartheta)
\end{align*}
\tag{B.19}
\]

and

\[
\begin{align*}
\chi(\rho, \vartheta) &= -\rho^\alpha \sin \vartheta Q_{\alpha-1}^1(\cos \vartheta) \\
\xi(\rho, \vartheta) &= \alpha \rho^\alpha \sin \vartheta Q_{\alpha-1}^0(\cos \vartheta)
\end{align*}
\tag{B.20}
\]

for \( \alpha \in \mathbb{R}, \alpha \neq \frac{1}{2} \). The limiting values of these solutions as \( \alpha \to 0 \) exist and are given by

\[
\begin{align*}
\chi(\rho, \vartheta) &= \cos \vartheta - 1 \\
\xi(\rho, \vartheta) &= \sin \vartheta
\end{align*}
\tag{B.21}
\]

and

\[
\begin{align*}
\chi(\rho, \vartheta) &= 1 \\
\xi(\rho, \vartheta) &= 0
\end{align*}
\tag{B.22}
\]

For \( \alpha = 1 \) the solutions are,

\[
\begin{align*}
\chi(\rho, \vartheta) &= 0 \\
\xi(\rho, \vartheta) &= \rho \sin \vartheta
\end{align*}
\tag{B.23}
\]

and

\[
\begin{align*}
\chi(\rho, \vartheta) &= \rho \\
\xi(\rho, \vartheta) &= -\rho \sin \vartheta \ln(\tan \frac{1}{2} \vartheta)
\end{align*}
\tag{B.24}
\]

Notice that for \( \alpha = n \) and \( n = 1, 2, 3, \ldots \) the solution pair defined by Eq. (B.20) coincides with the polynomial solutions \( (\chi_n, \xi_n) \) given earlier. This is the motivation for our normalization in Eq. (B.2).

In the case \( \kappa < -\frac{1}{4} \) one may set \( \kappa = (i\alpha + \frac{1}{2})(i\alpha - \frac{1}{2}) \) for some real \( \alpha > 0 \). The differential equation for \( f \) then admits the solutions \( f = \sin(\alpha \ln \rho)/\sqrt{\rho} \) and \( f = \cos(\alpha \ln \rho)/\sqrt{\rho} \).
Solutions to the equation for $g$ are the conical functions, viz. Legendre functions of degree $i\alpha - \frac{1}{2}$. We thus obtain the solutions,

$$\begin{align*}
\chi(\rho, \vartheta) &= -2\sqrt{\rho} \left(c_{1} \sin(\alpha \ln \rho) + c_{2} \cos(\alpha \ln \rho)\right) \sin \vartheta P_{i\alpha - 1}^{1}(\cos \vartheta) \\
\xi(\rho, \vartheta) &= \sqrt{\rho} \left(d_{1} \sin(\alpha \ln \rho) + d_{2} \cos(\alpha \ln \rho)\right) \sin \vartheta P_{i\alpha - 1}^{0}(\cos \vartheta)
\end{align*}$$

and

$$\begin{align*}
\chi(\rho, \vartheta) &= -2\sqrt{\rho} \left(c_{1} \sin(\alpha \ln \rho) + c_{2} \cos(\alpha \ln \rho)\right) \sin \vartheta Q_{i\alpha - 1}^{1}(\cos \vartheta) \\
\xi(\rho, \vartheta) &= \sqrt{\rho} \left(d_{1} \sin(\alpha \ln \rho) + d_{2} \cos(\alpha \ln \rho)\right) \sin \vartheta Q_{i\alpha - 1}^{0}(\cos \vartheta)
\end{align*}$$

(B.25) and (B.26)

where $\chi$ and $\xi$ are conjugate functions provided that $d_{1} = c_{1} - 2\alpha c_{2}$ and $d_{2} = c_{2} + 2\alpha c_{1}$.

The case $\kappa = -\frac{1}{4}$ can be dealt with by taking the appropriate limits of the solutions found earlier. A complete system of solutions for this case is given by,

$$\begin{align*}
\chi(\rho, \vartheta) &= -2\sqrt{\rho} \left(c_{1} + c_{2} \ln \rho\right) \sin \vartheta \left(c_{3} P_{-\frac{1}{2}}^{1}(\cos \vartheta) + c_{4} Q_{-\frac{1}{2}}^{1}(\cos \vartheta)\right) \\
\xi(\rho, \vartheta) &= \sqrt{\rho} \left(d_{1} + d_{2} (\ln \rho + 2)\right) \sin \vartheta \left(c_{3} P_{-\frac{1}{2}}^{0}(\cos \vartheta) + c_{4} Q_{-\frac{1}{2}}^{0}(\cos \vartheta)\right)
\end{align*}$$

(B.27)

Solutions from separation in toroidal coordinates. For the study of tokamak configurations the most generally useful family of analytical solutions is obtained by separation of variables in a toroidal coordinate system. These coordinates are defined by,

$$\begin{align*}
\rho &= r_{0} \sinh \zeta \cdot (\cosh \zeta - \cos \eta) \\
z &= z_{0} + r_{0} \sin \eta / (\cosh \zeta - \cos \eta)
\end{align*}$$

(B.28)

$(r_{0}, z_{0})$ being the location of the singularity of the coordinate system, at $\zeta = \infty$. The operator $\Delta^{*}$ is,

$$\Delta^{*} \chi = \cosh \zeta \cdot \cos \eta \left(\frac{\sinh \zeta}{r_{0}} \frac{\partial}{\partial \zeta} \left(\frac{\cosh \zeta - \cos \eta}{r_{0} \sinh \zeta} \frac{\partial \chi}{\partial \zeta}\right) + \frac{\partial}{\partial \eta} \left(\frac{\cosh \zeta - \cos \eta}{r_{0}} \frac{\partial \chi}{\partial \eta}\right)\right)$$

(B.29)

and the connection between the conjugate functions $\chi$ and $\xi$ is,

$$\begin{align*}
\frac{\partial}{\partial \zeta} (r^{-1} \xi) &= r^{-1} \frac{\partial \chi}{\partial \eta} \\
\frac{\partial}{\partial \eta} (r^{-1} \xi) &= -r^{-1} \frac{\partial \chi}{\partial \zeta}
\end{align*}$$

(B.30)

Appendix 3.B.
As $\Delta(r^{-1} \chi) = 0$ separates with the substitution $r^{-1} \chi = (\cosh \zeta - \cos \eta)^{\frac{1}{2}} f(\zeta) g(\eta)$, one is motivated to try for $\Delta' \chi = 0$ the ansatz,

$$\chi = \frac{r_0 \sinh \zeta}{\sqrt{\cosh \zeta - \cos \eta}} f(\zeta) g(\eta).$$

and a similar ansatz for $\xi$ in the equation $\Delta(r^{-1} \xi) = 0$. This leads to the separated equations,

$$\begin{aligned}
\left\{ \begin{array}{c}
\frac{1}{\sinh \zeta} \frac{\partial}{\partial \zeta} \left( \sinh \zeta \frac{\partial f}{\partial \zeta} \right) - \frac{m^2}{(\sinh \zeta)^2} f - \left( n^2 - \frac{1}{4} \right) f = 0 \\
\frac{\partial^2 g}{\partial \eta^2} + n^2 g = 0
\end{array} \right. \\
\text{(B.31)}
\end{aligned}$$

where $n^2$ is the separation constant, and $m^2 = 1$ for the solution $\chi$, $m^2 = 0$ for the solution $\xi$. We assume that the point $(r = r_0, z = z_0)$ should lie inside the domain of the solution, so that $\eta$ is a periodic coordinate, and $n$ must be an integer.

The solutions for $g(\eta)$ are known. Solutions to the equation for $f(\zeta)$, when $m = 0$ or $m = 1$, are of two forms:

$$f(\zeta) = \begin{cases}
P_n^{m-\frac{1}{2}}(\cosh \zeta) & \text{(notation of [18], [19])} \\
(-i)^m P_n^{m-\frac{1}{2}}(\cosh \zeta) & \text{(notation of [89])}
\end{cases}$$

$$= \frac{\Gamma(n + m + \frac{1}{2})}{2^m m! \Gamma(n - m + \frac{1}{2})} (\tanh \zeta)^m (\cosh \zeta)^{\frac{1}{2}}$$

$$\times F\left(\frac{n - m + \frac{1}{2}}{2}; \frac{n - m + \frac{1}{2}}{2}; m + 1; (\tanh \zeta)^2 \right)$$

$$= \frac{\Gamma(n + m + \frac{1}{2})}{2^m m! \Gamma(n - m + \frac{1}{2})} (1 - e^{-2\zeta})^m (\exp \zeta)^{\frac{n}{2}}$$

$$\times F\left(n + m + \frac{1}{2}; m + \frac{1}{2}; 2m + 1; 1 - e^{-2\zeta} \right)$$

which is singular at $\zeta \to \infty (r = a, z = 0)$, and

$$f(\zeta) = \begin{cases}
Q_n^{m-\frac{1}{2}}(\cosh \zeta) & \text{(notation of [18], [19])} \\
(-1)^m Q_n^{m-\frac{1}{2}}(\cosh \zeta) & \text{(notation of [89])}
\end{cases}$$

$$= (-1)^m \sqrt{\frac{\pi}{2}} \frac{\Gamma(n + m + \frac{1}{2})}{2^{m+\frac{1}{2}} \Gamma(n+1)} (\tanh \zeta)^m (\cosh \zeta)^{-n-\frac{1}{2}}$$

3. The Interpretation of Tokamak Magnetic Diagnostics
which is regular throughout $\Omega$. In the sequel we will stay with the notation of Refs. [18] and [19]. Notice that both $P_{n,-\frac{1}{2}}^m$ and $Q_{n,-\frac{1}{2}}^m$ are invariant under the replacement of $n$ by $-n$. Computer routines for the evaluation of the half-order Legendre functions are not presently available in any of the major scientific subroutine libraries (NAG, SLATEC, IMSL), but the partial set of approximations given in Ref. [90] may be useful.

It may appear strange that the definitions of $P_{n}^m$ in our references should differ by a factor $(-1)^m$ when the argument is $\cos \theta$ and by a factor $(-i)^m$ when the argument is $\cosh \zeta$. This is nevertheless the case, and is related to the presence of branch points at $\pm 1$. Other authors have argued the merits of (re-)defining the special functions in such a way that they are free of unnecessary singularities and branch points.

A complete family of solutions to $\Delta \chi = 0$ in any toroidal region of the form $\zeta = \zeta_0$ is therefore obtained with the functions,

$$\chi_n^c = \frac{r_0 \sinh \zeta}{\sqrt{\cosh \zeta - \cos \eta}} Q_{n,-\frac{1}{2}}^1 (\cosh \zeta) \cos(n\eta), \quad n \leq 0$$

$$\chi_n^s = \frac{r_0 \sinh \zeta}{\sqrt{\cosh \zeta - \cos \eta}} Q_{n,-\frac{1}{2}}^1 (\cosh \zeta) \sin(n\eta), \quad n \geq 1$$

The conjugate functions, $\xi$, must satisfy $\Delta (r^{-1} \xi) = 0$, and a complete family of solutions to this equation is provided by the functions,

$$\xi_n^c = \frac{r_0 \sinh \zeta}{\sqrt{\cosh \zeta - \cos \eta}} Q_{n,-\frac{1}{2}}^0 (\cosh \zeta) \cos(n\eta), \quad n \leq 0$$

$$\xi_n^s = \frac{r_0 \sinh \zeta}{\sqrt{\cosh \zeta - \cos \eta}} Q_{n,-\frac{1}{2}}^0 (\cosh \zeta) \sin(n\eta), \quad n \geq 1$$

It turns out, however, that the conjugacy relation, Eq. (B.30), does not lead to a simple one-to-one correspondence between the members of Eq. (B.34) and those of Eq. (B.35). The conjugate function to a single member of Eq. (B.34) can be an infinite series from

Appendix 3.B. 113
Eq. (B.35) and vice versa. The following definition for the basis functions \( \chi \) and \( \xi \) has been chosen in order to restore the symmetry.

\[
\begin{align*}
\chi_n^{(1)}(\zeta, \eta) & = \chi_n^c - \chi_{n+1}^c \\
\xi_n^{(1)}(\zeta, \eta) & = -(n + \frac{1}{2})(\xi_n^s - \xi_{n+1}^s)
\end{align*}
\]

(B.36)

and

\[
\begin{align*}
\chi_n^{(2)}(\zeta, \eta) & = \chi_n^s - \chi_{n+1}^s \\
\xi_n^{(2)}(\zeta, \eta) & = (n + \frac{1}{2})(\xi_n^c - \xi_{n+1}^c)
\end{align*}
\]

(B.37)

for \( n \geq 0 \), with the understanding that \( \chi_0^s = 0 \) and \( \xi_0^s = 0 \).

**Appendix 3.C. Determination of the Current Distribution From the Flux Surface Structure**

An interesting different approach to plasma current profile identification has been proposed by Christiansen and Taylor [68]. They discuss the possibility of determining the current profile from purely geometric information about the shape of the magnetic surfaces, as may be available from electron temperature measurements for instance. They conclude that for a toroidal configuration this determination is always possible, but the analysis below shows their argument to be incorrect, and clarifies under which circumstances the procedure of [68] will be well-conditioned. This analysis will be seen to be of interest for the question of uniqueness of a fit to the magnetic measurements, and is also relevant to MHD equilibrium determination from an extended set of diagnostics.

A differential equation for the flux function. Assume that some function \( \sigma \), known to be a flux-surface quantity, has been measured over the poloidal cross-section of the plasma. Thus the flux surface structure of the plasma is known as the isocontours of \( \sigma \), but the functional relationship between \( \psi \) and \( \sigma \) is as yet undetermined. Notice that \( \sigma \) must for consistency satisfy a certain differential equation: as

\[
\Delta^* \sigma = \frac{d\sigma}{d\psi} \Delta^* \psi + \frac{d^2\sigma}{d\psi^2} |\nabla \psi|^2
\]

\[
= -\frac{d\sigma}{d\psi} F \frac{dF}{d\psi} - \mu_0 \frac{d\sigma}{d\psi} \frac{dp}{d\psi} r^2 + \frac{d^2\sigma}{d\psi^2} |\nabla \sigma|^2 \left( \frac{d\sigma}{d\psi} \right)^2,
\]

(C.1)
where the equilibrium relation, Eq. (16), has been used, it follows that

\[ \Delta^* \sigma = -\alpha - \beta r^2 - \gamma |\nabla \sigma|^2, \]

where \( \alpha, \beta, \) and \( \gamma \) are flux surface quantities. Specifically, \( \alpha = \sigma'F'F' \), \( \beta = \mu_0 \sigma'p' \), and \( \gamma = -\sigma''/(\sigma')^2 \), where \( ' \) denotes \( \partial/\partial \psi \). The representation of \( \Delta^* \sigma \) in Eq. (C.2) is unique, and the functions \( \alpha, \beta, \) and \( \gamma \) are determinable from knowledge of the function \( \sigma \), provided that \( |\nabla \sigma|^2 \) is not, on any flux surface, linearly dependent on \( \sigma \) and \( r^2 \).

One may then consider \( \psi \) as a function of \( \sigma \), and derive from \( \gamma = -\sigma''/(\sigma')^2 \) the differential equation,

\[ \frac{d}{d\sigma} \ln \left( \frac{d\psi}{d\sigma} \right) = \gamma(\sigma). \]

After two integrations \( \psi \) is obtained as a function of \( \sigma \), and therefore also as a function of \( (r, z) \). Of the two free constants arising from the integrations, one corresponds to the indeterminacy of the total current (indeterminate for this geometric information on its own), and the other is the irrelevant constant term in \( \psi \). It follows that, except in the degenerate case to be discussed below, the current profile in a toroidal configuration can be determined from knowledge of the geometry of the flux surfaces together with a measurement of the total plasma current.

A degenerate configuration. The above procedure is related to the program presented in [68], although in that work a different and not unambiguous route is followed to obtain the function \( \gamma \) in Eq. (C.3). Clearly the procedure fails completely if over an extended range of values of \( \sigma \) there is a linear dependence of \( |\nabla \sigma|^2 \) on \( \sigma \) and \( r^2 \), that is, if \( |\nabla \sigma|^2 = c_1(\sigma) + c_2(\sigma)r^2 \) over a range of values of \( \sigma \), for some flux functions \( c_1 \) and \( c_2 \). If this relation holds only on isolated flux surfaces or only approximately, then the procedure may still be feasible, but will be badly conditioned. These restrictions are also implicit in the derivation employed in [68].

The fact that this degenerate case, \( |\nabla \psi|^2 \) linearly dependent on \( \sigma \) and \( r^2 \) over a range of flux surfaces, can actually occur in a toroidal system, was shown some years ago in a different context by Palumbo [49], [50] and recently rediscovered in the present context [51], [52]. An explicit construction of such an equilibrium is given in those references, and it is shown that the associated flux surface configuration is unique up to a scale factor. This construction is rather complicated; a much simpler argument [J.B. Taylor, private communication, August 1984] for the corresponding problem in

Appendix 3.C.
straight geometry shows that there it is only for concentric circular flux surfaces that $|\nabla \psi|^2$ can be constant on each surface.

Notice that the family of equilibria in [49]-[52] also provides an example to show that in toroidal geometry the external magnetic measurements need not define uniquely the current profile, even if these measurements are made to arbitrary accuracy. A set of measurements that is consistent with one current profile from this family is automatically consistent with any other current profile that gives the same flux surface structure and the same total current.

**Partial profile determination.** The procedure of [68] becomes badly conditioned when $|\nabla \sigma|^2 \simeq c_1(\sigma) + c_2(\sigma)r^2$, and it is of interest to investigate whether partial information about the current profile can then still be obtained from the geometric information. Indeed, in such a degenerate, or in a nearly degenerate case, it follows from Eq. (C.2) that what can be determined correctly is not the three functions $\alpha$, $\beta$, and $\gamma$, but only the combinations $\alpha + \gamma c_1$ and $\beta + \gamma c_2$, where $c_1$ and $c_2$ are known. Eliminating $\gamma$ it is seen that the combination $c_1 \beta - c_2 \alpha$ is always measurable, or equivalently,

$$\eta = \sigma'(c_1 \mu_0 p' - c_2 F F')$$  \hspace{1cm} (C.4)

is measurable.

A limited study of the use of geometric information for current profile determination has been reported [91], but the authors were unaware of the limitations on this method of analysis. On careful reading Ref. [91] does not indicate that for current profile determination, geometric information is a useful addition to the magnetic measurements.

**Final remarks.** We conclude this Appendix with some speculation as to what combination of diagnostic systems, magnetic and other, would be most suitable for an accurate current profile determination. The magnetic diagnostics measure the plasma current $I$, the plasma position $(r_c, z_c)$ as defined in Eqs. (35) and (36), the parameter $\beta_l + l/2$, the location of the plasma boundary, and in good approximation also the position of the magnetic axis. For sufficiently shaped cross-section or with the aid of a diamagnetic flux measurement it is also possible to obtain separate estimates for $\beta_l$ and $l$, but beyond this no information on the radial shape of the current profile is obtained.

The radial shape of the current profile does show up to some extent in Faraday rotation measurements [64]-[67]. Taken in isolation these are difficult to interpret, but it would seem that a limited number of such measurements in conjunction with the
usual magnetic diagnostics should suffice for a rough identification of the shape of the current profile in the interior plasma, and in particular for an accurate separation of $\beta_1$ and $h$. The Faraday rotation measurements can be shown to provide information on the quantity

$$h = r^2 \nabla \cdot (r^{-2} n_e \nabla \psi),$$

via the Radon transform. ($n_e$ is the electron density).

That leaves the problem of obtaining in more detail the separate contributions from the $\mu_0p'$ and the $FF'$ terms in the current profile. Measurements of the pressure would suffice for that purpose, but only the electron component of the pressure is accurately measured. In principle though, some data on the flux surface geometry (most likely from electron temperature measurements) in conjunction with magnetic measurements and Faraday rotation measurements, could serve to separate these two contributions to the current, with the aid of Eq. (C.4). These diagnostics (magnetic, FIR interferometry and polarimetry, and electron cyclotron emission) are all suitable for near continuous monitoring of the plasma, making a successful effort at an efficient consistent interpretation particularly worthwhile.

References


3. The Interpretation of Tokamak Magnetic Diagnostics


60. J. Blum, "Sur quelques Problèmes d’Analyse Numérique et de Contrôle Optimal en Physique des Plasmas" (Thesis). Université Pierre et Marie Curie, Paris 6, 1985; an English translation is to be published by Dunod.


120 3. The Interpretation of Tokamak Magnetic Diagnostics


3. The Interpretation of Tokamak Magnetic Diagnostics
4. FAST DETERMINATION OF PLASMA PARAMETERS THROUGH FUNCTION PARAMETRIZATION

Abstract

The method of function parametrization, developed by H. Wind for fast data evaluation in high energy physics, is demonstrated in the context of controlled fusion research. This method relies on a statistical analysis of a large data base of simulated experiments in order to obtain a functional representation for intrinsic physical parameters of a system in terms of the values of the measurements. Rapid determination of characteristic equilibrium parameters of a tokamak discharge is shown to be a particularly indicated application. The method is employed on the ASDEX experiment to determine the following parameters of the plasma: position of the magnetic axis, geometric centre, and current centre; minor radius, elongation, and area of the plasma column; a normalized safety factor at the plasma boundary; the Shafranov parameter $\delta_p \cdot b_p/2$; the flux difference between the plasma boundary and an external reference value; the position of the lower and upper saddle points, and the intersections of the separatrix with the four divertor plates. The relevant measurements consist of three differential poloidal flux measurements, four poloidal field measurements, the current through the multipole shaping coils, and the total plasma current. The model obtained by function parametrization provides a very accurate interpretation of these diagnostics, which is now used for online data analysis, and which is also sufficiently fast to be suitable for real-time control of the plasma.

