ASYMPTOTICS OF THE MODAL LINES OF SOLUTIONS OF
2-DIMENSIONAL SCHröDINGER EQUATIONS

M. Hoffmann-Ostenhof
Institut für Theoretische Physik
Universität Wien

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I. Introduction and Previous Results

In this paper we sharpen results on nodal properties of $L^2$-solutions of 2-dimensional Schrödinger equations, recently obtained in collaboration with T. Hoffmann-Ostenhof and J. Swetina in [1]. We shall consider real valued $W^{2,2}$