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1. Introduction

In the interpretation of tokamak diagnostics, as in many other areas of physics, the amount of experimental information that is utilized is often not limited by the rate at which measurements can be made, but more by the rate at which the raw diagnostics can be interpreted. Hence, efficient methods of data analysis are highly desirable. A very efficient procedure, function parametrization, was developed by H. Wind (CERN) for the purpose of momentum determination from spark chamber data \[1, 2\], and was recently proposed for application in tokamak physics \[3\]. Although this method has not previously been noticed outside the high energy physics field, it has a much wider range of applicability and can be considered whenever many measurements are to be made with the same diagnostic setup. The application described in this paper concerns the determination of characteristic equilibrium parameters from magnetic measurements on a tokamak.

Function parametrization relies on an analysis of a large data set of simulated experiments, aiming to obtain an optimal representation of some simple functional form for intrinsic physical parameters of a system in terms of the values of the measurements. Statistical techniques for dimension reduction and multiple regression are used in the analysis. The resulting function may be chosen to involve only low-order polynomials in only a few linear combinations of the original measurements: this function can therefore be evaluated very rapidly and needs only minimal hardware facilities.

Three steps have to be made for this method of experimental data interpretation. (1) A numerical model of the experiment is used to generate a data base of simulated states of the physical system, in which each state is represented by the values of the relevant physical parameters and of the associated measurements. (2) This data base is made the object of a statistical analysis, with the aim to provide a relatively simple function that expresses the physical parameters in terms of the measurements. (3) The resulting function is then employed for the fast interpretation of real measurements.

Determination of characteristic equilibrium parameters of a magnetically confined plasma is a particularly indicated application. The MHD model provides a well-defined and generally accepted connection between the unknown intrinsic plasma parameters, the externally applied fields, and the magnetic measurements. Identification of the position and the profile of the plasma column is the basis for interpreting practically all other diagnostics and therefore requires an efficient algorithm. Here we describe a successful application of function parametrization for the interpretation of magnetic
measurements on the ASDEX tokamak. It is shown that the method provides simple and accurate approximations for a range of geometric and other parameters characterizing the equilibrium configuration. These approximations are presently in use for fast data analysis between discharges, but they are also suitable for real-time control of the experiment.

The paper is organized as follows. A mathematical description of the method is given in Section 2, followed by a report on the ASDEX study in Section 3. The prospects for real-time applications of function parametrization are discussed in Section 4.

2. Mathematical Description

Basic concepts. A classical physical system is considered, of which \( \mathcal{P} \) denotes a typical state. The system may have any number of degrees of freedom, but interest will be restricted to a (partial) characterization by \( n \) intrinsic real parameters, represented collectively by a point \( \mathbf{p} \in \mathbb{R}^n \). In the experimental situation \( \mathbf{p} \) is to be estimated from the readings of \( m \) measurements, represented by a point \( \mathbf{q} \in \mathbb{R}^m \). It is assumed that \( \mathbf{p} \) is completely specified by \( \mathcal{P} \), but that \( \mathbf{q} \) may be a stochastic function of \( \mathcal{P} \), the stochasticity being due to random errors in the measurement process. We will write \( \mathbf{p} \Rightarrow \mathbf{p}(\mathcal{P}) \) and \( \mathbf{q} \Rightarrow \mathbf{q}(\mathcal{P}) \).

The aim of the function parametrization is to obtain some reasonably simple function, \( \mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^m \), such that for any state \( \mathcal{P} \) the associated \( \mathbf{p}(\mathcal{P}) \) and \( \mathbf{q}(\mathcal{P}) \) satisfy \( \mathbf{p} \Rightarrow \mathbf{F}(\mathbf{q}) \Rightarrow \mathbf{e} \) for a sufficiently small error term \( \mathbf{e} \). The functional form of \( \mathbf{F} \) may typically be chosen as a low-order polynomial in only a few linear combinations of the components of \( \mathbf{q} \). The unknown coefficients in \( \mathbf{F} \) are then determined by analysis of a data base containing the values of the parameters \( \mathbf{p}_i \) and of the measurements \( \mathbf{q}_i \) corresponding to \( N \) simulated states \( \mathcal{P}_n \) \( (1 \leq n \leq N) \). This is a problem of function fitting over scattered data in the high-dimensional space \( \mathbb{R}^m \), for which the use of methods from multivariate statistical analysis is appropriate. To a statistician the \( \mathbf{q}_i \) are the 'conditions', the \( \mathbf{p}_i \) are the 'responses', \( \mathbf{F} \) is a 'regression', and function fitting is called regression analysis. The terminology of conditions and responses is very unnatural in the present context, and instead we will refer to these as the independent and the dependent variables, or as the measurements and the physical parameters. Otherwise, the terminology of statistical analysis is retained.
The $n$ physical parameters need not all be independent, and even if no exact relations exist between them, the nature of the measurements may in practice allow only a limited number of combinations to be determined independently. Let $n_0$ denote that number of independently determinable combinations. Then it must be that $m \geq n_0$, and for a well-diagnosed experiment in fact $m \gg n_0$, while for an accurate statistical analysis it is furthermore required that $N \gg m$. For the applications that we have in mind, $P$ formally has infinitely many degrees of freedom, $n_0 \sim 10$. $n$ is arbitrary, $m \sim 10 - 100$, and $N \sim 10^3 - 10^4$.

**The data base.** In the first stage a code $G$ is employed to generate a data base. This code must be suited to compute possible states of the physical system over the whole of the system’s regime and must also contain a model for the measurements. The code will take certain numerically convenient parameters as input and produce $p$ and $q$ as results. The input parameters are varied, and for each successful calculation, indexed by $\alpha$, the values $p_\alpha$ and $q_\alpha$ are saved. As the subsequent automatic analysis will only detect dependencies that are reflected in the data base, one must ensure that every parameter or combination of parameters that can vary in the actual experiment is also varied when generating these data. At best, one employs a suitable pseudo-random variation of the code parameters in order to generate the data base.

It need hardly be stressed that the result of the function parametrization can never be expected to improve upon the physical model that is embodied in $G$, nor can it be employed to find the values of physical parameters that are not reasonably well reflected in the measurements. The usual least-squares method of data interpretation would actually rely on a code such as $G$, and attempt in an iterative procedure to optimize the free parameters for minimal deviation between calculated and actual measurements. Using function parametrization the aim is to obtain a direct and much simpler connection between measurements and physical parameters, without sacrificing too much accuracy.

**Dimension reduction.** Since the dimensionality $m$ of the space of the measurements may be of the order of several tens in many cases, and since a linear representation for $p$ in terms of $q$ is not expected to suffice, the dimensionality of the space of trial functions with which the physical parameters will be fitted can be very large. A polynomial model of degree $l$, for instance, has $\sim m^{l/2}$ degrees of freedom for each physical parameter. It is therefore necessary to first reduce the number of independent variables (the components of $q$) by means of a transformation to a lower-dimensional
space. A second aim for this transformation of variables must be to eliminate or reduce multicollinearity (near linear dependencies) between the data points, thereby improving the conditioning of the regression problem (e.g. [4, ch. 8]). Multicollinearity is likely to be present whenever the number of measurements is much larger than the number of independently measurable physical parameters; specific causes may be some underlying smoothness in the data, or any explicit physical constraint that relates different measurements.

A method for dimension reduction and elimination of multicollinearity that is widely used in statistics is based on principal component analysis [4, ch. 8], [5, ch. 8]. From the \( N \) suitably scaled pseudo-measurements \( \mathbf{q}_\alpha \), each of which is a point in \( \mathbb{R}^m \), the sample mean \( \overline{\mathbf{q}} = N^{-1} \sum \mathbf{q}_\alpha \) and the \( m \times m \) sample dispersion matrix

\[
S = N^{-1} \sum_{\alpha=1}^{N} (\mathbf{q}_\alpha - \overline{\mathbf{q}})(\mathbf{q}_\alpha - \overline{\mathbf{q}})^T
\]  

are calculated. \( S \) is symmetric and positive semi-definite. An eigenanalysis yields \( m \) eigenvalues, \( \lambda_1^2 \geq \cdots \geq \lambda_m^2 \geq 0 \), with corresponding orthonormal eigenvectors \( \mathbf{a}_1, \ldots, \mathbf{a}_m \). Any measurement vector \( \mathbf{q} \) may be resolved along these eigenvectors to obtain a set of transformed measurements, \( x_i = \mathbf{a}_i^T \cdot (\mathbf{q} - \overline{\mathbf{q}}) \). This is the principal component transformation, and the \( x_i \) are the principal components of the measurement vector \( \mathbf{q} \).

The transformed measurements \( (x_i)_{1 \leq i \leq m} \) are linearly independent within the sample, have zero mean and standard deviation \( \lambda_i \). One of the aims, the reduction of multicollinearity, has therefore been achieved, but if all \( m \) components \( x_i \) are retained, the dimensionality of the problem is not reduced. The assumption underlying a regression analysis based on principal component analysis is that the most significant information will be contained in the first few principal components, \( (x_i)_{1 \leq i \leq m_0} \), where \( m_0 \leq m \), and preferably \( m_0 \ll m \). These \( m_0 \) components are called the 'significant components', and the associated first \( s \) eigenvectors \( \mathbf{a}_s \) are the 'significant variables'. The desired dimension reduction is thus achieved through the transformation \( \mathbb{R}^m \rightarrow \mathbb{R}^{m_0} \) defined by \( \mathbf{x} = \mathbf{A}^T \cdot (\mathbf{q} - \overline{\mathbf{q}}) \), where \( \mathbf{A} \) is the matrix that has columns \( \mathbf{a}_i \) (1 \( \leq i \leq m_0 \)). The selection of \( m_0 \) will be based on an inspection of the behaviour of the sequence of eigenvalues, but no universally accepted prescription exists. One hard criterion is that one must choose \( m_0 \geq m_0 \). On the other hand, the principal components corresponding to smaller eigenvalues are more difficult to measure accurately, and a study of the effect of measurement errors will provide an upper bound on the reasonable values for \( m_0 \).

2. Mathematical Description
The use of principal component analysis for dimension reduction is motivated by the idea that those linear combinations of the measurements ("affine" combinations in mathematical terminology) that display the largest variation are also the best suited for the interpretation of the data. In practice, however, significant information may well be concealed in linear combinations of the measurements that show relatively little variation, or alternatively, some measurements may vary widely without much correlation with the physical parameters that are to be determined. A preliminary linear or nonlinear transformation of variables on the basis of physical insight will then be beneficial. Principal component analysis is invariant only under orthogonal transformations; it is not invariant under scaling of variables, or under more general linear transformations.

An alternative procedure for dimension reduction is to employ canonical correlation analysis. Using this procedure, two transformations, \( x = A^T(q - q) \) and \( y = B^T(p - p) \), are obtained simultaneously. \( A \) and \( B \) are \( m \times s \) and \( n \times s \) matrices respectively, with columns \( a_i \) and \( b_i \) (\( 1 \leq i \leq s \)). \( a_1 \) and \( b_1 \) are chosen so that \( x_1 = a_1 \cdot (q - q) \) and \( y_1 = b_1 \cdot (p - p) \) have the largest possible linear correlation within the data set \((q_0, p_0), i = 1 \ldots N\). Therefore, \( y_1 \) is that linear function of the parameters which is best predicted by a linear function of the measurements, and \( x_1 \) is the corresponding linear function of the measurements. Generally, each pair \((a_i, b_i)\) is chosen to maximize the correlation between \( x_i \) and \( y_i \), subject to \( a_i \) being linearly uncorrelated with \( a_1 \ldots a_{i-1} \) and \( b_i \) being linearly uncorrelated with \( b_1 \ldots b_{i-1} \). We refer to Ref. 5, ch. 10 for the mathematical development.

Canonical correlation analysis may be a better technique for dimension reduction than principal component analysis in those cases where the relation between measurements and physical parameters is approximately linear. It selects those linear transformations of the measurements that provide the best prediction for some linear transformation of the physical parameters, and disregards combinations that may show a large variance, but are irrelevant to the data interpretation. On the other hand, if a certain linear combination of the measurements enters only quadratically in the regression, then canonical correlation analysis would not select this combination, although it may be important and could be found by principal component analysis. There does not exist any automatic procedure which can be considered best in all cases.

One further remark related to canonical correlation analysis is appropriate: For the present application it will usually be advisable to employ some form of ridge analysis [4, ch. 8], [5, ch. 10], as the data may be highly co-linear.
Regression analysis. Having defined the linear transformation \( q \rightarrow x \), it is necessary to face the task of fitting the relation (nonlinear in general) between \( x \) and \( p \). The problem has been simplified by the dimension reduction obtained with the transformation \( q \rightarrow x \), and it is expected to be better conditioned through the elimination of multicollinearity.

It is desired to find for each component \( p_j \ (1 \leq j \leq n) \) a regression, \( p_j = f_j(x) + \varepsilon_j \), to fit the data \((x_\alpha, p_\alpha)_{1 \leq \alpha \leq N}\). A polynomial model, of the form

\[
p_j = \sum_k c_{kj} \prod_{i=1}^{m_k} \phi_k(x_i/r_i) - \varepsilon_j,
\]

is suitable. Here, the multi-index \( k \) has \( m_0 \) components \( k_1, \ldots, k_{m_0} \) in the non-negative integers, the \( c_{kj} \) are the unknown regression coefficients, which are determined by a least-squares fitting procedure over the data base, \((\phi_k)_{k \geq 0}\) is some family of polynomials, \( r_i \) is a suitable scaling factor for the component \( x_i \), and \( \varepsilon_j \) is the error term. Wind [1], [2] employs Chebyshev polynomials and sets \( r_i = \max_\alpha |x_\alpha| \), whereas we employ Hermite polynomials and set \( r_i = \lambda_i \).

An upper bound on some norm of \( k \) must be supplied in order to make the model finite, and in addition one can employ with the above model some form of subset regression, the objective being to retain in the final expression only the terms which make a significant contribution to the goodness-of-fit. A variety of algorithms exists for deciding which terms to retain and which to discard; see for instance Ref. [4, ch. 7], or Ref. [6, ch. 6].

This completes the construction of the function parametrization, \( p = F(q) + e \), which is thus given by Eq. (2) together with the relation \( x = A^T \cdot (q - \bar{q}) \). Notice that even though a significant effort may be involved in generating and analyzing the data base, the evaluation of the final function—and this is the operation that has to be performed many times—is almost trivial. In order to test the accuracy of the regression function it is proper to generate a new, independent collection of simulated experiments, and to evaluate the standard magnitude of the error term \( e \) from this second set of data.

Treatment of erroneous signals. An attractive feature of the principal component analysis, also pointed out by Wind [1], [2], is that the least significant components can be used to obtain constraints on the data. These constraints make it possible to test whether actual measurements are consistent with the model that was used to generate the data base, and also to automatically correct failing signals.
Specifically, let us assume that the principal component analysis has been performed on simulated measurements that have been scaled and transformed in such a way that they are assumed in the experiment to suffer independent random errors coming from a normal distribution having mean 0 and width \( \sigma \). Now for each actual measurement \( q \) we define

\[
\chi^2(q; \sigma) = \sum_{i=1}^{m} \frac{x_i^2}{\lambda_i^2 + \sigma^2},
\]

where \((x_i)_{1 \leq i \leq m}\) is the complete set of principal components (significant and insignificant) associated with \( q \). In this case, \( \chi^2 \) will have average value \( \approx m \), and much larger values of \( \chi^2 \) indicate either an error in the measurements or a violation of some assumption that was employed in generating the data base. If it is known that one or more specific components of the measured \( q \) are in error, then these components can be restored to that set of values by which the quadratic form \( \chi^2 \) is minimized. This requires only the solution of a system of linear equations of which the dimension is equal to the number of failing signals. (See also this thesis, Section 3.5).

3. Application to the ASDEX Magnetic Data Analysis

**Measurements and parameters.** As an initial study we applied function parametrization to the determination of a limited set of characteristic equilibrium parameters for the ASDEX experiment, using only magnetic signals measured outside the plasma. Important features of the geometry of ASDEX are displayed in Fig. 1, which shows the location of the vacuum vessel, the divertor plates, the poloidal field and flux measuring coils, and an example equilibrium plasma configuration.

The relevant measurements for our study consist of three differential flux measurements, four measurements of the component of the poloidal field tangential to the measuring contour, the current through the multipole shaping coils, and the toroidal plasma current. However, the plasma current is scaled out of the problem, so that 8 independent measurements remain. With reference to Fig. 1, these are \( \psi_3 - \psi_1, \psi_4 - \psi_2, \psi_3 - \psi_4, B_1, B_2, B_3, B_4, \) and \( I_{\text{mpc}} \), in each case normalized to unit plasma current.

The physical parameters to be determined are

\[
(t_{\text{axis}}, z_{\text{axis}}) \quad \text{position of the magnetic axis}
\]

\[
(t_{\text{curr}}, z_{\text{curr}}) \quad \text{position of the current centre}
\]
Fig. 1. Poloidal cross-section through the ASDEX experiment. Measurements of the poloidal field and flux, $B_1$, $B_3$ and $\psi_1$, $\psi_4$ are made at the locations shown. The measurement $I_m\psi_1$ is the current through each of the two main multipole shaping coils, which are marked with the symbol +. Each of the four smaller multipole coils next to the divertor entrances carries a return current $-\frac{1}{2}I_m\psi_1$. Notice that the equilibria in our data base are in general not up-down symmetric.
The current centre is defined by \( r_{\text{curr}} I_{\text{pl}} = \int r^2 j_t \, dS \) and \( z_{\text{curr}} I_{\text{pl}} = \int z_j \, dS \), where \( I_{\text{pl}} = \int j_t \, dS \), \( j_t \) is the toroidal current density, and the integrals are taken over the poloidal cross section of the plasma. These integrals can be rigorously evaluated from the external magnetic measurements [7]. Furthermore, \( \psi_b \) is the flux at the plasma boundary and \( \psi_r = \frac{1}{4}(\psi_1 + \psi_2 + \psi_3 + \psi_4) \). These fluxes, as well as \( \psi_{x1} \) and \( \psi_{x2} \), have been scaled to correspond to unit plasma current. The normalized safety factor \( q_{\text{norm}} \) is the value of the safety factor at the contour defined by \( \psi - \psi_a = 0.95 \cdot (\psi_b - \psi_a) \), scaled to unit plasma current and unit toroidal field coil current (\( \psi_a \) denotes the flux at the magnetic axis). The subscripts indicating the divertor plates are mnemonic for bottom-inner, bottom-outer, top-inner, and top-outer.

The data base. Simulated measurement were generated using the Garching equilibrium code [8], which, after significant optimization in view of these calculations, now computes an equilibrium on our 64 \times 128 grid in \( \sim 0.6 \) seconds on the Cray 1. The free parameters of the code were randomly varied in order to cover the operating regime of the ASDEX experiment in the divertor mode, and for each of the \( \sim 4000 \) calculated equilibria the corresponding magnetic measurements and physical parameters were recorded.

Dimension reduction. A principal component analysis of the correlation matrix obtained from the simulated measurements gave the following sequence of eigenvalues: 3.67, 1.91, 1.39, 0.99, 2.01 \times 10^{-2}, 1.12 \times 10^{-2}, 1.90 \times 10^{-3}, 8.93 \times 10^{-4}. (Using the correlation matrix instead of the dispersion matrix is equivalent to scaling all measurements in such a way that they have unit variance). Considering that the first four eigenvalues
are all of order unity and that the fifth eigenvalue is much smaller, it appeared that 4 significant variables should be retained for the regression analysis.

Of these four significant variables, the first two are even under up-down reflection of the equilibrium, and together they already provide a measure of the radial position and of $\beta_p + k/2$. The third significant variable is odd under reflection, and is related to the vertical position of the plasma, and the fourth one (again even under reflection) is essentially the multipole current. The remaining four least significant variables cannot be recovered accurately from beneath the error level of realistic measurements.

**Regression analysis.** By experimenting with various regression models it was found that the fourth significant variable (essentially the multipole current) is of little overall importance, but is mainly relevant to the determination of the intersection of the separatrix with the divertor plates. Good results were obtained using a model that is of second order in the first three significant variables and of first order in the fourth. For each of the physical parameters $p_j$:

$$
p = c_0 + c_1 x_1^j + c_2 x_2^j + c_3 x_3^j + c_4 x_4^j + c_5 H_2(x_1^j) + c_6 x_1^j x_2^j + c_7 x_1^j x_3^j + c_8 H_2(x_2^j) + c_9 x_2^j x_3^j + c_{10} H_2(x_3^j) + \epsilon$$

where $x_i^j = x_i/\lambda_i$ and $H_2(x) = (x^2 - 1)\sqrt{2}$. The coefficients $c_0, \ldots, c_{10}$ are to be determined, and $\epsilon$ is the error term. The dependence on the index $j$ is suppressed in Eq. 4. The particular set of basis functions in this model was chosen in order to obtain an approximately orthonormal family.

Besides Eq. (4) we tried several other regression models, up to a complete third order polynomial model in the first four principal components, and a complete quadratic model in six principal components. The third order polynomial model provides only marginally more accurate results than does Eq. (4), whereas the model involving six principal components is more accurate for ideal measurements, but less accurate at realistic error levels. In the analysis of a different, related set of data we also used an approach in which the (infinitely many) terms in a multivariate Taylor series model were ranked in a particular linear order, and added one by one to the model equation until an asymptotic error level was attained. The criterion for defining the linear order was the eigenvalue product associated with the Taylor series term; e.g. associated with the term $x_1^2 x_2$ is the eigenvalue product $\lambda_1^2 \lambda_2$. Clearly all eigenvectors must be $\prec 1$ for this ranking to make sense, but this is easily achieved by scaling. We envisage that this

3. Application to the ASDEX Magnetic Data Analysis
approach could be useful for situations in which more complicated models than Eq. (4) are required, in particular when it is used in conjunction with some form of stepwise regression.

The choice for Eq. (4) was in the end motivated by its simplicity. However, a slight modification was necessary for a reason that is discussed in the following subsection.

A special complication. The physical parameters that we seek to determine exhibit a first-derivative discontinuity in their dependence upon the measurements, which is due to the fact that the x-point defining the plasma boundary can lie either in the lower or in the upper half plane. One can visualize this first-derivative discontinuity by considering the behaviour of the plasma minor radius as the centre of gravity of the plasma cross-section is moved smoothly from a position in the lower half plane to a position in the upper half plane. At first the plasma boundary is defined by the lower x-point, and the minor radius increases as the plasma centre moves away from this x-point. Upon crossing the meridian plane, the plasma boundary becomes dependent upon the upper x-point, and the minor radius decreases as the plasma is shifted further upwards. The relation between plasma minor radius and vertical position is non-smooth as the meridian plane is crossed, and this lack of smoothness is also present in the relation between the minor radius and the measured data. Furthermore, all other physical parameters exhibit a similar non-smooth behaviour.

For the ASDEX study the problem of this non-smoothness was easily handled by making use of the up-down symmetry of the experiment. By taking sums and differences where necessary, the physical parameters were transformed to a form where each is either symmetric or antisymmetric under up-down reflection of the equilibrium. For the antisymmetric parameters the discontinuity occurs only in the second derivative, and it was ignored in the regression analysis. One of the antisymmetric parameters is $\psi_{x2} - \psi_{x1}$. The sign $s$ of this parameter determines in which half-plane the x-point is located. The regression analysis for the symmetric physical parameters was then performed not in terms of $(x'_1, x'_2, x'_3, x'_4)$, as expressed in Eq. (4), but actually in terms of $(x'_1, x'_2, sx'_3, x'_4)$. The antisymmetric significant variable $x'_3$ is multiplied by $s$.

A similar complication occurs for a limiter experiment when an inside and an outside limiter are both present, or for an experiment in which both a limiter and a divertor geometry is possible. In such cases it will usually be necessary to employ different fits, depending upon the estimated value of some discrete indicator variable.
Results. The results obtained by using the model described previously are shown in Table I. For each parameter, Table I shows first the average, minimum and maximum values occurring in the data base, and the standard deviation from the average. The last two columns show the standard error, \( \bar{\epsilon} \), of the fitted values, first for exact measurements (\( \delta = 0.0 \)), and then for measurements that have been randomly perturbed by a term coming from a normal distribution of zero mean and width equal to a fraction \( \delta = 0.1 \) of the standard deviation of the measurements. SI units are employed throughout. Notice that the column of standard deviations shows the standard error associated with a very naive method of data interpretation, viz. ignoring the data and assuming that each physical parameter always has its average value. The quality of any other method of data interpretation may be evaluated not just in absolute terms, but also in relation to this naive procedure.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Average</th>
<th>Minimum</th>
<th>Maximum</th>
<th>St. Dev.</th>
<th>( \bar{\epsilon}(\delta = 0.0) )</th>
<th>( \bar{\epsilon}(\delta = 0.1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_{\text{axis}} )</td>
<td>1.722</td>
<td>1.630</td>
<td>1.830</td>
<td>5.6 \times 10^{-2}</td>
<td>2.2 \times 10^{-3}</td>
<td>4.1 \times 10^{-3}</td>
</tr>
<tr>
<td>( r_{\text{curr}} )</td>
<td>1.710</td>
<td>1.612</td>
<td>1.824</td>
<td>5.6 \times 10^{-2}</td>
<td>3.1 \times 10^{-4}</td>
<td>3.4 \times 10^{-3}</td>
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<tr>
<td>( r_{\text{geom}} )</td>
<td>1.661</td>
<td>1.515</td>
<td>1.807</td>
<td>5.7 \times 10^{-2}</td>
<td>2.8 \times 10^{-3}</td>
<td>5.1 \times 10^{-3}</td>
</tr>
<tr>
<td>( z_{\text{axis}} )</td>
<td>0</td>
<td>0.100</td>
<td>0.100</td>
<td>5.4 \times 10^{-2}</td>
<td>2.3 \times 10^{-3}</td>
<td>5.8 \times 10^{-3}</td>
</tr>
<tr>
<td>( z_{\text{curr}} )</td>
<td>0</td>
<td>0.102</td>
<td>0.102</td>
<td>5.4 \times 10^{-2}</td>
<td>2.5 \times 10^{-3}</td>
<td>6.0 \times 10^{-3}</td>
</tr>
<tr>
<td>( z_{\text{geom}} )</td>
<td>0</td>
<td>0.134</td>
<td>0.134</td>
<td>7.3 \times 10^{-2}</td>
<td>9.8 \times 10^{-3}</td>
<td>1.2 \times 10^{-2}</td>
</tr>
<tr>
<td>( a )</td>
<td>0.367</td>
<td>0.290</td>
<td>0.463</td>
<td>3.0 \times 10^{-2}</td>
<td>3.8 \times 10^{-3}</td>
<td>6.2 \times 10^{-3}</td>
</tr>
<tr>
<td>( b )</td>
<td>0.358</td>
<td>0.295</td>
<td>0.438</td>
<td>2.5 \times 10^{-2}</td>
<td>1.9 \times 10^{-3}</td>
<td>4.7 \times 10^{-3}</td>
</tr>
<tr>
<td>( b/a )</td>
<td>0.978</td>
<td>0.796</td>
<td>1.033</td>
<td>3.2 \times 10^{-2}</td>
<td>8.1 \times 10^{-3}</td>
<td>8.5 \times 10^{-3}</td>
</tr>
<tr>
<td>( A )</td>
<td>0.416</td>
<td>0.271</td>
<td>0.592</td>
<td>6.1 \times 10^{-2}</td>
<td>5.5 \times 10^{-3}</td>
<td>1.2 \times 10^{-2}</td>
</tr>
<tr>
<td>( \beta_p + \ell_n/2 )</td>
<td>1.791</td>
<td>0.563</td>
<td>3.428</td>
<td>6.3 \times 10^{-1}</td>
<td>1.2 \times 10^{-2}</td>
<td>4.4 \times 10^{-2}</td>
</tr>
<tr>
<td>( q_{95n} )</td>
<td>5.07 \times 10^{-2}</td>
<td>3.26 \times 10^{-2}</td>
<td>8.62 \times 10^{-2}</td>
<td>9.1 \times 10^{-3}</td>
<td>1.8 \times 10^{-3}</td>
<td>2.4 \times 10^{-3}</td>
</tr>
<tr>
<td>( \psi_b - \psi_r )</td>
<td>8.39 \times 10^{-7}</td>
<td>3.64 \times 10^{-7}</td>
<td>1.31 \times 10^{-6}</td>
<td>1.7 \times 10^{-7}</td>
<td>1.3 \times 10^{-8}</td>
<td>3.0 \times 10^{-8}</td>
</tr>
<tr>
<td>( r_{x2} )</td>
<td>1.550</td>
<td>1.530</td>
<td>1.570</td>
<td>1.1 \times 10^{-2}</td>
<td>5.3 \times 10^{-3}</td>
<td>5.4 \times 10^{-3}</td>
</tr>
<tr>
<td>( z_{x2} )</td>
<td>0.447</td>
<td>0.417</td>
<td>0.483</td>
<td>1.2 \times 10^{-2}</td>
<td>4.6 \times 10^{-3}</td>
<td>4.7 \times 10^{-3}</td>
</tr>
<tr>
<td>( \psi_{x2} - \psi_{x1} )</td>
<td>0</td>
<td>-8.86 \times 10^{-7}</td>
<td>8.86 \times 10^{-7}</td>
<td>4.4 \times 10^{-7}</td>
<td>2.0 \times 10^{-8}</td>
<td>4.9 \times 10^{-8}</td>
</tr>
<tr>
<td>( z_{ti} )</td>
<td>0.836</td>
<td>0.685</td>
<td>0.934</td>
<td>4.6 \times 10^{-2}</td>
<td>3.2 \times 10^{-3}</td>
<td>7.1 \times 10^{-3}</td>
</tr>
<tr>
<td>( z_{lo} )</td>
<td>0.862</td>
<td>0.764</td>
<td>0.965</td>
<td>4.1 \times 10^{-2}</td>
<td>3.3 \times 10^{-3}</td>
<td>6.4 \times 10^{-3}</td>
</tr>
</tbody>
</table>

3. Application to the ASDEX Magnetic Data Analysis
It is seen that the model obtained by function parametrization accurately reconstructs the physical parameters as they are computed in the MHD equilibrium code. Furthermore, the method is not particularly sensitive to measurement errors. These results demonstrate the effectiveness of function parametrization for the analysis of magnetic measurements within the context of static, ideal MHD equilibrium theory.

For use in online data analysis, the coefficients defining the principal component transformation and the regression functions have been inserted into a program (called FP), which runs on the ASDEX VAX. As presently incorporated in the ASDEX data analysis, FP is used to provide a complete picture of the time evolution of the quantities \( r_{\text{geom}}, z_{\text{geom}}, r_{\text{axis}} - r_{\text{geom}}, q_{55}, \) and \( \beta_p - k_i/2 \) immediately after every shot. Furthermore, the time derivative of the quantity \( v_0 \) is used to correct the loop voltage measured by the flux loops to the actual value prevailing at the plasma surface; this is important for an accurate evaluation of the electromagnetic power flux into the plasma column during phases with strong dynamics, like during pellet injection. Additional parameters characteristic of the field configuration and needed for the interpretation of other diagnostics can be evaluated at arbitrary instances, or their evolution can be displayed.

Figure 2 shows the time evolution of the parameters \( \beta_p \cdot k_i/2 \) and \( r_{\text{axis}} \) as determined by FP for an ohmically heated, pellet fuelled discharge \( \# 14669, q = 3.4, B_i = 2.2 \, \text{T} \). The parameters are evaluated at 1 ms intervals and plotted for the time range 800 ms \( t \leq 1000 \, \text{ms} \). The longer wavelength oscillations correspond to the 35 ms period of the pellet injection. This shot has been described in detail in Ref. 9.

It should be emphasized that the present application of function parametrization relies on the validity of the static, ideal MHD model as implemented in the equilibrium code \( \mathcal{A} \). Deviations from axisymmetry, anisotropic pressure, and plasma rotation will cause more or less severe errors in the interpretation of the magnetic measurements. Recent X-ray tomography data obtained by Smeulders \( \dagger \) for ASDEX discharges that involved high-power neutral injection and apparently strong toroidal plasma rotation, are therefore cause for concern. In some cases the magnetic axis position as determined from the X-ray data lies 5 cm or more inward of the position computed by a method based on static, ideal MHD equilibrium analysis. This problem is of course not specific to the function parametrization method of data interpretation. In order to determine accurately some of the plasma parameters in the presence of strong toroidal rotation it will be necessary to include this effect in the data base, and also to include appropriate additional diagnostics.
Fig. 2 Evolution of $\beta_p + 1/2$ (upper) and $r_{axis}$ (lower) as determined by FP for ASDEX shot # 14669.
Comparison with other methods. Until now, the online analysis of magnetic measurements on ASDEX has been done using the FILM code, in which these measurements are fitted using a model that represents the plasma by a single wire current in an unknown location, and that contains two free parameters to describe the horizontal and vertical external fields. FILM computes the location of the plasma boundary and the associated geometric parameters. Furthermore, the Shafranov parameter $\beta_p + l_i/2$ and the radial shift of the geometric centre of the plasma with respect to the machine centre ($\Delta = r_{\text{geom}} - r_{\text{mach}}$) are computed on the basis of Shafranov's method [11], which relies on the assumptions of large aspect ratio, small radial shift, up-down symmetry, and circular cross-section. Given an estimate of the plasma minor radius $a$, the theory of [11] provides linear expressions for $\beta_p + l_i/2$ and $\Delta$ in terms of the measurements.

In the implementation on ASDEX the FILM code employs a set of magnetic diagnostics that is different from the set used in FP, so we do not have a direct comparison between the single wire current model together with Shafranov's theory on the one hand, and function parametrization on the other. However, within the bounds set by measurement errors, the results obtained by the FILM and FP codes are in agreement. We have also compared the best possible linear formulae for $r_{\text{axis}}$, $r_{\text{curr}}$, $r_{\text{geom}}$, and $\beta_p + l_i/2$ with the formulae obtained by function parametrization as described earlier. The linear formulae were obtained from a regression analysis using a linear model in terms of all the original measurements. This comparison showed that for all these parameters the optimal linear fit gives about twice as large an error as the fit obtained by function parametrization. This discrepancy is primarily due to the fact that in our application of function parametrization we have different fits depending upon the location of the separatrix x-point, thereby obtaining a more accurate representation for asymmetric configurations: the quadratic terms in Eq. (4) are small for these parameters, and do not explain the higher accuracy of the function parametrization method.

There exist codes more sophisticated than FILM for the fast analysis of magnetic measurements [12], [19]. In these codes the plasma boundary is computed using either a filament current model, an expansion in toroidal harmonics, or a single-layer potential method. The parameter $\beta_p + l_i/2$ is then computed by using the theory of Zakharov and Shafranov [7], which does not rely on the assumption of circular plasma cross-section and is therefore more general than the theory of Ref. [11]. Such magnetics analysis codes proceed in two stages. In the first stage the expansion coefficients in the representation of the plasma current (and sometimes also of external currents) are computed. This computation can be arranged so as to require only a matrix-vector
multiplication and can therefore be done very rapidly. In the second stage the plasma boundary is found, and the required integrals are evaluated. This computation is more expensive. The relative gain in efficiency obtained by function parametrization is due to it avoiding the explicit computation of the plasma boundary contour.

4. Discussion

The principal advantage of the use of function parametrization in comparison with other methods for rapid data interpretation is that the computationally intensive part of the procedure—the elaborate model calculations and the statistical analysis—has to be carried out only once during the lifetime of a diagnostic set-up or device, whereas the calculations needed in the application involve only simple algebra. An additional advantage is that function parametrization is designed to find an optimal representation within a rather large class of algebraic functions and may therefore well be more accurate than a method that relies on analytic approximations derived by consideration of exactly solvable special models. In situations where many measurements are made with the same experimental set-up, the ultimate gain in diagnostic capabilities can obviously be very large.

An important problem where fast data analysis is essential is the real-time control of machine operation. Present-day tokamak operation still proceeds in many parameters via trial and error, giving completely reproducible results and the quasi-stationary discharge behaviour required for a reactor only under rather narrowly defined conditions. Particularly the experience on ASDEX has shown that a dramatic improvement in this respect can be obtained by feedback control, at least for those parameters included in the control loop, like plasma current, position, and density.

In order to illustrate the feasibility of using function parametrization for real-time control we will estimate the number of floating point operations (flops) required at each time slice for the analysis of the magnetic measurements on the JET tokamak. JET has 32 axisymmetric field and flux measurements versus 8 on ASDEX, and it also has more freedom in plasma shaping. Projecting $m$ measurements onto 6 principal components requires $\sim 12m$ flops. (Six significant variables for JET, as compared to four for ASDEX, seems reasonable in view of the extra freedom in plasma shaping). There are many different ways of coding the algorithm for correcting the failing signals, but for the present discussion a cost of again $\sim 12m$ flops is assumed. A complete
quadratic model in the 6 principal components has 28 terms, so to approximate \( n \) physical parameters using such a model requires \( \sim 56n \) flops. Then assuming \( m = 32 \) and \( n = 25 \), and adding 30% overhead, one arrives at \( \sim 3000 \) flops, of which the majority occur in matrix-vector multiplication. The storage requirement for the matrix defining the principal component transformation and the matrix of coefficients in the regression is \( m^2 + 28n \) words, which is trivial.

For real-time use it will be desirable to evaluate the approximations in a time that is less than the skin-time of the vacuum vessel, although there can be no need to stay far below that bound. On JET the skin-time is about 3 ms (on many other experiments it is longer). It is seen that a small processor, having a few kilowords of 32-bit store and a speed of about 5 Mflops/sec for modest-size matrix-vector multiplication, will meet the requirements for real-time control with ample room to spare.

It is intended to employ function parametrization on the ASDEX Upgrade tokamak, due to commence operation in late 1988, for real-time analysis of a wide spectrum of diagnostic signals whose use in the feedback loop is necessary or desirable. An inherent advantage of the function parametrization method of data analysis is its ability to mix different types of diagnostics. Important is principally that these diagnostics complement each other to give together a unique interpretation of the discharge conditions. The objective is to have on ASDEX Upgrade an integrated package of diagnostics giving real-time information about the flux surface geometry, the plasma energy content, the profiles of electron temperature, electron density and current density, the impurity concentration, and the plasma rotation. The diagnostics to be used for this include, besides magnetic measurements, also electron temperature and line-integral electron density measurements, a thermography camera observing the divertor plates, and soft X-ray pinhole cameras.

Acknowledgements

The present text is based on a joint paper with W. Jilge and K. Lackner, whose contributions are gratefully acknowledged. This author learnt of function parametrization while working as a summer student at CERN under the guidance of Henk Wind (on a different project). Valuable conversations with Dr. Wind, and the hospitality of the CERN organization are gratefully acknowledged. Mr. P. Martin is thanked for optimizing the equilibrium code and implementing the ASDEX user interface. This work
was performed under the Euratom-IPP and Euratom-FOM association agreements with financial support from ZWO and Euratom.

References

10. P. Smeulders. Tomography of Quasi-Static Deformations of Surfaces of Constant Emission of High-Beta Plasmas in ASDEX, submitted for publication; and private communication.


PART II: MODELLING OF THE EDGE PLASMA
5. MODELLING OF THE BOUNDARY PLASMA OF LARGE TOKAMAKS

B.J. Braams, P.J. Harbour*, M.F.A. Harrison*, E.S. Hotston*, and J.G. Morgan*

Abstract

Fluid models developed at Culham for the boundary plasma of tokamaks are surveyed and their application to devices such as INTOR or NET is described. The simplest model is based upon transport of a single fluid and considers parallel conduction along the scrape-off layer with a convective sheath boundary condition: its versatility is demonstrated in a comparison of the INTOR divertor and limiter. A two-fluid, one-dimensional model has been critically compared with the simple model; it has also been used in a study of impurity transport, and predictions for a highly recycling plasma are described. A two-fluid, two-dimensional model of the plasma is presented: it has provided predictions for the thermal load on the INTOR divertor plate. We conclude by showing the need for a careful physical description of a two-dimensional plasma boundary.

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5.1. Introduction

This paper surveys fluid models developed at Culham for the boundary plasma of tokamaks and describes their application to devices such as INTOR or NET. Both for the design of NET/INTOR and for the operation of large machines, an understanding of the transport of particles and energy through the plasma scrape-off layer is increasingly important. In particular, the present knowledge of the profile of energy deposition on the walls and target plates, of helium pumping characteristics, and of the control of wall-produced impurities is insufficient. A numerical fluid model for the edge plasma must match the radial transport processes, which are dominant inside the separatrix, to the convection and the thermal conduction along open field lines onto a material boundary, and so the edge model must be two-dimensional. Furthermore, the model should incorporate details of the recycling of neutral particles from the boundary surfaces and energy loss by radiation.

In view of the complexity of a two-dimensional model we have found it valuable to develop a series of one-dimensional, parallel flow models which we have used to provide insight into the two-dimensional problem. The simplest model describes a single fluid \( \rho = \frac{1}{T_r} \) in which radial variations are ignored except in prescribing the characteristic e-folding thickness of the scrape-off layer \([1]\). The transport of energy parallel to the magnetic field is by electron conduction with a convective sheath boundary condition. Neutral particles generated at the target (D, T, DT, He and sputtered atoms) are followed using a random walk analysis, leading to predictions of the probability of pumping or recycling and of the sputtering at the target and walls of the divertor chamber. This model has been applied to the INTOR divertor \([1, 2]\) and limiter \([3]\).

A two-fluid, one-dimensional plasma model, \(Z\)\_phys, based on the Braginskii equations, has been applied to a poloidal divertor of INTOR size. The results were compared with those from the simple single-fluid model and the requirements for a physically realistic fluid treatment in pure hydrogen (D, T) have been assessed. Trace impurities and helium “launched” into the hydrogen plasma obtained with the two-fluid model are shown to be entrained in the flow and all are dragged into the divertor chamber where they may accumulate in the plasma within the divertor chamber. The balance of forces is shown to be delicate, and with present knowledge it is difficult to assess the importance of this process.

One-dimensional models, whether radial or parallel, cannot adequately describe the transition region where both radial and parallel transport is important. Recent approaches to this problem include the development of a radially stacked version of the
single-fluid model, allowing the variation of scrape-off layer thickness to be investigated. A fully two-dimensional numerical model has also been evolved; this allows perpendicular particle diffusion and the convection and conduction of energy to be coupled self-consistently to the processes described in the one-dimensional, parallel flow, two-fluid model. It has been used to obtain predictions for the energy transport and the temperature profiles in the INTOR scrape-off layer, demonstrating quantitatively the need for a high edge density if the temperature of the plasma in the divertor is to be maintained at a suitably low value ($\lesssim 50$ eV).

5.2. Application of the Simple, One-Dimensional Model

The single-fluid, conductive model for flow parallel to the magnetic field in a scrape-off layer has been fully described in Ref. [1]. The geometry used is one in which the plasma is assumed to flow in a channel of constant area to a divertor target plate. The assumption of constant channel area is consistent with the weak entrance mirror of a poloidal divertor. The flow is assumed to be supplied by volume sources of particles and energy in the scrape-off, which represent radial diffusion. It is taken to be steady-state because timescales in the edge region are short compared with the global timescales of the whole plasma.

The principle advantage of the single-fluid conductive model lies in its simplicity, because analytical solutions of the plasma equations readily lead to a detailed description of the plasma-wall interaction in the divertor chamber. Given the magnitude of the power which crosses the separatrix, an estimate of the plasma density $n(0)$ at the separatrix, and the geometry of the scrape-off/divertor region, it is possible to describe the following processes: sputtering and erosion; the transport of sputtered particles; the recycling of neutral particles; the power balance via radiation from various species and via particle interactions with walls and target; and the pumping of helium and hydrogen. Thus the use of a simple plasma model has permitted us to incorporate a detailed and self-consistent description of the atomic physics.

Figure 1 shows the configurations of INTOR that have been investigated. In the poloidal divertor case (Fig. 1a), Harrison, Harbour & Hotston [1] considered the flow of 75 MW of power, $2Q_n$, to a single-null divertor with tungsten targets and stainless steel walls. In this work the e-folding width for power flow in the scrape-off layer at the outer mid-plane was $\Delta = 3.3$ cm, a value specified for INTOR, Phase I [4], which meant that
Fig. 1. Alternative boundary configurations of INTOR (a) with single-null divertor and (b) with pumped limiter.
the effective area $A_{\|,t}$ of each divertor entrance was $0.35 \text{ m}^2$. More recently Harrison and Hotston [2] considered two different values of $\Delta$, 3.5 cm and 1.6 cm, and three different power flows, $2Q_{\|}$, of 100, 80 and 40 MW. In addition they investigated three different scrape-off densities ($2.5$ to $7.5 \times 10^{19} \text{ m}^{-3}$), and they were able to compare the predicted performances for tungsten and beryllium divertor targets.

Sputtering rates of the divertor target are indicated in Fig. 2a by the effective yield $Y_{\text{eff}}$ (target atoms/incident ion of the plasma plus its impurity content). For the case of a tungsten target when $n = 5 \times 10^{19} \text{ m}^{-3}$ and $\Delta = 3.5 \text{ cm}$ the sputtering yield is negligibly small. Reduction of $\Delta$ to 1.6 cm causes an increase by a factor of about 40, but this can be more than offset if the boundary density is increased. If the scrape-off width $\Delta$ should be any narrower than 1.6 cm, the plasma temperature will be higher than in Ref. [5] and the sputter-rate must rise, especially if the self-sputter limit is approached. For beryllium the self-sputter does not exceed unity, so that the sensitivity to both $\Delta$ and $n(0)$ is less marked. However, $Y_{\text{eff}}$ is substantial, although it may be reduced because of redeposition of sputtered material. The pumping requirements were also considered [2, 5], a beryllium target being about twice as demanding as a tungsten target because fewer energetic atoms are backscattered from its surface.

The pumped limiter configuration (Fig. 1b) was modelled using two flow channels to each of the upper and lower surfaces of the limiter [3], and the dimensions of the flow channels were chosen such that the power flux to the limiter tip was 1 MW m$^{-2}$. A wide range of powers $2Q_{\|}$ was considered in order to allow for impurity radiation inboard of the separatrix. Figure 2b shows the variation of the sputter yield with the power $2Q_{\|}$ to the upper surfaces of the limiter ($n(0) = 5 \times 10^{19} \text{ m}^{-3}$ and $\Delta = 1.5 \text{ cm}$). For Be, radiation is expected to be low, and so only a large value of $2Q_{\|}$ (80 MW) was considered. The beryllium sputter yield was too high to be acceptable unless local redeposition occurs, while for tungsten it was only acceptable with $2Q_{\|} \leq 40 \text{ MW}$. The power $Q_{\|}$ flowing to the lower surface of the limiter was no more than a few megawatts and two different plasma regimes were obtained with the same input conditions (Fig. 2c). For one case the plasma temperature, $T_i$, near the limiter is very low and the density high; in the other case $T_i$ is higher and density is lower. Various densities were considered [5] and predicted pumping speeds are very sensitive to temperature, cf. Refs. [3], [5]. A similar two-valued solution may exist for the divertor and this is being investigated. The low temperature, high density solution appears to offer advantages of low sputtering and low pumping speed for NET/INTOR.

5.2. Application of the Simple, One-Dimensional Model
5.3. The Two-Fluid, One-Dimensional Model and its Applications

The geometry used in the two-fluid, parallel-flow model is similar to that of the simple model (Fig. 3a). Uniform volume sources of particles and energy were prescribed in the scrape-off layer, and an exponential neutral profile was used in the divertor [6]. The two-fluid Braginskii transport equations were solved numerically. The physics in this model includes ion and electron thermal conduction, convective energy transport, equipartition of energy between ions and electrons, viscosity, and sources and sinks of particles, momentum and energy due to interactions with neutral hydrogen atoms.

Harbour and Morgan [6] have made a critical comparison between the two-fluid and one-fluid models, and in Ref. [7] they have used the hydrogen background plasma obtained with the two-fluid model to study the behaviour of trace impurities. One such background plasma is described in Fig. 3b. Here the recycling coefficient was \( \sim 0.985 \) in the divertor (\( \sim 0.90 \) in Ref. [7]), and so the flow velocity was much lower than in...
Ref. [7]. A single particle description was used to follow the dynamics of impurity ions as they move subject to the forces due to electric field, $F_E$, electron and ion drag, $F_u$, and forces arising from non-uniformities of the hydrogen background (i.e. thermal diffusion and viscous forces, $F_{TD}$ and $F_{\eta}$).

The hydrogen plasma solutions sometimes have regions where the electric field points upstream and, with $F_{TD}$ pointing towards regions of higher temperatures, a balance with $F_{\eta}$ is possible. This occurs for heavy impurities for the highly recycling plasma considered here. The viscosity provides an additional force $F_{\eta}$ towards the target. The balance of forces is delicately poised and is shown in Fig. 3c for a W ion at rest in the plasma of Fig. 3b. Between 0.32 and 5.2 m from the divertor target the net force, $F$, is directed upstream, but it is directed downstream elsewhere. If $F$ is small or negative, an additional force, $F_p$, due to the gradient in impurity ion pressure, which acts on the
impurity ions, may be significant. Figure 3d shows the variation with distance from the
target plate of the velocity of W\(^+\) ions moving under the influence of \(F_\mu - F_{TD} + F_E\) (i.e. ignoring \(F_p\) and \(F_n\)). There is a stable stagnation point at 5.2 m from the target,
upstream of which \(F_\mu\) is dominant. An important question is whether the pressure of
the impurity ions in the neighbourhood of the stagnation point can become sufficient to
drive them upstream and/or towards the target [7], [8]. For example, Fig. 3d shows that
a W\(^+\) ion at the stagnation point would require a velocity of \(+1.5 \times 10^4\) m s\(^{-1}\) to explore
upstream to the divertor throat (15 m from the target), whereas if heated to \(~50\) eV,
its thermal speed would be \(0.7 \times 10^4\) m s\(^{-1}\), and most W\(^+\) ions would be retained within
the divertor. For higher charge states (which would predominate) impurity retention
by the divertor would be greater. This favourable divertor performance may depend
on our choice of hydrogen plasma profile (Fig. 3b) which exhibits a maximum in \(T_i\)
at 10 m from the divertor target, and therefore outside the divertor \(F_{TD}\) is directed
downstream. This maximum in \(T_i\) is a consequence of our input assumptions \(T_e \gg T_i\)
and of uniform volume sources outside the divertor, despite the variation in \(T_i\) and
\(T_e\) in this region. Additional uncertainties arise because Roussel-Dupré has recently
shown [9] that the magnitude of \(F_{TD}\) used here and in Refs. [7], [8] is up to \(30\) too
high and a more rigorous theoretical approach is needed; it will be necessary to include
two-dimensional effects.

5.4. Calculations with the Two-Dimensional Model

For the two-dimensional description of the scrape-off plasma we employ [10] a sys-
tem of equations governing the ion density \(n_i\), parallel flow velocity \(u\), radial diffusion
velocity \(v\), and temperatures \(T_e\) and \(T_i\):

\[
\frac{\partial}{\partial t} n_i + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left( \frac{\sqrt{g}}{h_x} n_i u \right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial y} \left( \frac{\sqrt{g}}{h_y} n_i v \right) = S_n, \tag{1}
\]

\[
\frac{\partial}{\partial t} (pu) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left( \frac{\sqrt{g}}{h_x} p u^2 \cdot \frac{\sqrt{g}}{h_z} \eta_x \frac{\partial u}{\partial x} \right)
\]

\[
+ \frac{1}{\sqrt{g}} \frac{\partial}{\partial y} \left( \frac{\sqrt{g}}{h_y} p v u - \frac{\sqrt{g}}{h_z} \eta_y \frac{\partial u}{\partial y} \right) = S_{nu} - \frac{1}{h_x} \frac{\partial p}{\partial x}, \tag{2}
\]

\[
v = -\frac{D}{h_y} \frac{\partial}{\partial y} (\ln n_i), \tag{3}
\]

5. Modelling of the Boundary Plasma of Large Tokamaks
\[
\frac{\partial}{\partial t} \left( \frac{3}{2} n_e T_e \right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left( \frac{\sqrt{g}}{h_x} \frac{3}{2} n_e u T_e - \frac{\sqrt{g}}{h_x} \kappa_x \frac{\partial T_e}{\partial x} \right) \\
+ \frac{1}{\sqrt{g}} \frac{\partial}{\partial y} \left( \frac{\sqrt{g}}{h_y} \frac{3}{2} n_e v T_e - \frac{\sqrt{g}}{h_y} \kappa_y \frac{\partial T_e}{\partial y} \right) \\
= S_E^E - k (T_e - T_i) + \frac{u}{h_x} \frac{\partial p_e}{\partial x} - \frac{v}{h_y} \frac{\partial p_e}{\partial y}. \tag{4}
\]

\[
\frac{\partial}{\partial t} \left( \frac{3}{2} m_i T_i \cdot \frac{1}{2} \rho u^2 \right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left( \frac{\sqrt{g}}{h_x} \left( \frac{5}{2} m_i u T_i + \frac{1}{2} \rho u^2 \right) \right) \\
+ \frac{1}{\sqrt{g}} \frac{\partial}{\partial y} \left( \frac{\sqrt{g}}{h_y} \left( \frac{5}{2} m_i v T_i - \frac{1}{2} \rho v^2 \right) \right) \\
= S_i^i + k (T_e - T_i) - \frac{u}{h_x} \frac{\partial p_i}{\partial x} - \frac{v}{h_y} \frac{\partial p_i}{\partial y}. \tag{5}
\]

\(\sqrt{g}, h_x, h_y\) and \(h_y\) are metric coefficients; the coordinate system may be curvilinear although it must be orthogonal. \(S_n, S_{im}, S_E^E\) and \(S_i^i\) are volume sources of ions, momentum, electron and ion energy. Furthermore, \(\rho = m_i n_i, \rho = p_e - p_i, p_e = n_e T_e, p_i = n_i T_i, n_e = Z n_i,\) and \(k(T_e - T_i)\) is the rate of energy transfer from electrons to ions. Equations (1) - (5) are presented here in greater detail than in Ref. [10] and we now include the contribution of the ion viscosity in the ion energy equation as well as in the ion momentum equation. Note added in this printing: The energy transfer term \(\rho (v, h_y) \partial p_e, \partial y\) was incorrectly omitted in the published version of this paper. Visually the energy transfer between electrons and ions is dominated by \(k(T_e - T_i),\) and the other terms have little effect.

We have used this model to obtain predictions for the energy thickness of the NET/INTOR scrape-off layer and for the temperature profile of the plasma impinging upon the divertor plate. In the anticipated high recycling regime of NET/INTOR the energy thickness of the scrape-off layer will be much less than the density thickness. We modelled an idealized case by assuming 100% recycling of particles and 0% recycling of energy, and also ignoring all atomic physics. The main limitation of this work in comparison with that of Petravic, Post et al. [11] is the lack of a good neutral particle model, but, on the other hand, the important two-dimensional fluid dynamic aspects of the problem are included.

The geometry, mesh and boundary conditions are shown in Fig. 4. The field lines are along the \(x\)-coordinate. AB is the separatrix surface between the scrape-off layer and the main plasma, BD is the side of the divertor plasma. EH is the torus wall.

5.4. Calculations with the Two-Dimensional Model
the divertor plate and AE is a symmetry edge. BFHD is the divertor and CGHD is our recycling region. With \( \sqrt{y} \) and \( h_y \) constant but \( h_y \) a function of \( x \), the width of the scrape-off layer varies with \( x \), but the flow experiences no area change.

The computations were carried out for three input radial power fluxes \( Q_y^{\text{in}} \) of 60, 120 and 240 kW m\(^{-2}\) which for INTOR correspond to 20, 40 and 80 MW into the scrape-off layer, respectively. The coefficients \( D, \kappa_y^e/m_e \) and \( \kappa_i^e/m_i \) were taken to be 1, 2 and 0.2 m\(^2\) s\(^{-1}\), respectively, and \( \eta_y^e/\rho \) was taken to be 0.2 m\(^2\) s\(^{-1}\) in the momentum equation but was neglected in the energy equation. All parallel coefficients, \( \kappa_y^e, \kappa_i^e \) and \( \eta_x^e \) were classical, \( m_i = 2.5 \text{ amu} \) and \( Z_i = 1 \). It was found that the energy thickness of the scrape-off layer was typically 5 to 8 mm, measured along the symmetry edge AE. Details of this will be published elsewhere. Figure 5 shows the variation of plasma temperatures with the ion density at the separatrix, \( n_i^{\text{sep}} \). The separatrix temperature \( T_i^{\text{sep}} \) is a weak function of \( n_i^{\text{sep}} \), but due to the \( T_1^{\text{max}} \) dependence of thermal conductivity, the characteristic temperature at the target, \( T_i^{\text{max}} \), decreases rapidly with increasing \( n_i^{\text{sep}} \). (\( T_i^{\text{max}} \) is the highest of \( T_i \) or \( T_1 \) anywhere on the divertor plate. Usually this is \( T_i \) at corner D).

The high power level, \( Q_y^{\text{in}} = 240 \text{ MW m}^{-2} \) corresponds to INTOR putting 80 MW into the scrape-off layer. The design power of INTOR is 120 MW, so this is realistic if a highly radiation edge is not established. It is seen that at this power level an edge
density of $\gtrsim 1 \times 10^{20} \text{ m}^{-3}$ will be required in order to obtain temperatures $T_e$ and $T_i$ at the plate below about 50 eV.

5.5. Conclusion

We have discussed the application of one-dimensional models for parallel flow to a problem which is inherently two-dimensional. The one-dimensional models have been of considerable value in predicting the cool, high-density regime in divertors (see for example Tenney [12] and Brunelli and Harbour [13]) which was subsequently found experimentally on ASDEX, PDX and D-III. Moreover, we have shown here that simple plasma models are readily used with quite advanced neutral particle and impurity models. However, we should add a strong note of caution, that these models are no better than the physical assumptions that are put into them. For example, if we select a physical regime in which the scale thickness, $\Delta$, of the scrape-off plasma is large, then the heat flux parallel to the field will be low and the temperature will be controlled by the finite thermal conductivity: it might be typically 100 eV at the separatrix and very much lower at the divertor target or limiter. If we change our assumptions, e.g. by assuming a smaller diffusion coefficient for radial transport, then the heat flux parallel to the field will be high and so the temperature both at separatrix and target will be high.

5.5. Conclusion
as shown in Fig. 5 for the highest power at the lowest density. The implications for the
designer of a divertor or limiter system for a large tokamak such as NET or INTOR are
serious: in one regime the heat flux is low and the erosion and contamination rates may
be reasonably small, but in the other regime both the heat flux and the rates of erosion
and contamination may be excessively large. Pending the full, routine availability of our
two-dimensional model, we have adopted a quasi-two-dimensional approach by solving
a series of the simple models for parallel flow, such as described in Sec. 2, and linking
the solutions together to allow for the radial flow of energy. This quasi-two-dimensional
model appears to have all the advantages of the simple one-dimensional model on which
it is based, and it is being applied to study the causes and consequences of variations
in scrape-off layer thickness.

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References

1. M.F.A. Harrison, P.J. Harbour and E.S. Hotston, Plasma Characteristics and Gas
Transport in the Single-Null Poloidal Divertor of the International Tokamak Reactor,
2. M.F.A. Harrison and E.S. Hotston, Predicted Behaviour of the Single-Null Divertor
3. M.F.A. Harrison and E.S. Hotston, Predicted Behaviour of the Pumped Limiter of
INTOR, Report CLM R232, UKAEA Culham Laboratory, 1982.
5. M.F.A. Harrison and E.S. Hotston, Exhaust of Helium and Impurity Release in IN-
TOR with a Divertor or Pumped Limiter (11th European Conference on Controlled
Fusion and Plasma Physics, Aachen, 1983), Europhysics Conference Abstracts 7D-
II (1983), 439-442.
6. P.J. Harbour and J.G. Morgan, Models and Codes for the Plasma Edge Region,
Report CLM-R234, UKAEA Culham Laboratory, 1982.
7. P.J. Harbour and J.G. Morgan, The Transport of Impurity Ions in a Scrap-off
Plasma (11th European Conference on Controlled Fusion and Plasma Physics,

5. Modelling of the Boundary Plasma of Large Tokamaks


References
6. MODELLING OF A TRANSPORT PROBLEM IN PLASMA PHYSICS

Abstract

A two-dimensional, two-fluid model for the edge plasma in a tokamak is presented, and the numerical methods employed for its solution are described. Emphasis is given to those aspects of the numerical treatment that are generally relevant to the solution of coupled systems of convection-conduction equations.

6.1. Introduction

An important problem in plasma physics for nuclear fusion is how to extract from the magnetic confinement device the large amounts of energy and alpha particles that are produced in the inner plasma. In the interior of the toroidal confinement region the main transport process is diffusion across magnetic surfaces, while near the edge it is flow along those field lines that intersect a material boundary \[1\] \[3\]. Until recently these two processes were only modelled separately, using sophisticated codes for the one-dimensional radial transport problem and fairly simple models for the one-dimensional flow along the field lines.

The subject of this presentation is the two-dimensional, two-fluid modelling of the edge plasma region \[4\] \[7\]. In this region both the radial diffusion of particles and energy, and the convection and conduction along field lines are important processes. A
two-fluid model is employed because the electrons and the ions satisfy separate, coupled energy balance equations.

The code that has been developed for these studies is based on a finite-volume discretization of the conservation equations on a topologically rectangular mesh, using methods that were largely developed by D.B. Spalding and co-workers [8], [9]. The discrete coefficients depend continuously on the local cell Péclet number, and give central differencing and pure convective upwind differencing in the limits of small and large Péclet number, respectively. A nonlinear modification enhances stability in the presence of strong gradients. The discretization is fully implicit in time. A distributive relaxation method, leading to an elliptic equation, is employed to obtain the pressure correction at each iteration. The discretized equations are solved with the aid of the Strongly Implicit Procedure, which is based on incomplete L*U decomposition.

The paper is organized as follows: Section 2 presents a simplified version of the governing differential equations of our model. The simplified set exhibits all the features that are relevant for the choice of the numerical methods employed for the solution of the full set, while omitting many of the details. The full set of equations has been given elsewhere [e.g. this thesis, Sections 5.4 and 7.4]. Section 3 reviews the methods that are employed for the discretization and solution of the equations. In Section 4 some calculations are shown.

The present paper is meant to be self-contained for an audience with an interest in computational fluid dynamics but not necessarily interested in the plasma physics context of our work. The governing equations are presented with a minimum of physical justification, and the generally relevant aspects of the numerical treatment are emphasized. The physical aspects of these edge plasma studies are presented in greater detail in Refs. [4]–[7] and in several contributions to the IN'TOR workshop.

6.2. A Two-Dimensional Model of the Edge Plasma

Background to the problem. In a magnetized plasma the charged particles are constrained, in lowest approximation, to follow helical orbits 'tied' to a magnetic field line. Parallel and perpendicular transport processes therefore differ in character. Transport of matter and energy along the magnetic field is governed by a system of equations of the Navier-Stokes form, whereas perpendicular transport is governed by diffusion
equations. Such a plasma is a highly anisotropic medium, in which the coefficients for particle and energy transport parallel to the field are much larger than those for perpendicular transport.

In a magnetic confinement configuration of the tokamak type each magnetic field line in the main plasma is confined to a toroidal 'flux' surface, and because of the strong parallel coupling it can be assumed that the density, the electron temperature and the ion temperature are all constant over the flux surfaces. The transport equations may then be reduced to a one-dimensional system, in which only the radial coordinate enters. The assumption of uniformity along field lines breaks down in the edge region of the plasma, where the field lines intersect material boundaries. In an axisymmetric toroidal device the transport in this region requires a two-dimensional description.

The principal reasons for wanting to model the edge plasma are to understand and predict the heat load on material surfaces and the transport of helium and other minority ions. The issues of energy removal and composition control pose serious problems for a fusion reactor, which have to be resolved if fusion is to become a viable source of energy.

**Simplified mathematical model.** The appropriate set of equations is made up of a continuity equation governing the ion density \( n_i \), a momentum balance equation governing the parallel flow velocity \( u_i \), a diffusion equation for the radial velocity \( r \). and convection–conduction equations governing the electron- and ion temperatures \( T_e \) and \( T_i \). Auxiliary physical quantities are the electron density, \( n_e = Z_i n_i \), the mass density, \( \rho = n_i m_i \), the electron and ion pressures, \( p_e = n_e T_e \) and \( p_i = n_i T_i \), the total pressure, \( p = p_e + p_i \), and the poloidal flow velocity, \( u = (B_\phi / B) u \), where \( B_\phi \) is the poloidal component of the magnetic field and \( B \) is the total magnetic field. The complete system has been given elsewhere (e.g., this thesis, Chapter 7, Appendix). For the purpose of the present discussion, which concentrates on the numerical aspects of the modelling, some simplifications can be made. We consider therefore the following model equations:

\[
\frac{\partial}{\partial t} n_i + \frac{\partial}{\partial x} (n_i u_i) + \frac{\partial}{\partial y} (n_i v_i) = S_n
\]

\[
\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2) + \frac{\partial}{\partial y} (\rho v u) = S_{nu} - \frac{\partial p}{\partial x}
\]

\[
v = -D \frac{\partial}{\partial y} (\ln n_i)
\]

6.2. A Two-Dimensional Model of the Edge Plasma

161
\[
\frac{\partial}{\partial t}\left(\frac{3}{2}n_eT_e\right) + \frac{\partial}{\partial x}\left(\frac{5}{2}n_euT_e - \kappa_x^e \frac{\partial T_e}{\partial x}\right) \\
+ \frac{\partial}{\partial y}\left(\frac{5}{2}n_euT_e - \kappa_y^e \frac{\partial T_e}{\partial y}\right) = S_e^e - k(T_e - T_i) 
\] (4)

\[
\frac{\partial}{\partial t}\left(\frac{3}{2}n_iT_i + \frac{1}{2}\rho u^2\right) + \frac{\partial}{\partial x}\left(\frac{5}{2}n_iuT_i + \frac{1}{2}\rho uu^2 - \kappa_x^i \frac{\partial T_i}{\partial x}\right) \\
+ \frac{\partial}{\partial y}\left(\frac{5}{2}n_iuT_i + \frac{1}{2}\rho uu^2 - \kappa_y^i \frac{\partial T_i}{\partial y}\right) = S_i^i - k(T_e - T_i) 
\] (5)

The terms \( S_n \), \( S_{mz} \), \( S_E^e \), and \( S_E^i \) are volume sources of ions, momentum, electron and ion energy, \( \eta_x^i \) and \( \eta_y^i \) are the poloidal and radial ion viscosity coefficients, \( D \) is a radial diffusion coefficient, \( \kappa_x^e \) and \( \kappa_y^e \) are thermal conductivities, and \( k(T_e - T_i) \) is the electron-ion energy equilibration term.

In this simplified set of equations the distinction between the parallel direction and the poloidal direction has been ignored, the equations have been written in a cartesian coordinate system instead of in a curvilinear orthogonal system, and some contributions have been left out of the energy equations. These aspects do not influence the choice of numerical procedure in a significant way.

The source terms in Eqs. (1) (5) are associated with ionization of neutral particles, recombination and radiation, and have a complicated nonlinear and non-local dependence on the unknowns of the system. The transport coefficients are nonlinear local functions of the solution and of its gradients. For cases of interest the flow velocity is usually subsonic but it increases to sonic velocity at an outflow boundary. Our interest is restricted for the present to steady solutions of the equations. It is to be noted that, although the parallel transport coefficients are much larger than the perpendicular coefficients, also the parallel length scale is larger than the perpendicular scale, so the problem really is two-dimensional. The nonlinearity of the transport coefficients (e.g. \( \kappa_x^e \sim T_e^{5/2}, \kappa_x^i \sim T_i^{5/2}, k \sim T_e^{-3/2} \)) makes it difficult to predict whether any particular term will dominate any other, and even within one calculation the relative magnitude of the transport terms usually varies by several orders of magnitude over the geometrical domain. However, in general the electron energy transport does dominate over the ion energy transport.
Boundary conditions. Counting derivatives, one sees that a total of seven conditions is required on the boundaries perpendicular to the $x$-coordinate, and that eight conditions are required on boundaries perpendicular to the $y$-coordinate. For the momentum equation (2) and for each of the two energy equations (4) and (5), conditions must be given on each segment of the boundary. For the continuity equation (1) and the diffusion equation (3) together, only one condition must be specified on the boundaries perpendicular to the $x$-coordinate, and two conditions on the boundaries perpendicular to the $y$-coordinate.

The boundary conditions appropriate to the two energy equations can specify either the energy fluxes or the temperatures, or more generally they can specify the energy fluxes as function of the temperatures and density. For the parallel momentum equation we usually impose a sonic flow condition on one boundary segment perpendicular to the $x$-coordinate, and zero flow or zero shear elsewhere. For the continuity equation and the diffusion equation together, on one face perpendicular to the $x$-direction and on both faces perpendicular to the $y$-direction, either the density, or the particle flux, or some relation between the two may be specified.

6.3. Numerical Treatment

Outline. Following Ref. [8] it was decided to employ a finite-volume spatial discretization on a non-uniform staggered mesh, and a fully implicit discretization in time. The continuity equation is treated by the implicit method developed by Patankar and Spalding [8], [9], through which it is replaced by a pressure correction equation of standard convection-conduction form. The discrete coefficients for each of the convection-conduction equations are computed using a modification of the power law scheme of Patankar [8]. This scheme is formally second-order accurate and is stable at all values of the cell Péclet or Reynolds number. The modification promotes stability in cases when strong gradients arise. The resulting five-point equations are relaxed separately in a cyclic order; the Strongly Implicit Procedure (SIP) of Stone [10] is employed. The possibility of strong coupling between the two energy equations is dealt with by relaxing in turn the electron energy balance, the ion energy balance, and the total energy balance. The following subsections present the numerical methods in more detail.
Pressure correction procedure. The need for a special treatment of the continuity equation may be seen most clearly by considering the system of equations that governs the incompressible flow of a simple fluid. If one would consider the momentum balance equation to govern the velocity field and the energy equation to govern the temperature, then the continuity equation would have to govern the pressure. But the pressure does not even appear in that equation.

For compressible flow the pressure is a derived quantity, and the density is one of the primary variables. The continuity equation then appears suitable in principle for relaxation of the density field, but severe problems appear for low Mach number flows, when the fluid is effectively incompressible. The traditional recommendation, given for instance in Ref. '11', is to employ an explicit discretization in time for the continuity equation, irrespective of the treatment of the other equations in the system. The time step used for the continuity equation is then governed by the CFL condition based on the velocity of sound.

Patankar and Spalding’s method is to satisfy the continuity equation through simultaneous changes to the density, pressure and velocity fields. We present the method here with reference to a steady-state equation of the form \( \partial(nu)/\partial x + \partial(nv)/\partial y = S_n \), noting that it is equally well applicable to an implicit treatment of a time-dependent equation. At each iteration on the continuity equation the following coupled adjustments are made:

\[
\begin{align*}
  p & \rightarrow p + \xi \\
  n & \rightarrow n + \kappa \xi \\
  u & \rightarrow u - c_x \frac{\partial \xi}{\partial x} \\
  v & \rightarrow v - c_y \frac{\partial \xi}{\partial y}
\end{align*}
\]

Inserting these changes into the continuity equation and retaining only the terms that are linear in \( \xi \), one sees that \( \xi \) is to be obtained as solution to a standard convection-conduction equation:

\[
\frac{\partial}{\partial x} \left( \kappa u \xi - n c_x \frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial y} \left( n v \xi - n c_y \frac{\partial \xi}{\partial y} \right) = r
\]

where \( r \) is the residual before relaxation, \( r = S_n - \partial(nu)/\partial x - \partial(nv)/\partial y \). Thus, through the relations (6) a relaxation procedure that is suitable for Eq. (7) is turned into a relaxation procedure for the continuity equation (1). The local coefficients \( \kappa \),
\( c_x \) and \( c_y \) are chosen in order to minimize the damage that the replacements (6) do to the other equations of the system, i.e., the equation of state and the two equations that govern the velocities.

In order not to upset the equation of state the assignment \( \kappa = (\partial n/\partial p)_T \) is appropriate, so that for our application \( \kappa = 1/(Z_i T_e + T_j) \). The choice of the coefficients \( c_x \) and \( c_y \) is less straightforward and requires consideration of the discretized momentum equation. The discrete equation for the \( x \)-velocity \( u \) has the form

\[
A \cdot u = S_{mu} - \frac{\delta p}{\delta x}.
\]  

We assume that \( A \) is a diagonally dominant operator of five-point form, and that any term that does not fit into \( A \) has been moved into the right hand side. The prescription of Patankar and Spalding for the coefficient \( c_x \) is now to set \( c_x = 1/\alpha \) at each point, where \( \alpha \) is the diagonal coefficient in the matrix \( A \). In this way the effects of the two adjustments \( u \rightarrow u - c_x \partial z/\partial x \) and \( p \rightarrow p + \xi \) approximately cancel out in the \( x \)-component of the momentum equation. With the usual fluid flow problems the prescription for \( c_y \) is similar to that for \( c_x \), but in the system (1)-(5) the \( y \)-velocity is governed by a diffusion equation instead of by a momentum balance equation. In order to let the two adjustments \( v \rightarrow v - c_y \partial z/\partial y \) and \( p \rightarrow p - \xi \) cancel approximately in the diffusion equation the assignment \( c_y = \kappa D/\nu \) is indicated.

Relaxation procedures analogous to this implicit pressure correction procedure have been discussed by Brandt and others in a more general setting (see for instance Refs. [12] and [13]). There the method is referred to as 'distributive relaxation', and is recommended generally as a relaxation procedure for systems of equations that are not separately elliptic: the Cauchy–Riemann system, compressible and incompressible Navier–Stokes, and the Euler equations.

**Spatial discretization.** This subsection is concerned with finite-volume discretization schemes for a convection-conduction equation in conservation form:

\[
\mathcal{L} \phi = \nabla \cdot (\rho u \phi - \nabla \cdot \Gamma \phi) = S.
\]  

We assume that \( \Gamma \) is diagonal when \( \mathcal{L} \) is expanded on coordinates, so that

\[
\mathcal{L} \phi = \frac{1}{\sqrt{\gamma}} \sum_{\alpha} \frac{\partial}{\partial x_\alpha} \left[ \frac{\sqrt{\gamma}}{h_\alpha} (\rho u_\alpha \phi - \gamma_\alpha \frac{\partial \phi}{\partial x_\alpha}) \right].
\]  

6.3. **Numerical Treatment**
Let us consider the three-dimensional case. The region is divided into rectangular cells (control volumes), with $\phi$ discretized at cell centers and $u$ at cell faces. For each interior mesh point $P$ the differential equation can be integrated over the control volume surrounding $P$. Let the neighbors of $P$ be denoted by $E$, $W$, $N$, $S$, $T$, $B$ (east, west, north, south, top, bottom), and let the corresponding cell faces be denoted by subscripts $e$, $w$, $n$, $s$, $t$, $b$. The volume integral can be expressed as a sum of six surface integrals:

$$\iiint S \, dV = J_e - J_w + J_n - J_s + J_t - J_b,$$

where, e.g., for the 'east' face:

$$J_e = \iint \left( \rho u_1 \phi - \gamma_1 \frac{\partial \phi}{\partial x_1} \right) h_2 h_3 \, dx_2 dx_3$$

(11)

In order to arrive at a discretization scheme of seven-point molecule form (five-point form in two dimensions) the expression (11) is to be approximated by

$$J_e \approx \beta \phi_E - \alpha \phi_P$$

(12)

where the coefficients $\alpha$ and $\beta$ depend on approximations $F_e$ and $D_e$ to the strength of flow through the surface and the conductance between the mesh points:

$$F_e \approx \iint \rho u_1 h_2 h_3 \, dx_2 dx_3$$

$$D_e \approx \frac{1}{d_e} \iint \gamma_1 h_2 h_3 \, dx_2 dx_3$$

(13)

in which $d_e$ is the distance between the points $P$ and $E$.

Let $A_e$ denote an approximate area of the cell face, and understand $h_1$, $\rho u_1$ and $\gamma_1$ to be some local average of the corresponding continuous quantity; then $d_e \approx h_1(x_1(P), x_2(P))$, $F_e \approx \rho u_1 A_e$ and $D_e \approx \gamma_1 A_e / d_e$. Two discretization schemes used often are the central difference scheme, which employs

$$\alpha = -D_e - F_e / 2, \quad \beta = D_e + F_e / 2$$

(14)

and the upwind scheme, for which

$$\alpha = \min(-F_e, 0) - D_e, \quad \beta = \min(F_e, 0) - D_e$$

(15)
As is well known, the central difference scheme is second-order accurate but unstable at high cell Peclet number, \( P_e = F_e / D_e \), whereas the upwind scheme is always stable but is only first-order accurate.

Through consideration of the exact solution to the one-dimensional convection-conduction equation with constant coefficients Patankar [8] is led to define two intermediate schemes, both of which approximate central differencing at low cell Peclet number and upwind differencing for zero diffusion at high cell \( P \). The general form of the coefficients in these and other schemes is

\[
\alpha = \min(-F_e, 0) - D'_e, \quad \beta = \min(F_e, 0) - D'_e.
\]

For the piecewise linear scheme \( D'_e = \max(0, D_e - |F_e|/2) \), and for the power law scheme \( D'_e = D_e \max(0, (1 - P_e/10)^3) \). These schemes reduce to upwind differencing for zero conduction at \( |P| \leq 2 \) and at \( |P| \geq 10 \), respectively. In practice there is very little difference between the results obtained by using the power law scheme or the piecewise linear scheme. All the calculations that we reported in Refs. [4]-[7] were done using the power law scheme.

At present we employ for the temperature equations a modification of the piecewise linear scheme which is more robust in cases when strong gradients are present. We now use Eq. (16) together with the assignment

\[
D'_e = \max \left( 0, D_e - F_e \min(\phi_P, \phi_E) \right).
\]

where \( \phi \) is the strictly positive quantity that is being discretized. In comparison with the piecewise linear scheme this discretization enhances the strength of conduction whenever the values of \( \phi \) vary widely over one mesh spacing. The modification helps to avoid convergence to a solution in which the temperature is negative at some point in the grid.

The continuity equation is purely convective. In order to evaluate, e.g., the mass flux through the 'east' cell face, it is natural to employ the assignment \( F_e = \rho_e u_e A_e \), where \( \rho_e = (\rho_P + \rho_E)/2 \). (Remember that \( \rho \) is discretized on cell centers and \( u \) on cell faces.) That is in fact what was done for all the calculations reported in Refs. [4]-[7]. At present, however, we employ

\[
\rho_e = \begin{cases} 
(\rho_P + \rho_E)/2, & \text{if } u_e(\rho_P - \rho_E) \geq 0 \\
2\rho_P\rho_E/(\rho_P + \rho_E), & \text{if } u_e(\rho_P - \rho_E) < 0
\end{cases}
\]

6.3. Numerical Treatment
Notice that \((\rho P + \rho E)/2 \geq 2\rho P \rho E/(\rho P + \rho E)\), with equality holding only if \(\rho P = \rho E\). In comparison with the standard assignment, \(\rho_c = (\rho P + \rho E)/2\), the mass flux computed by using Eq. (18) is reduced when the flow is from a cell of lower \(\rho\) into a cell of higher \(\rho\). This helps to keep \(\rho\) positive. The procedure bears some relation to upwind differencing but it is effective only for large differences in \(\rho\) between neighbouring cells. Where steep gradients do not arise, second-order accuracy is maintained. Notice that the mass flux \(F_e = \rho_e u_e A_e\) computed by using Eq. (18) is a continuous function of \(\rho P, \rho E, \) and \(u_e\).

**Relaxation procedure.** In order to obtain a steady solution to the discretized system of equations (1)–(5), a procedure is employed in which each equation is relaxed in turn in a cyclic order until convergence is achieved. Time-stepping is employed, but only to obtain some under-relaxation; the discretization is fully implicit and within any single timestep the equations are not relaxed to convergence. Each cycle consists of the following actions:

1. The source terms \(S_m, S_m u, S_E\) and \(S_E\) are computed.
2. The momentum balance equation (2) is relaxed by changes to the field \(u\), and the coefficient \(c_T\) is computed for each point in the mesh.
3. The field \(v\) is adjusted to satisfy the diffusion equation (3), and the coefficient \(c_T\) is computed for each point in the mesh.
4. The continuity equation (1) is relaxed through simultaneous changes to \(u, v, r\), and \(p\).
5. The electron and ion energy equations (4) and (5) are relaxed separately by changes to the fields \(T_e\) and \(T_i\).
6. The total energy equation (4) + (5) is relaxed by coupled changes to \(T_e\) and \(T_i\).
7. The continuity equation is relaxed again, as in step 4.

Relaxation of each of the five-point equations is done by means of one or two iterations of the Strongly Implicit Procedure of Stone [10], as implemented in the NAG library code D03UAF [14]. The residuals of all equations are monitored in order to decide whether a converged solution has been achieved.

A special complication in the system (1)–(5) is the presence of two energy equations which can be strongly coupled over at least part of the domain through the term \(k (T_e - T_i)\). Relaxing these equations separately will then lead to very slow convergence. (Analogous problems occur in the modelling of chemically reacting flow.) As seen above,
this problem is dealt with by relaxing not just the separate energy equations but also the total energy equation. The total energy equation is relaxed through identical changes to $T_e$ and $T_i$, and in this process the energy coupling term can be ignored.

The heuristic behind the repeated iteration on the continuity equation in step (7) is that this equation can be unfavourably affected by the pressure change associated with the temperature relaxation in steps (5) and (6). Although we did not consider this heuristic particularly convincing at first, the inclusion of step (7) in the relaxation procedure did lead to a marked improvement in the rate of convergence.

### 6.4. Example Calculations

Calculations made with the edge plasma code described above have been presented in Refs. 4-7 and in several contributions to the INTOR workshop. In those references the emphasis has been on parametric studies, involving variation of the assumptions made about the outer core plasma density and about the energy transport coefficients. The principal concern is with the peak temperature near material surfaces and the power load on these surfaces as function of the free parameters in the model.

In this Section we will not present such parametric studies, but show instead the results of two specific calculations in more detail. The first calculation involves a geometry and plasma parameters representative of the ASDEX divertor experiment, and the second calculation is directed at the TFCX conceptual limiter fusion experiment. The present discussion refers to the set of equations given in the appendix to Chapter 7 of this thesis, which concerns parametric studies of the TFCX edge plasma.

**The ASDEX model.** Figure 1 shows a poloidal cross-section of the ASDEX experiment, with the domain of the calculation indicated. This region is mapped to the rectangular mesh shown in Fig. 2. The size of the region is $1.0 \text{ m} \times 0.04 \text{ m}$ and the mesh contains 32 x 24 cells, which are strongly concentrated in front of the divertor target plate. The metric coefficients $\sqrt{g}$, $h_r$ and $h_\varphi$ were taken as constants in this calculation, and $B_n/B$ was assumed to have the constant value 0.06. The ion mass and the charge number were taken as $m_i = 1.5 m_p$ and $Z_i = 1$, respectively.

The parallel transport coefficients $\eta^i$, $\kappa^i$ and $k$ were assumed to have the classical values given by Spitzer and Braginskii [15]. For $\kappa^e$ we assumed a flux-limited classical
Fig. 2. The computational mesh for the AVDEX study.

Fig. 1. Geometry of the AVDEX separator tray and diverter.
value, which was computed according to the formula
\[ \kappa = \kappa_{\text{SH}} \left[ 1 + \frac{q_{\text{SH}}}{q_{\text{FL}}} \right]^{-1/\gamma}, \] (19)
where \( \kappa_{\text{SH}} \) is the classical (Spitzer-Härm) electron heat conduction coefficient, \( q_{\text{SH}} = -\kappa_{\text{SH}} \frac{\partial T_e}{\partial x} \) is the corresponding classical conductive electron energy flux density, \( q_{\text{FL}} = \alpha n_e T_e \sqrt{T_e}/2m_e \) is the flux limit, and the parameters \( \alpha = 0.12 \) and \( \gamma = 1 \) were chosen in accordance with Refs. [16] and [17].

The radial transport coefficients were assigned anomalous values: \( D = 2 \text{ m}^2/\text{s} \), \( \eta_y^i/\rho = 0.2 \text{ m}^2/\text{s} \), \( \kappa_y^i/n_e = 4 \text{ m}^2/\text{s} \) and \( \kappa_y^i/n_i = 0.2 \text{ m}^2/\text{s} \). These values are near the upper end of the range found empirically for the ASDEX edge plasma.

The boundary conditions were chosen as follows:

On the interface with the main plasma, the ion density, parallel flow velocity and two temperatures were prescribed: \( n_i = 1.8 \times 10^{13} \text{ m}^{-3}, u = 0, T_e = 80 \text{ eV} \) and \( T_i = 80 \text{ eV} \).

On the outer wall we prescribed zero transverse particle flux, zero shear, \( T_e = 2 \text{ eV} \) and \( T_i = 2 \text{ eV} \).

- On the upstream boundary, symmetry conditions were specified: \( \partial n_i/\partial x = 0, u = 0, \partial T_e/\partial x = 0 \) and \( \partial T_i/\partial x = 0 \).

- On the divertor plate we required sonic flow: \( u = \sqrt{\rho \cdot \rho} \) and the following two conditions on the temperatures:

\[ Q_e \quad \delta_e n_e u T_e, \] \[ Q_i \quad \delta_i n_i u T_i - \frac{1}{2} \rho n_i u_{\text{e}}^2, \] (20)

where \( \delta_e = 4.0 \) and \( \delta_i = 2.5 \). Conditions of the form (20) and the numerical values of \( \delta_e \) and \( \delta_i \) are obtained from kinetic models of the plasma sheath.

To obtain the volume sources \( S_n, S_{nu}, S_{nT_e}^i \) and \( S_{nT_i}^i \) we relied on a simple model for the hydrogen recycling. The flux of plasma ions impinging on the divertor plate was assumed to be completely converted into a flux of neutrals flowing along the poloidal coordinate and away from the plate at a velocity \( u = \sqrt{10 \text{ eV} \cdot m_i} \). (Recycling coefficient \( R > 1 \)). These neutrals were ionized according to an approximation to the collisional radiative cross-section, which we took as \( (\sigma v) = 3 \times 10^{-14} \cdot a^2 \cdot (3 \cdot a^2) \text{ m}^3/\text{s} \), in which \( a = T_e/10 \text{ eV} \). With each ionization event was associated an electron energy loss of \( 25 \text{ eV} \) and an ion energy gain of \( 5 \text{ eV} \); the momentum source from ionization processes was ignored in view of the small pitch angle of the field lines.

6.4. Example Calculations
Results of the ASDEX calculation. The results of this calculation are shown in the contour plots, Figs. 3-6.

Figure 3 shows the density field \( n_i \). In the recycling zone in front of the hot end of the divertor plate the density rises to a value \( n_i = 5.8 \times 10^{19} \text{ m}^{-3} \), which is to be compared with the prescribed value of \( 1.8 \times 10^{19} \text{ m}^{-3} \) on the interface with the main plasma.

In Figure 4 the Mach number associated with the parallel flow velocity is displayed. The flow is subsonic everywhere, but increases to sonic velocity on the plate. The global balance equations cause the flow to be directed away from the plate in the vicinity of the outer wall.

Figures 5 and 6 show the electron temperature \( T_e \) and the ion temperature \( T_i \). It can be seen that the two temperatures are not the same. In the hotter region of the scrape-off layer, outside the recycling zone, the strong parallel conductivity causes the temperatures to be nearly constant along the magnetic field. This effect is more pronounced for the electron temperature than for the ion temperature.

The TFCX model. Figure 7 shows a poloidal cross-section of the TFCX design, with the domain of the calculation indicated (see also Chapter 7). This region is mapped to the rectangular mesh shown in Fig. 8. The size of the region is \( 4.0 \text{ m} \times 0.2 \text{ m} \) and it is divided into 32 \( \times \) 24 cells. As in the ASDEX example, the metric coefficients \( g \), \( h_x \) and \( h_y \) are constants, and in this calculation, \( B_0/B = 0.2 \). The ions are assumed to have mass \( m_i = 2.5 m_p \) and charge number \( Z_i = 1 \).

The parallel transport coefficients were again assumed to have the classical values, with the flux limit included as in the ASDEX example. (A flux limit was not used in the earlier studies reported in Chapter 7, but for the high density regime studied there the flux limit has only a small influence on the solutions. Having implemented the flux limit in view of the ASDEX modelling, we now prefer to use it for all calculations.) For the radial transport coefficients we assumed the following anomalous values: \( D = 2 \text{ m}^2 \text{ s}^{-1} \), \( \eta_{ij}/\rho = 0.2 \text{ m}^2 \text{ s}^{-1} \), \( \kappa_{ij} n_e = 4 \text{ m}^2 \text{ s}^{-1} \) and \( \kappa_{ij} n_i = 0.2 \text{ m}^2 \text{ s}^{-1} \).

The boundary conditions were the following:

On the interface with the main plasma, the ion density, parallel flow velocity and two temperatures were prescribed: \( n_i = 7 \times 10^{19} \text{ m}^{-3} \), \( u \), \( 0 \), \( T_e = 150 \text{ eV} \) and \( T_i = 150 \text{ eV} \).
**Fig. 3.** Contour plot of the ion density. The increment between the dotted contours is $1 \cdot 10^{18}$ m$^{-3}$; between the solid contours it is $5 \cdot 10^{18}$ m$^{-3}$.

**Fig. 4.** Contour plot of the Mach number of the flow parallel to the field. The increment between the dotted contours is 0.02; between the solid contours it is 0.10.
**Fig. 5.** Contour plot of the electron temperature. The increment between the dotted contours is 1 eV; between the solid contours it is 5 eV.

**Fig. 6.** Contour plot of the ion temperature. The increment between the dotted contours is 1 eV; between the solid contours it is 5 eV.
Fig. 7. Geometry of the TFCX edge plasma.

Fig. 8. The computational mesh for the TFCX study.

6.4. Example Calculations
- On the outer wall, we prescribed zero transverse particle flux, zero shear, and temperature pedestals $T_e = T_i = 2 \text{ eV}$.

- On the midplane and on the downstream boundary inward from the limiter, symmetry conditions were specified: zero parallel flow and zero poloidal gradients of $n_i, T_e$ and $T_i$.

- On the limiter, sonic flow is required: $u_\parallel = \sqrt{p/p_i}$ and the energy fluxes were specified as $Q_e = \delta_e n_e u T_e$ and $Q_i = \delta_i n_i u T_i + \frac{1}{2} \rho u u^2$, in which $\delta_e = 4.0$ and $\delta_i = 2.5$.

The hydrogen recycling model used to calculate the volume sources was similar to the one used for the ASDEX example, but it proceeded in two stages. The recycled neutral flux travels first along the poloidal coordinate for a distance calculated from the local ionization rate coefficient, and then it travels radially inward for again such a distance. In this way we take into account to some extent the effect of the inclination of the limiter surface, which causes the neutral flux coming from the limiter to move preferentially inwards. For this calculation the recycling coefficient was set to 0.98.

**Results of the TFCX calculation.** The outcome of this calculation is shown in Figs. 9-12.

Figure 9 shows the ion density. The maximum density at the limiter reaches $3.7 \times 10^{20} \text{ m}^{-3}$, which is to be compared with the density of $7.0 \times 10^{19} \text{ m}^{-3}$ prescribed at the interface with the main plasma. The recycling is seen to be much more localized than in the ASDEX example. This is due to the higher edge density assumed for TFCX.

Figure 10 shows the Mach number of the parallel flow. As in this calculation the neutral particles are preferentially displaced inwards, a compensating flow must occur in the ion channel. Figure 10 indeed shows a large circulating flow region, with the ions moving away from the limiter in the vicinity of the main plasma. The consequences of this flow pattern for the penetration into the main plasma of impurities released from the limiter tip, have not yet been adequately studied.

Figures 11 and 12 show the electron and ion temperatures respectively. The dominating effect of parallel electron heat conduction can again be seen. A large electron temperature gradient along the field occurs only in the high-recycling zone.
Fig. 9. Contour plot of the ion density. The increment between the dotted contours is $5 \times 10^{16} \text{ m}^{-3}$; between the solid contours it is $2.5 \times 10^{19} \text{ m}^{-3}$.

Fig. 10. Contour plot of the Mach number. The increment between the dotted contours is 0.02; between the solid contours it is 0.10.
Fig. 11. Contour plot of the electron temperature. The increment between the dotted contours is 2 eV; between the solid contours it is 10 eV.

Fig. 12. Contour plot of the ion temperature. The increment between the dotted contours is 2 eV; between the solid contours it is 10 eV.
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References


7. LOW-TEMPERATURE PLASMA NEAR A TOKAMAK REACTOR LIMITER

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Abstract

Analytic and two-dimensional computational solutions for the plasma parameters near a toroidally symmetric limiter are illustrated for the projected parameters of a Tokamak Fusion Core Experiment (TFCX). The temperature near the limiter plate is below 20 eV, except when the density 10 cm inside the limiter contact is \(8 \times 10^{13} \text{ cm}^{-3}\) or less and the thermal diffusivity in the edge region is \(2 \times 10^4 \text{ cm}^2/\text{s}\) or less. Extrapolation of recent experimental data suggests that neither of these conditions is likely to be met near ignition in TFCX, so a low temperature plasma near the limiter should be considered a likely possibility.

7.1. Introduction

To have an adequate safety margin against sputtering and erosion, the plasma near a material boundary in a tokamak should have an electron temperature of \(\sim 10\) eV or less. This is relatively easy to achieve with an expanded boundary or divertor configuration [1], where the large volume in the boundary allows for the possibility of radiating a significant power. A large scrape-off thickness can be achieved by radial

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neoclassical and/or charge-exchange (CX) transport because the competing process of heat condition parallel to the magnetic field occurs over an extended connection length near the magnetix separatrix.

The requirements for achieving low temperature near a limiter plate have not been well defined. Earlier work used poloidally averaged transport models to investigate "cold plasma mantle" solutions to this problem [2]. These solutions required a sizeable poloidally averaged impurity density on the outermost flux surfaces in order to radiate the power flow from an ignited (or nearly ignited) plasma core. This required limited inward plasma transport near the plasma periphery in order to avoid mixing impurities into the plasma core. Subsequent investigations of microinstabilities due to the strong radial temperature gradients have cast doubt on whether such a cold plasma mantle can be obtained in practice [3].

Recent experience has shown that poloidal asymmetries play a dominant role in the energy flows in the peripheral plasma in a tokamak. We have therefore reinvestigated the question of tokamak boundary conditions by including consideration of poloidal asymmetries in an analysis of the temperature expected near the limiter of a tokamak with an ignited plasma core. First, we adapt a simple "two-point model" of the poloidal asymmetries [4] in order to illustrate the important contributions to the energy and particle flows. Then we present numerical solutions of two-dimensional particle, momentum, and energy balances.

The emphasis here is not on working out in great detail the consequences of a specific set of assumptions about boundary density, radial transport coefficients, etc. Rather, we survey a variety of assumptions to see which give an acceptable temperature near the limiter plate. We then go on to review briefly the present state of knowledge about energy and particle transport, and we argue that favourable solutions are reasonably likely to be realized in practice. This has significant implications for the range of options that must be considered in designing limiters for ignition experiments.

7.2. Two-Point Model

The geometry of the two-point model [4], [5] is shown in Fig. 1. For the particular example worked out here, the limiter is positioned at the bottom, but the diagram can be rotated ±90° for inboard or outboard limiters. The power from the plasma core is

182 7. Low-Temperature Plasma Near a Tokamak Reactor Limiter
Fig. 1. Geometry for analytic and two-dimensional models of the TFCX scrape-off.

7.2. Two-Point Model
assumed to leave the upper scrape-off regions by heat conduction along magnetic field lines. The D-shaped toroidal plasma is first modelled as an equivalent circular cylinder and the scrape-off regions are then elongated to rectangular boxes. Half of the power flows through each of two such boxes, only one of which is shown here and modelled below. The width of the scrape-off region is taken as twice the half-width \( L_\psi \) estimated from the power balance for the first region,

\[
\frac{n \chi T A_{\text{core}}}{L_\psi} = Q_{\text{core}},
\]

where

\[
n = \text{density in region 1},
\]

\[
\chi = \text{temperature in region 1},
\]

\[
T = \text{radial thermal diffusivity in region 1},
\]

\[
A_{\text{core}} = 4\pi^2 R_0 a_{eq} = \text{surface area of the plasma core, where } R_0 \text{ is the major radius and } a_{eq} \text{ is the circular equivalent minor radius},
\]

\[
Q_{\text{core}} = \text{power outflux from the core}.
\]

The energy balance for each of the scrape-off regions near the limiter in the analytic model includes conduction from region 1, radiative losses in region 2, and transport to the limiter through an electrostatic sheath,

\[
\frac{1}{2} Q_{\text{core}} = Q_{\text{cond}} = P_{\text{rad}} + Q_{\text{sheath}},
\]

where

\[
Q_{\text{cond}} = \frac{B_\theta}{\beta q_i A},
\]

\[
q_i = -\kappa \frac{T_2 - T_1}{L_1},
\]

\[
\kappa = \left( \frac{\gamma_0}{Z_{\text{eff}}} \right) \frac{3T^{5/2}}{4m_e^{1/2} (2\pi)^{1/2} \lambda c^4},
\]

\[
T = \frac{T_1 + T_2}{2},
\]

and where \( \gamma_0 \approx 12.2 - 9Z_{\text{eff}}^{-1/2} \) and \( \lambda \) are an approximate non-Lorentz gas correction and Coulomb logarithm given by Braginskii [6]. (Note that we cannot conveniently integrate.
this energy balance [7] when there is a substantial contribution from radiation, so we have taken care to use the boundary-centered value \( T = (T_1 + T_2)/2 \) when computing the parallel conductivity.) We include carbon radiation,

\[
P_{\text{rad}} = n_2^2 f_c L_Z V_2,
\]

where \( f_c = (Z_{\text{eff}} - 1)/Z_C (Z_C - 1) \) relates the fractional carbon concentration to its mean charge, \( Z_C \), and the effective charge, \( Z_{\text{eff}} \); and \( L_Z \approx 5 \times 10^{-19} \text{ erg cm}^3/\text{s} \) is the coronal equilibrium radiation rate for carbon near 10 eV [8]. The geometric parameters in the energy balance are the effective length \( L_l \) of region 1, the area \( A = 2\pi R_0 \Delta \) of the boundary between the regions (with \( \Delta = 2L_\psi \)), the volume \( V_2 = 2\pi R L_2 \Delta \) of region 2 where \( L_2 \) is the poloidal length of region 2, and the magnetic field inclination \( B_\theta / B \approx \epsilon/q \) where \( \epsilon = a_{\text{eq}}/R \) is the inverse aspect ratio and \( q \) is the equivalent circular safety factor. The energy flow to the sheath is \( Q_{\text{sheath}} = 8n_2 T A B_\theta / B \) where \( v \leq (2T/m_i)^{1/2} \);

\[
m_i = \bar{A}_i m_p; m_p \text{ is the proton mass; and } \bar{A}_i \text{ is the average ion atomic mass.}
\]

These equations allow one to determine the upstream temperature, \( T_1 \), as a function of downstream density \( n_2 \) and temperature \( T_2 \), given assumptions about \( Q_{\text{core}}, \chi, Z_{\text{eff}}, \) and the geometric quantities \( L_l, L_2, R, a_{\text{eq}}, \) and \( q \). The upstream density is determined by assuming that half the upstream pressure is lost to kinetic motion in the downstream region to obtain \( n_1 T_1 = 2n_2 T_2 \) (consistent with the usual assumption of sonic flow through the sheath boundary). It is also of interest to compute the ratio \( R \) of the plasma flux on the plate to the particle outflux from region 1:

\[
R^{-1} = 1 - (1 - f_{\text{leak}})(1 - e^{-\kappa_0}).
\]

where

\[
\kappa_0 = L_0 / \lambda_{\text{mfp}}^0 = \text{neutral opacity of region 2,}
\]

\[
\lambda_{\text{mfp}}^0 = v_0/(n_2 \langle \sigma v \rangle_2) = \text{neutral mean-free-path to ionization,}
\]

\[
v_0 = [2E_0/(A_0 m_p)]^{1/2} = \text{neutral velocity,}
\]

\[
E_0 = \text{neutral energy,}
\]

\[
\bar{A}_0 = \text{mean atomic mass of the neutrals.}
\]

\[
\langle \sigma v \rangle_2 = \text{Electron impact ionization rate coefficient evaluated at the temperature } T_2 \text{ (see Sec. 7.3.A).}
\]

\[
L_0 = 2^{1/2} L_\phi = \text{effective path length for neutrals in region 2,}
\]

\[
f_{\text{leak}} = e^{-\kappa_0} = \text{transverse leakage of neutrals from the recycling region.}
\]

7.2. Two-Point Model
(Note that the recycling near the limiter plate is reduced by two factors, one for poloidal transmission of neutrals through the recycling region and one for leakage in the radial direction. The peculiar notation here is adapted from treatments of divertor plasma where the transverse leakage and penetration through the recycling region are not identical; see Ref. [4].) Note that we have neglected CX, which will generally mean that we somewhat underestimate the deceleration of the parallel flow. The Mach number of the flow just before the recycling region for \( M^2 < 1 \) is then approximately:

\[
M = \frac{1}{2R} \left( \frac{T_1}{T_2} \right)^{1/2}
\]

A numerical example of the results of these equations for \( a_{eq} = 120 \times 1.6^{1/2} = 150 \) cm, \( R_0 = 300 \) cm, \( Q_{core} = 50 \) MW, \( \chi = 5 \times 10^4 \) cm\(^2\)/s, \( E_0 = T_2 = 12\) eV, \( n_2 = 2 \times 10^{14} \) cm\(^{-3}\), \( L_2 = a_{eq} \pi/4 = 120 \) cm, \( q = 2.3, \tilde{A}_i - \tilde{A}_0 = 2.5, Z_C = 6, \) and \( f_C = 0.012 \) (\( Z_{eff} = 1.36 \)) gives \( L_\psi = 1.4 \) cm, \( P_{rad} = 15 \) MW, \( Q_{sheath} = 10 \) MW, \( T_1 = 62 \) eV, \( n_1 = 0.8 \times 10^{14} \) cm\(^{-3}\), \( R = 2.0, \) and \( M = 0.56 \). That is to say, reasonable upstream plasma conditions \((n_1, T_1)\) appear to be compatible with reactor-relevant power flux, low temperature near the limiter, and a moderately high recycling region preceded by subsonic flow. To test and elaborate on the predictions of this simple model, two-dimensional transport simulations were performed.

### 7.3. Two-dimensional Flows

#### 7.3.A. Fixed parameters.**

The geometry described above was extended to include 10 cm of the plasma region on closed flux surfaces. The computational mesh is related to the corresponding Tokamak Fusion Core Experiment (TFCX) geometry in Fig. 1. A typical result [9] is projected onto a magnetohydrodynamic (MHD) equilibrium for TFCX in Fig. 2. The transport equations were solved in rectangular geometry and included balances for electrons, momentum, electron energy, and ion energy, with classical bulk viscosity and parallel heat conduction, and anomalous radial transport of particles, parallel momentum (i.e., anomalous shear viscosity), and ion and electron energy [7], as described in detail in the Appendix (see also Refs. [9] and [10]). Viscous heating was also included in these calculations, but had no visually noticeable effect on graphs of the solutions. The radial ion thermal diffusivity and shear viscosity were \( \chi_i = 0.2 \times 10^4 \) cm\(^2\)/s and \( n_i \tilde{A}_i \eta_{x_i} \) = \( n_i \tilde{A}_i m_p \chi_i \), where the mean ion mass number was
Fig. 2. (a) Electron density and (b) electron temperature projected onto TFCX geometry for $\chi_e = 5 \times 10^4 \text{ cm}^2/\text{s}$.
\[ A_i = 2.5. \] The mean ion charge was \( Z_i = 1 \), and therefore \( n_e = n_i (= n) \). We denote the radial distance on the computational mesh by \( y \) and the poloidal distance by \( x \).

For this application, we employed a simple model of particle and energy sources based on exponential attenuation of recycling from the limiter and wall. Plasma striking the boundaries was recycled as 10-eV neutrals attenuated first along the field lines (for the limiter) and then radially inward. The attenuation of neutral flux through the plasma was computed from the rate of electron impact ionization. The ionization source rate coefficient was approximated by the expression \( \langle \sigma v \rangle = 3 \times 10^{-8} \times a^2/(3 + a^2) \) \( \text{cm}^3/\text{s} \), with \( a = T_e/(10 \text{eV}) \). Each ionization event removed 25 eV from the electron energy and added 5 eV to the ion energy.

Fixed boundary conditions at the outer wall were \( n_i = 10^{13} \) \( \text{cm}^{-3} \), \( T_e = T_i = 2 \text{eV} \), \( \partial u / \partial y = 0 \), where \( u \) is the flow speed along the magnetic field. At the upstream end, and at the downstream end inward from the limiter, we prescribed zero poloidal flow and zero poloidal gradients. On the limiter the conditions were \( u = (p/\rho)^{1/2} \), and heat fluxes \( Q_e = \delta_e n_e u T_e \), \( Q_i = \delta_i n_i u T_i + 1/2 \rho u^3 \), with \( \delta_e = 4 \) and \( \delta_i = 5/2 \), where \( p = n_e T_e - n_i T_i \) and \( \rho = m_i A_i m_p \). The conditions on the temperatures at the inner boundary were \( T_e = T_i = T_m \), with \( T_m \) computed in the code in order to obtain an average power flux into the scrape-off of 135 kW/m². The parallel flow velocity was zero along the inner boundary.

### 7.3.B. Results of Parameter Variation.

The radial energy and particle transport coefficients were varied, as was the density on the inner boundary. The energy and particle fluxes were

\[ Q_j = \chi_e n_j \partial T_j / \partial y, \quad j = e, i, \]

and

\[ \Gamma = D \partial n / \partial y, \]

where \( \chi_e \) was set to \( 2 \times 10^4 \) \( \text{cm}^2/\text{s} \) or \( 5 \times 10^4 \) \( \text{cm}^2/\text{s} \), with \( D = \chi_e / 2 \) and \( \chi_i = 2 \times 10^3 \) \( \text{cm}^2/\text{s} \).

Results of these parameter variations are given in Fig. 3 where we show the electron temperature in front of the limiter tip. For all but the lowest diffusivities and inner boundary densities, the temperature in front of the limiter is below 20 eV. Upstream from this location is a transition region (to use terminology used for a similar phenomenon in the solar atmosphere) to a point where electron heat conduction along
field lines maintains nearly constant temperature. When the boundary density drops below a critical value, a cold recycling region fails to develop in front of the limiter, and the transition to low temperatures no longer occurs. The existence of such a critical density is compatible with the results of Petravic et al. [11] who predicted a maximum $T_e > 60$ eV near the TFCX limiter plate with an upstream core boundary density of $n_e = 0.4 \times 10^{14}$ cm$^{-3}$. The critical density depends on radial transport. Enhanced radial transport dilutes the parallel heat flux density and facilitates formation of the temperature transition. When volumetric energy losses are included, as here, then the sensitivity to radial transport rates is stronger than obtained from simpler analytic estimates. This is because increasing the volume of the scrape-off plasma soon leads to a situation where volume losses are significant, until the temperature in the recycling region becomes too low to support further volume losses.

Of particular interest is the poloidal power flow shown in Fig. 4. Peak poloidal flow is 8 MW/m$^2$, fading to 4 MW/m$^2$ in a radial distance of 1.6 cm.

7.3.C. Uncertainties in $n_e$ and $\chi_e$. To appreciate the significance of the above results requires an understanding of how scrape-off parameters have recently been chosen for reactor designs such as the International Tokamak Reactor (INTOR). To the best of our knowledge these parameters have been chosen as described below, first for the boundary density and then for the boundary diffusivity.

The boundary density is generally obtained from an estimated volume average or peak core plasma density and a ratio of the core plasma density to the boundary value. The ratio of core to separatrix density was found to be $\sim 3$ for a variety of conditions in the first studies of INTOR plasma parameters [12], and later studies did not contradict this conclusion. We now know, however, that very broad density profiles can develop in cases where intense edge-localized recycling can occur. One should therefore ask whether there is experimental evidence to support the expectation of peaked density profiles in ignited plasmas. It is true that many present tokamak experiments show peaked density profiles, but these experiments have substantial contributions from deep fuelling profiles and/or the Ware pinch. By contrast, the present calculations show extremely shallow fuelling profiles. The contribution of the Ware pinch in an ignition plasma is negligible, as shown by the flat central density profiles obtained with the Ware pinch included in early INTOR modelling studies. An assumption that ignition plasmas will have peaked density profiles must therefore be based on an extrapolation of an anomalous pinch. However, the evidence that an anomalous hydrogen pinch exists at
all is based either on the dubious assumption of a flat or monotonic diffusion coefficient and/or a rather dubious analogy with impurity transport. There is therefore very little basis for extrapolating such an effect to ignited plasmas, as discussed elsewhere in detail [13]. To the extent that we have density profiles from high-power experiments with shallow fuelling profiles, such as the Poloidal Divertor Experiment (PDX) and Axially Symmetric Divertor Experiment (ASDEX) H-mode and possibly the PDX scoop experiment, the experimentally observed trend is toward very flat density profiles, with the density at 90% of the separatrix minor radius approaching one-half or more of the line-averaged density. These experiments still have an appreciable loop voltage and contribution from the Ware pinch. It therefore seems reasonable that we should investigate a case with relatively flat density profiles—with the obvious qualification that all such extrapolations are difficult and other assumptions should be pursued in
parallel. Using an extrapolation based on a purely empirical formula such as \( \chi_e \sim 5 \times 10^{17}/n_e \) (INTOR) leads to larger radial gradients at the boundary in one-dimensional radial transport simulations than have been observed in experiments. It seems dangerous to use such an empirical extrapolation of existing data that does not take into account the effect of such gradients. When one is forced to make an extrapolation of edge transport rates to construct a baseline case for point modelling studies, the best one can do is to use available extrapolations of the effect of gradients on transport. The only such model now available is a preliminary semiempirical model based on a very limited set of experimental simulations [14]. This model is consistent with the values at the radius of the limiter tip set to \( \chi_e = 2D = 5 \times 10^4 \text{ cm}^2/\text{s} \), as used to obtain the results described in Fig. 3.

According to the considerations given above, the value \( n_e = 10^{14} \text{ cm}^{-3} \) used here could conceivably be a rather low value for the inner boundary density, but a higher boundary density leads only to a proportionally higher density near the limiter plate with no qualitative difference in the patterns of temperature distribution, Mach number, or power flows. A higher \( \chi_e \) is not necessary for achieving low \( T_e \) near the limiter in this density range.

### 7.4. Conclusions

We have demonstrated solutions of two-dimensional plasma flows with low temperature and intense recycling near a limiter plate, and have explained why these solutions may be compatible with currently possible extrapolation from the limited existing experimental data. We have also investigated the dependence of these solutions on edge plasma transport and shown that low-temperature solutions occur under a variety of conditions. Of course, there are many other uncertainties about edge plasma. As one example, we set \( \Gamma = (\chi_e/3)\partial n/\partial y - (\chi_e/6)(n/p)\partial p/\partial y \) and got virtually identical results for Fig. 3 as with \( \Gamma = -\chi_e/2\partial n/\partial y \). There are also significant problems with our neutral transport model, which can only be rectified by including a complete treatment of CX effects. Since CX reactions are the dominant effect in neutral transport below \( \sim 10 \text{ eV} \), the temperatures below this value in our model are very uncertain. A separate investigation of these effects suggests, however, that CX should not lead to a major increase in limiter sputtering [15]. Finally, we note that radiative losses from the plasma boundary depend sensitively on the distribution of electron and impurity.
density near the limiter. While the other features of our analytic model are relatively insensitive to its simple geometry, the total radiative loss is only an order of magnitude estimate without a detailed two-dimensional multispecies transport analysis. It is therefore possible that impurity radiation losses not included in our two-dimensional model could make a significant contribution to formation of a low-temperature plasma near the limiter plate. It therefore seems reasonable, but not certain, that a solution exists with low plasma temperature near the limiter, but the exact plasma temperature that may be obtained is uncertain. The questions remain of whether our solutions are accessible and stable.

As to the question of accessibility, we find a continuum of solutions as the density is gradually raised. We therefore expect the high-density solutions to be approached merely by reducing the pumpout rate. Only if the radial transport were increasing at higher density might one possibly expect a bifurcation, with low- and high-recycling solutions simultaneously possible at a given pumpout rate. (This may be the case in a divertor where neoclassical and/or CX effects can significantly affect radial transport at very high density.) In such a case, the high-density solution would be obtained by temporarily reducing the pumpout rate or by depositing a fuelling pellet in the scrape-off. A transient increase in impurity content may also help to reach to the high-density state. For example, eliminating the carbon from our above analytic calculation gives a temperature $T_2 = 21$ keV instead of 12 keV near the limiter. If this higher temperature resulted in significant generation of impurities from the limiter plate, it should help drive the temperature down. Next consider the question of stability. Since high-density solutions are strongly influenced by volume losses, they tend to evolve toward even higher density. This will continue until the temperature is too low to sustain further volume losses (for pure hydrogen). The high-density state should therefore be a thermally stable equilibrium (except perhaps with sufficient impurities to cool to hydrogen recombination temperatures by $\Delta n = 0$ transitions). Provided the low-temperature region near the limiter occupies a small fraction of the radial current profile, as in Fig. 2, it may not have a significant effect on stability of potentially disruptive low MHD mode numbers. More detailed analysis of solutions such as those presented here may therefore be fruitful.
Appendix. Equations for the Two-Dimensional Model

We employ a system of equations governing the ion density \( n_i \), the parallel flow velocity \( u_{\parallel} \), the radial diffusion velocity \( v \), and electron- and ion temperatures \( T_e \) and \( T_i \), respectively. Auxiliary physical quantities are the electron density, \( n_e = Z_i n_i \); the mass density, \( \rho = m_i n_i \); the electron and ion pressures, \( p_e = n_e T_e, p_i = n_i T_i \); the total pressure, \( p = p_e + p_i \); and the poloidal flow velocity, \( u = (B_\theta/B) u_{\parallel} \). The coordinates \( x \) and \( y \) correspond to the poloidal and radial directions, respectively. The \( \sqrt{g}, h_x \) and \( h_y \) are metric coefficients; the coordinate system may be curvilinear, although it must be orthogonal. The equations are:

\[
\frac{\partial}{\partial t} n_i + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left( \frac{\sqrt{g}}{h_x} n_i u \right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial y} \left( \frac{\sqrt{g}}{h_y} n_i v \right) = S_n. \tag{A.1}
\]

\[
\frac{\partial}{\partial t} \left( \rho u_{\parallel} \right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left( \frac{\sqrt{g}}{h_x} \rho u_{\parallel} u - \frac{\sqrt{g}}{h_x^2} n_e \frac{\partial u_{\parallel}}{\partial x} \right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial y} \left( \frac{\sqrt{g}}{h_y} \rho u_{\parallel} u - \frac{\sqrt{g}}{h_y^2} n_i \frac{\partial u_{\parallel}}{\partial y} \right) = S_{mu_{\parallel}} - \frac{B_\theta}{B} \frac{1}{h_x} \frac{\partial p}{\partial x}, \tag{A.2}
\]

\[
v = - \frac{D}{h_y} \frac{\partial}{\partial y} (\ln n_i), \tag{A.3}
\]

\[
\frac{\partial}{\partial t} \left( \frac{3}{2} n_e T_e \right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left( \frac{\sqrt{g}}{h_x} \frac{5}{2} n_e u T_e - \frac{\sqrt{g}}{h_x^2} \kappa_x \frac{\partial T_e}{\partial x} \right) - \frac{1}{\sqrt{g}} \frac{\partial}{\partial y} \left( \frac{\sqrt{g}}{h_y} \frac{5}{2} n_e T_e - \frac{\sqrt{g}}{h_y^2} \kappa_y \frac{\partial T_e}{\partial y} \right) = S_e^i - k (T_e - T_i) + \frac{u}{h_x} \frac{\partial p_e}{\partial x} + \frac{v}{h_y} \frac{\partial p_e}{\partial y}, \tag{A.4}
\]

and

\[
\frac{\partial}{\partial t} \left( \frac{3}{2} n_i T_i + \frac{1}{2} \rho u_{\parallel}^2 \right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left[ \frac{\sqrt{g}}{h_x} \frac{5}{2} n_i u T_i + \frac{1}{2} \rho u_{\parallel}^2 \right] - \frac{\sqrt{g}}{h_x^2} \left( \kappa_x^i \frac{\partial T_i}{\partial x} + \frac{1}{2} n_x^i \frac{\partial u_{\parallel}^2}{\partial x} \right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial y} \left[ \frac{\sqrt{g}}{h_y} \frac{5}{2} n_i T_i + \frac{1}{2} \rho u_{\parallel}^2 \right] - \frac{\sqrt{g}}{h_y^2} \left( \kappa_y^i \frac{\partial T_i}{\partial y} + \frac{1}{2} n_y^i \frac{\partial u_{\parallel}^2}{\partial y} \right) = S_i^i + k (T_e - T_i) + \frac{u}{h_x} \frac{\partial p_i}{\partial x} - \frac{v}{h_y} \frac{\partial p_i}{\partial y}. \tag{A.5}
\]
where

\[ S_n, S_{m\parallel}, S_E, S_i^e, S_i^i = \text{volume sources of ions, momentum, electron, and ion energy, respectively,} \]

\[ \eta_z^i, \eta_y^i = \text{poloidal and radial ion viscosity coefficients,} \]

\[ \kappa_z^{e,i}, \kappa_y^{e,i} = \text{thermal conductivities.} \]

The poloidal coefficients are related to classical parallel coefficients according to \( \eta_z^i = (B_y^2/B^2) \eta_z^i \), and similarly for \( \kappa_z^{e,i} \). The radial coefficients, including \( D \), are anomalous. The term \( k(T_e - T_i) \) is the electron-ion energy equilibration term, and the \( \mathbf{u} \cdot \nabla p_e \) term on the right hand side of the energy equations represents work done by the electric field.

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**References**


8. A MULTI-FLUID CODE FOR THE STUDY OF HELIUM TRANSPORT IN THE EDGE PLASMA

Abstract

A multifluid code for the simulation of two-dimensional transport processes in the tokamak edge plasma is presented. The code models an electrically neutral and current-free plasma containing several ion fluids and an electron fluid. Coupling between the plasma species occurs through ionization and recombination processes, interspecies friction, electric and thermal forces, and temperature equilibration. The effectiveness of the model is illustrated in two example calculations. One calculation concerns differential transport of hydrogen and deuterium in an ASDEX-like scrape-off layer, and another models the transport of helium in the edge plasma of the conceptual TFCX experiment. The TFCX calculation indicates that significant helium dilution may occur in the edge plasma.

8.1. Introduction

A detailed understanding of multispecies transport through the tokamak edge plasma is required in order to assess the pumping requirements for helium exhaust from a reactor experiment and to predict the penetration into the plasma core of impurities released from the wall and plates. To date, most of the work on modelling of helium and impurities in tokamaks was concerned with the radial transport in the inner plasma. Issues of plasma-wall interaction and the associated impurity release can be modelled to some
extent by using such radial transport codes, but only in so far as poloidal asymmetries and transport along magnetic field lines do not play a significant role. This excludes the study of transport through the edge plasma under conditions of high recycling on divertor plates or limiters. Multispecies plasma transport along the magnetic field in the tokamak boundary layer has been studied numerically by Neuhauser, Schneider et al. [1], and by Harbour and Morgan [2]. Their work demonstrated the importance of the longitudinal electric field and of thermal forces, which act differently on the various ionic species, for calculating the transport of minority ions.

Using a one-dimensional model, whether radial or parallel, it is not possible to treat accurately the transport through an edge plasma in which there is a large region of recirculating flow, such as found in the calculations in the preceding Chapters. In the present work, therefore, the previous single-ion edge plasma code has been extended to provide a multifluid description. Although the need for a consistent two-dimensional transport model of a multispecies edge plasma has been felt for some years now, the work described here seems to be the first actual implementation of such a model. We consider a multifluid edge plasma without imposing any restriction on the relative concentrations of the ion fluids. The ion fluids have distinct flow velocities but a common temperature, which may be different from the electron temperature. Coupling between the species is through ionization and recombination processes, interspecies friction, electric and thermal forces, and temperature equilibration. The numerical methods used here are well able to handle these coupling terms.

The outline of the paper is as follows. Section 2 presents the equations of the model. In Sec. 3 the numerical approach is discussed, and in Sec. 4 two applications are treated. Conclusions are given in Sec. 5.

8.2. Equations for Describing the Multi-Fluid Edge Plasma

The present equations form a direct extension of the two-fluid (electrons and ions) edge plasma model that has been described in Chapters 5–7 of this thesis. The plasma is governed by a system of equations that is of Navier-Stokes form as regards the parallel flow and of a diffusive form in the radial direction. There are $N$ ion fluids, which may have different velocities but have a common temperature. For each ion fluid $a$ ($1 \leq a \leq N$) there is a continuity equation governing the particle density $n_a$, a momentum equation governing the parallel velocity $u_{\parallel a}$, and a diffusion equation.
governing the radial velocity $v_a$. The electron and ion temperatures $T_e$ and $T_i$ are governed by convection–conduction equations. The poloidal flow velocity $u_a$ is computed as $(B_0/B)u_{||a}$, therefore ignoring a diamagnetic contribution. The electron density and velocity follow from the assumptions of charge neutrality and absence of an electric current.

Classical multispecies plasma transport theory [3], [4] gives rise to a rather complicated system of force-friction relations, both for the parallel and for the radial transport. We have used the classical theory as a guide to obtain a simplified set of equations for the parallel transport. These equations are consistent with the standard classical theory in the limit when one fluid is dominant and all others are trace impurities, and they remain mathematically sound also at finite relative concentrations. The radial transport is taken to be anomalous in our model. Specifically, we solve the following system of equations:

Continuity of species $a$:

$$\frac{\partial n_a}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left( \frac{\sqrt{g}}{h_x} n_a u_a \right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial y} \left( \frac{\sqrt{g}}{h_y} n_a v_a \right) = S^a_n,$$  \hspace{1cm} (1)

Momentum balance of species $a$:

$$\frac{\partial}{\partial t} (m_a n_a u_{||a}) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left( \frac{\sqrt{g}}{h_x} m_a n_a u_{||a} - \frac{\sqrt{g}}{h_x^2} \eta_x \frac{\partial u_{||a}}{\partial x} \right)$$

$$+ \frac{1}{\sqrt{g}} \frac{\partial}{\partial y} \left( \frac{\sqrt{g}}{h_y} m_a n_a v_{a y} - \frac{\sqrt{g}}{h_y^2} \eta_y \frac{\partial u_{a y}}{\partial y} \right)$$

$$= \frac{B_0}{B h_x} \left[ - \frac{\partial p_a}{\partial x} - \frac{Z_a n_a}{n_e} \frac{\partial p_e}{\partial x} + c_r \left( \frac{Z_a}{Z_{\text{eff}}} - 1 \right) Z_a n_a \frac{\partial T_e}{\partial x} \right]$$

$$+ c_l \left( \frac{Z_a^2}{Z_{\text{eff}}} - 1 \right) n_a \frac{\partial T_i}{\partial x} + \sum_{b=1}^{N} F_{ab} + S^a_{\text{nu}},$$ \hspace{1cm} (2)

Diffusion of species $a$:

$$v_a = - \frac{D^a_n}{h_y} \frac{\partial}{\partial y} (\ln n_a) - \frac{D^a_p}{h_y} \frac{\partial}{\partial y} (\ln p_a),$$ \hspace{1cm} (3)

Electron energy balance:

$$\frac{\partial}{\partial t} \left( \frac{3}{2} n_e T_e \right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left( \frac{\sqrt{g}}{h_x} \frac{5}{2} n_e u_e T_e - \frac{\sqrt{g}}{h_x^2} \kappa_x \frac{\partial T_e}{\partial x} \right)$$

$$+ \frac{1}{\sqrt{g}} \frac{\partial}{\partial y} \left( \frac{5}{2} n_e v_{e y} T_e - \frac{\sqrt{g}}{h_y^2} \kappa_y \frac{\partial T_e}{\partial y} \right)$$

8.2. Equations for Describing the Multi-Fluid Edge Plasma 199
Ion energy balance:

\[
\frac{\partial}{\partial t} \left( \frac{3}{2} n_i T_i + \frac{1}{2} \sum_a \rho_a u_{\|a}^2 \right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left[ \frac{\sqrt{g}}{h_x} \left( \sum_a \frac{5}{2} n_a v_a T_i + \frac{1}{2} \sum_a m_a n_a v_a u_{\|a} \right) \right] \\
+ \frac{1}{\sqrt{g}} \frac{\partial}{\partial y} \left[ \frac{\sqrt{g}}{h_y} \left( \sum_a \frac{5}{2} n_a v_a T_i + \frac{1}{2} \sum_a m_a n_a v_a u_{\|a} \right) \right] \\
= \frac{u_e}{h_x} \frac{\partial p_c}{\partial x} + \frac{v_e}{h_y} \frac{\partial p_e}{\partial y} - k (T_e - T_i) + S_{E}^c,
\]

(4)

where

\[\sqrt{g}, h_x, h_y = \text{metric coefficients},\]
\[Z_a, m_a = \text{charge number and mass of an ion of species} \ a,\]
\[S_{n}^a, S_{mu}^a = \text{volume sources of ions and momentum for species} \ a,\]
\[S_{E}^c, S_{E}^i = \text{volume sources of electron and ion energy},\]
\[\eta^x, \eta^y = \text{poloidal and radial viscosity coefficients for species} \ a,\]
\[F_{ab} = \text{friction force on ion species} \ a \text{ due to species} \ b,\]
\[c_e, c_i = \text{coefficients in the thermal force for electrons and ions},\]
\[D_n^a, D_p^a = \text{diffusion coefficients for species} \ a,\]
\[\kappa^x, \kappa^y = \text{heat conduction coefficients},\]
\[k = \text{energy equipartition coefficient}.
\]

The poloidal transport coefficients are related to parallel coefficients according to \[\eta^x = (B_0^2/B^2) \eta_{\|}^a, \ \kappa^x = (B_0^2/B^2) \kappa_{\|}^a\] and \[\kappa^x = (B_0^2/B^2) \kappa_{\|}^a.\] The radial coefficients \[\eta^y, D_n^a, D_p^a, \ k^y, \ k_y = \text{anomalous}.\] The term \[u \cdot \nabla p_c\] represents work done by the electric field.
The friction force $F_{ab}$ (which is proportional to $u_{ab} - u_{ba}$) is taken from Ref. [5]. The equipartition coefficient $k$, parallel heat conduction coefficients $\kappa_i^e$ and $\kappa_i^i$, and the parallel viscosity coefficients $\eta_i^e$ are computed using the formulae given in Ref. [5] for the case of a simple plasma (one ionic species), with the following replacements:

**Equipartition coefficient:**

- Simple plasma: $k \propto Z_i^2 m_i^{-1} n_i n_e$
- Multiple ion species: $k \propto \sum_a Z_a^2 m_a^{-1} n_a n_e$

**Parallel electron heat conduction coefficient:**

- Simple plasma: $\kappa_i^e \propto Z_i^{-1}$
- Multiple ion species: $\kappa_i^e \propto \sum_a Z_a n_a / \sum_b Z_b^2 n_b$

**Parallel ion heat conduction coefficient:**

- Simple plasma: $\kappa_i^i \propto Z_i^{-4} m_i^{-1/2}$
- Multiple ion species: $\kappa_i^i \propto \sum_a Z_a^{-2} n_a / \sum_b Z_b^2 n_b \sqrt{2 m_a m_b / (m_a + m_b)}$

**Parallel ion viscosity coefficient:**

- Simple plasma: $\eta_i^i \propto Z_i^{-4} m_i^{1/2}$
- Multiple ion species: $\eta_i^i \propto Z_a^{-2} n_a / \sum_b Z_b^2 n_b \sqrt{2 / (m_a + m_b)}$

Although these expressions are a simplification of the complete multispecies transport theory, they have the correct limit in the case when one species dominates, and are considered adequate for the present application.

### 8.3. Numerical Solution

In comparison with the previous two-fluid code the principal difficulty in the numerical treatment of the system (1)–(5) lies in the strong coupling between the ion flow velocities, due to the terms $F_{ab}$ in the momentum equations. A less severe problem is the possibility of strong coupling between the densities of ions of neighbouring charge states in situations where both ionization and recombination are important processes.

The present code has been designed to follow closely the approach that was established in the previous two-fluid modelling work and that is described in Sec. 6.3. The
discretization method for all equations is the same as used earlier, and the basic iteration procedure has the same seven steps as described in Sec. 6.3. Thus, first all source terms are computed; then all momentum equations are relaxed; then the diffusion equations are applied; next all continuity equations are relaxed; next the temperature equations are relaxed separately; then the total energy equation is relaxed; and finally all continuity equations are relaxed again.

The method of Patankar and Spalding (see Eq. (6.6) and the subsequent discussion) is again used to relax the continuity equations. Ionization has been treated implicitly in the calculations that are shown in Sec. 4, but it could have been treated explicitly or one can alternate between an explicit and an implicit treatment as suggested by Lackner et al. [6].

For the $N$ momentum balance equations we now employ $N + 1$ relaxation sweeps. $N$ sweeps are employed to relax each momentum equation separately, using an implicit treatment of the friction terms. In the final relaxation sweep an identical change is made to all velocities in order to correct the total momentum balance. For this final sweep the interspecies friction can be ignored, since it contributes no net momentum source. For each individual species the coefficient $c_2$ needed in the distributive relaxation of the continuity equation is the sum of two terms: one term corresponding to the momentum relaxation for that species individually, and one term corresponding to the total momentum relaxation.

This procedure is remarkably successful in handling the strongly coupled system of equations for the multifluid plasma, as will be demonstrated in the following Section. Evidently an alternative approach would be to rely on a locally one-dimensional splitting method, where the basic component of the iterative process is the solution of the coupled system of equations involving all unknowns along one poloidal or radial coordinate line. However, a comparison between the two approaches has not been made for the present system of equations.

8.4. Example calculations

In this Section the results of two specific applications are presented, one of which is representative for the ASDEX divertor and scrape-off plasma and the other for the limiter edge of the conceptual TFCX experiment. The ASDEX simulation pertains to
a two-ion-species plasma containing similar amounts of hydrogen (protons) and deuterium. The TFCX simulation is concerned with helium transport and models three ion fluids: a hydrogen fluid with mass number 2.5 (deuterium-tritium), a He\(^{1+}\) fluid, and a He\(^{2+}\) fluid. The transport coefficients and boundary conditions were chosen in order to stay as close as possible to the cases presented in Sec. 6.4 of this thesis, so that an immediate comparison is possible.

The ASDEX simulation. The purpose of this simulation is to demonstrate the ability of the code to handle a situation in which there exist large concentrations of more than one ionic species—a case for which some numerical methods would fail to converge because of the strong frictional coupling between the ion species. In order for the simulation to have some physical interest as well it was decided to model an edge plasma containing similar amounts of hydrogen and deuterium.

The geometry and mesh were the same as in Sec. 6.4. The two ion species were \(^1\)H\(^1+\) and \(^2\)H\(^1+\). For the parallel transport coefficients \(\eta^p\), \(\kappa^p\), \(\kappa^d\), and \(\kappa^t\) we took the classical values as described in Sec. 8.2, with a flux limit on \(\kappa^p\) as in Sec. 6.4. The radial transport coefficients were assigned the following anomalous values: for both species \((a = 1, 2)\) \(D^a_n = 2\ m^2/s\), \(D^a_e = 0\) and \(\eta^a_n, m_a n_a = 0.2\ m^2/s\), while furthermore \(\kappa^e/n_e = 4\ m^2/s\) and \(\kappa^d/n_i = 0.2\ m^2/s\). These are the same values as were employed in Sec. 6.4.

The boundary conditions were chosen as follows:

- On the interface with the main plasma, the ion densities, parallel flow velocities and two temperatures were prescribed: \(n_1 = 9 \times 10^{19}\ m^{-3}\), \(n_2 = 9 \times 10^{19}\ m^{-3}\), \(u_{11} = 0\), \(u_2 = 0\), \(T_e = 80\ eV\) and \(T_i = 80\ eV\).

- On the outer wall we prescribed zero transverse particle flux, zero shear, and pedestal temperatures \(T_e = 2\ eV\) and \(T_i = 2\ eV\).

- On the upstream boundary, symmetry conditions were specified: \(\partial n_a/\partial x = 0\), \(u_{ia} = 0\) \((a = 1, 2)\), \(\partial T_e/\partial x = 0\) and \(\partial T_i/\partial x = 0\).

- On the divertor plate we specified sonic flow for both species: \(u_{i1} = u_2 = \sqrt{\rho_e/\rho_i}\) and energy fluxes \(Q_e = \delta_e n_e u_e T_e\) and \(Q_i = \sum_a (\kappa^a n_a u_a T_i - \frac{1}{2} m_a n_a u_a^2)\), in which \(\delta_e = 4.0\) and \(\delta_i = 2.5\).

The recycling model for each of the two ion species was the same as that used for the single fluid calculation in Sec. 6.4. There is a dependence on mass in this model.
Fig. 1. Contour plots of the density of the two ion species in the ASDEX scrapeoff model. The increment between the dashed contours is $5 \times 10^{17}$ m$^{-3}$; between the solid contours it is $2.5 \times 10^{18}$ m$^{-3}$. Fig. 1a (upper): hydrogen (H) density. Fig. 1b (lower): deuterium density.
through the assumed neutral velocity, so the two fluids may be expected to behave
differently.

Figure 1 shows the density fields $n_1$ and $n_2$ (hydrogen and deuterium, respectively). The hydrogen and deuterium density profiles remain virtually identical through most of the scrape-off layer. In the recycling zone in front of the target plate the density rise for deuterium takes place in a narrower region and reaches a higher value than that for hydrogen. The reason is that the recycling deuterium neutrals are slower than the hydrogen neutrals, by a factor that is the square root of the mass ratio, and therefore they become ionized closer to the target plate.

Figure 1 here may be compared with Fig. 3 of Chapter 6 of this thesis, which shows the density for the single fluid (mass number 1.5) ASDEX simulation. The single fluid density profile clearly resembles very much the sum of the two density profiles in Fig. 1, although it does not have to be exactly equal to the sum. The profiles of electron and ion temperature obtained from the present calculation (not shown) are virtually indistinguishable from those given in Figs. 5 and 6 of Chapter 6.

**The TFCX simulation.** The immediate aim of the present multifluid plasma code is to aid in the study of helium transport in the edge plasma of a fusion reactor. The concentration of helium near the pumping ducts adjacent to the limiter or divertor plate, specifically also in comparison with the concentration of helium in the main plasma, is of interest for reactor design, since it determines the amount of gas that must be pumped in order to keep the central helium density at an acceptably low level. This factor of helium enrichment or dilution depends in a complicated way on the plasma flow in the scrape-off layer and on the helium recycling process, and cannot be assessed in a realistic way except through detailed numerical simulation.

The geometry and mesh for this simulation were the same as in Sec. 6.4. The three ion species were $H^1$, at mass number 2.5 (D-T), $^4\text{He}^{1+}$ and $^4\text{He}^{2+}$. For the parallel transport coefficients we took again the classical values, including the flux limit on $\kappa^c$. The anomalous values for the radial transport coefficients were the same as used for ASDEX, and therefore also the same as those used for the single-fluid TFCX study in Sec. 6.4. The boundary conditions were the following:

- On the interface with the main plasma, the densities of D-T and He$^{2+}$, the flux of He$^{1+}$, the parallel flow velocities of D-T and He$^{2+}$, the radial momentum flux associated with He$^{1+}$, and two temperatures were prescribed: $n_1 = 7 \times 10^{18} \text{ m}^{-3}$, $\Gamma_2 = 0$ (where $\Gamma_a$ denotes the radial flux density of species $a$), $n_3 = 5 \times 10^{18} \text{ m}^{-3}$.  

8.4. Example calculations
Fig. 2. Contour plot of the density of the D-T fluid in the TFCX scrape-off layer. The increment between the dashed contours is $5 \times 10^{18}$ m$^{-3}$; between the solid contours it is $2.5 \times 10^{19}$ m$^{-3}$.

$u_{\|a} = 0$ (a = 1 or a = 3), zero momentum flux for a = 2, $T_e = 150$ eV, and $T_i = 150$ eV.

- On the outer wall we prescribed zero transverse particle flux, zero shear, and pedestal temperatures $T_e = 2$ eV and $T_i = 2$ eV.

- On the upstream boundary, symmetry conditions were specified: $\partial n_a / \partial x = 0$, $u_{\|a} = 0$ (for a = 1, 2, 3), $\partial T_e / \partial x = 0$, and $\partial T_i / \partial x = 0$.

On the limiter plate we specified sonic flow at the common sound velocity: $u_{\|a} = \sqrt{p/\rho}$ (for a = 1, 2, 3); and energy fluxes $Q_e = \delta_e n_e u_e T_e$ and $Q_i = \sum_a (\beta_a n_a u_a T_i + \frac{1}{2} m_a n_a u_a u_{\|a}^2)$, with $\delta_e = 4.0$ and $\beta_i = 2.5$.

The D T recycling model was the same as the one used in Sec. 6.4. The helium recycling model has the same structure, but of course both He$^1+$ and He$^2+$ recycle into He$^1+$. The collisional radiative cross-section for ionization of neutral He was approximated by $\langle \sigma v \rangle = 2.5 \times 10^{-14} \times \alpha^2/(20 - \alpha^2)$ m$^3$/s, in which $\alpha = T_e/(10$ eV$)$. The cross-section for ionization of He$^1+$ to He$^2+$ was approximated by $\langle \sigma v \rangle = 4 \times 10^{-15} \times \alpha^2/(50 + \alpha^2)$ m$^3$/s, and the cross-section for ionization of neutral...
Fig. 3. Contour plot of the density of He$^{1+}$ in the TFCX scrapeoff model. The increment between the dashed contours is $5 \times 10^{16}$ m$^{-3}$; between the solid contours it is $2.5 \times 10^{17}$ m$^{-3}$.

Fig. 4. Contour plot of the density of He$^{2+}$ in the TFCX scrapeoff model. The increment between the solid contours is $5 \times 10^{17}$ m$^{-3}$.
hydrogen by \( \langle \sigma v \rangle = 3 \times 10^{-14} \times \alpha^2/(3 + \alpha^2) \text{ m}^3/\text{s} \). These approximations are based on data in Ref. [7]. With each ionization event, whether involving D-T or He, was associated an electron energy loss of 25 eV (representing both the ionization energy and the radiation in multistep processes) and an ion energy gain of 5 eV (representing the kinetic energy of the neutral particle).

The results of this calculation are shown in Figs. 2-4. The ion density shown in Fig. 2 is not much different from the result of the single fluid calculation (Fig. 9 in Chapter 6). Figure 3 shows that singly charged helium exists in significant concentrations only close to the limiter plate. In comparing Figs. 2-4 it is notable that the density rise of He\(^{2+}\) takes place over a much wider region than that of either D-T or He\(^{1+}\). This is a combined effect of the flow in the plasma (cf. Fig. 10 of Chapter 6) and the relatively low cross-section for ionization to He\(^{2+}\).

In the present calculation the prescribed ratio of He to D-T on the outer boundary of the main plasma was 7.1%, while for the ratio of the flux of He (both 1+ and 2+) to D-T on the target plate we find a value of only 2.6%. This represents a significant and unfavorable reduction of the helium concentration in the region where pumping has to take place. Much more detailed recycling models than the present one are required, however, in order to predict accurately the pumping performance of any configuration. The present result must therefore be taken as indicative only.

8.5. Conclusions

The modelling of helium pumping and impurity penetration requires accurate and effective models both for transport in a multispecies plasma as well as for plasma-wall interaction, and the results of such calculations may be quite sensitive to details of the recycling model employed. A code such as presented here is an indispensable tool in such studies, but without an adequate description of the recycling process it cannot do much more than show trends, such as the helium dilution found in the TFCX simulation. Adding an improved treatment of neutral particles, whether based on a diffusion model or on Monte Carlo calculations, is therefore important.

The present multifluid code can also be used to study impurity transport in the edge plasma—an issue which is of great importance already for present experiments. Again, an accurate and workable model for the release of impurities from material surfaces is important for performing physically meaningful calculations.
It should be noted, finally, that the calculation times required for the present code are comparatively modest (e.g. less than 3 minutes of Cray-1 time for each of the calculations presented above), so that an extension of the simulation to more than three fluids is feasible from a computational viewpoint.

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References


References
Algemene inleiding. Het doel van het onderzoek naar gecontroleerde kernfusie is een technologie te ontwikkelen die, naast of in plaats van kernspijting en zonnestraling, kan bijdragen aan de wereldenergievoorziening op langere termijn. Dit onderzoek wordt gedaan in laboratoria verspreid over de geïndustrialiseerde wereld, met een grote mate van internationale samenwerking.

De belangrijkste fusiereaktie voor het huidige onderzoek is die tussen de waterstof-isotopen deuterium (D) en tritium (T):

\[ D + T \rightarrow ^4\text{He} + n + 17.6 \text{ MeV}. \]

Om deze reaktie mogelijk te maken moet de Coulomb-afstoting tussen de D- en T-kernen overwonnen worden, waarvoor een hoge botsingsenergie vereist is: in een mengsel van deuterium en tritium wordt de kans op spontane reacties pas groot genoeg bij een temperatuur van tenminste 10 keV (ongeveer \(10^6\) K). Bij een dergelijke temperatuur is het D-T gas volledig geioniseerd. Men noemt dit geioniseerde medium een plasma, en spreekt van thermonucleaire fusie.

De twee hoofdlijnen van het onderzoek naar beheerste thermonucleaire fusie betreffen de magnetische opsluiting en de traagheidsopsluiting. Bij de eerste methode wordt een plasma van betrekkelijk lage dichtheid opgesloten in een magnetisch veld, meestal in de vorm van een torus. Bij de tweede methode wordt een plasma van zeer hoge dichtheid gedurende een zeer korte tijd door zijn eigen traagheid bijeen gehouden.

In dit proefschrift worden enkele problemen bestudeerd die betrekking hebben op het plasma in magnetische opsluitsystemen van het Tokamak type. De problemen liggen op het terrein van de numerieke behandeling van vraagstukken betreffende plasma-evenwicht en -transport, met speciale nadruk op de bepaling van het plasma-evenwicht.
Fig. 1. Een impressie van de Asdex-Upgrade tokamak, die in het Max-Planck-Institut für Plasmaphysik in aanbouw is.

Aan de hand van externe metingen en op de studie van plasmadynamica en transportverschijnselen in de buitenste laag van een tokamak. De hoop is dat dit onderzoek bijdraagt aan een beter begrip van bestaande experimenten en aan het optimaliseren van het ontwerp van toekomstige fusie-experimenten.

De tokamak. De electromagnetische kracht die werkt op een geladen deeltje in een magnetisch veld verhindert de vrije beweging in de richting loodrecht op het veld: in eerste benadering beweegt het deeltje zich in een spiraalvormige baan rond een enkele veldlijn. Op deze manier wordt een plasma door een magnetisch veld gedeeltelijk opgesloten. In de richting langs het veld is de beweging echter ongehinderd. Men maakt daarom gebruik van configuraties waarbij de magnetische veldlijnen een begrensdtoroïdaal gebied vullen. De tokamak is zo'n configuratie (zie Fig. 1).

Het magnetisch veld in een tokamak is axiaal symmetrisch. Het wordt opgewekt door een combinatie van stromen in externe windingen en een stroom in het plasma zelf. De plasmastroom wordt opgewekt door een geïnduceerd electrisch veld, zoals
de stroom in de secundaire winding van een transformator. De verhouding tussen de
sterkte van het uitwendige veld en de plasmastroom moet zorgvuldig gekozen worden
om een configuratie te verkrijgen die een hoge plasmadruck toestaat. Ook de vorm
van de plasmadoorsnede is daarvoor belangrijk. De tien à twintig grote tokamaks die
thans wereldwijd in bedrijf of in aanbouw zijn onderscheiden zich door grootte en vorm,
en verder vooral door de verhittingsmethoden en de methode voor beheersing van de
plasma-wand wisselwerking.

De typische levensduur van een tokamak experiment bedraagt tien jaar. In deze
periode wordt geprobeerd de 'performance' van het experiment (dichtheid, tempera-
tuur, opsluittijd) te optimaliseren, en tegelijkertijd een beter begrip te krijgen voor de
optredende instabiliteiten, de turbulentie, en de transportverschijnselen. De rol van de
theoretische plasmafysica in dit kader is de modellen te ontwikkelen waarmee de insta-
biliteiten, turbulentie en transport beschreven worden, en deze modellen toe te passen
ter ondersteuning van het experimentele onderzoek en van het ontwerp van nieuwe ex-
perimenten. Hiertoe dient ook gerekend te worden het ontwikkelen van methoden voor
de interpretatie van plasmadiagnostieken, omdat vele belangrijke eigenschappen van
een thermonucleair plasma niet voor directe meting toegankelijk zijn (e.g. het stroom-
profiel). Een nauwe band met de numerieke wiskunde is bij het theoretisch werk veelal
een vereiste.

Indeling van het proefschrift.

Hoofdstuk 1 geeft een inleiding in de beheerste kernfusie, toegespitst op de twee
hoofdthema's van deze dissertatie: magnetohydrodynamisch evenwicht en de plasma-
grenslaag in een tokamak.

In Hoofdstuk 2 wordt een toepassing beschreven van de meerrooster (multigrid)
methode op de berekening van het plasma-evenwicht in axiaal-symmetrische opsluit-
systemen zoals de tokamak. De meerrooster methode blijkt voor dit doel optimaal
efficiënt te zijn. Voorts wordt onderzocht hoe de meerrooster methode gebruikt zou
cummen worden voor de efficiënte berekening van drie-dimensionale evenwichten. Voor
dit doel zal het noodzakelijk zijn adaptieve technieken te ontwikkelen, waarbij de vorm
van het rooster zich aan de gezocht oplossing aanpast.

In Hoofdstuk 3 worden de analytische en numerieke methoden voor de interpretatie
van magnetische diagnostieken kritisch samengevat en vergeleken. Nieuw in deze ver-
handeling is onder meer de uitbreiding van een bekende klasse van integrale betrekkin-
gen tot situaties waarbij plasmarotatie en drukanisotropie van belang zijn; een ontwerp
voor de toepassing van standaard multivariate en robuuste statistische methoden bij
deur werking van de betreffende integralen aan de hand van de onvolmaakte meetingen;
een snel algorithme voor de bepaling van plasma-evenwicht; een formulering voor het
randwaarde probleem voor de evenwichtsvergelijking bij gegeven uitwendig veld; en een
kritiek op een gepubliceerde methode voor de bepaling van het plasma-evenwicht aan
de hand van alleen de geometrie van de flux-oppervlakken.

In Hoofdstuk 4 wordt de functieparametrisatie-methode van H. Wind toegepast voor
de interpretatie van magnetische metingen aan een tokamak. Bij deze methode wordt
een eenvoudig funktioneel verband tussen meetingen en fysisch interessante parameters
gevonden aan de hand van een statistische analyse van gesimuleerde experimenten.
Hierbij wordt gebruik gemaakt van technieken voor dimensiereductie en voor regressie-
analyse. De methode blijkt uitermate effectief te zijn, en heeft mogelijk toepassingen
voor de actieve besturing van een tokamak experiment.

In Hoofdstuk 5 worden enige modellen besproken waarmee het transport van deeltjes en energie in de plasmagrenslaag numeriek onderzocht kan worden. De ervaring met nul- en één-dimensionale modellen leert dat een volledig twee-dimensionaal model gewenst is, en zo een model wordt dan ook gebruikt. Toepassingen op het randplasma van een tokamak reactor-experiment worden behandeld.

In Hoofdstuk 6 worden de numerieke methoden besproken die in het twee-dimensionale model gebruikt worden, en wordt hun effectiviteit gedefinitieerd. De methoden berusten grotendeels op werk van Patankar en Spalding op het gebied van warmte-transport in vloeistoffen. Enige aanpassingen zijn nodig voor de huidige toepassing op een compressibel plasma met verschillende temperaturen voor electronen en ionen.

In Hoofdstuk 7 wordt een toepassing behandeld van de twee-dimensionale code op het randplasma in het TFCX ontwerp-fusieexperiment. De veronderstellingen voor het radiaal energietransport en voor de plasmadichtheid worden gevarieerd, en er wordt onderzocht voor welke waarden van de parameters de temperatuur op de limiterplaat een acceptabel lage waarde heeft.

In Hoofdstuk 8 tenslotte wordt een uitbreiding besproken van de eerder beschreven
code tot een code waarmee het transport in een plasma met meerdere ionensoorten
bestudeerd kan worden. Dit werk is in het bijzonder van belang voor de studie van
heliumtransport in een fusiereaktor en voor de studie van het transport van veront-
reinigingen. Er worden voorbeelden gegeven van berekeningen aan ASDEX en TFCX.

Samenvatting
CURRICULUM VITAE

De auteur van dit proefschrift werd geboren op 13 November 1954 te Boston, Massachusetts (U.S.A.). De schoolloopbaan in Nederland werd in 1972 afgesloten met het behalen van het diploma H.B.S.-b aan het Herman Jordan Lyceum te Zeist. Na een verblijf in de Verenigde Staten in het kader van een uitwisseling was in 1971 ook een diploma van Bozeman (Mt) Senior High School verkregen.


Voor het werk beschreven in het tweede deel van dit proefschrift werd in 1986 de Otto-Hahn-Medaille van de Max-Planck-Gesellschaft verleend.
Many people have contributed to the research that is described in this thesis. First of all I wish to extend my sincere appreciation to Professor F. Engelmann, both for the valuable advice and guidance throughout the course of this work and for allowing me the room to pursue scientific interests that ventured outside the original plan of the thesis.

The early work on this thesis was carried out during an 18 months stay in the exhaust physics group at the UKAEA Culham Laboratory. I am grateful to M.F.A. Harrison and the members of the exhaust physics group for their hospitality, and in particular to P.J. Harbour, M.F.A. Harrison, E.S. Hotston and J.G. Morgan for sharing with me their insight into the physics of the edge plasma.

Most of the thesis was elaborated during a stay of nearly 3 years in the tokamak physics group (Theory Division III) at the Max-Planck-Institut für Plasmaphysik. During this period, many discussions with K. Lackner, J. Neuhäuser, W. Schneider and other members of Theory Division III have been very helpful.

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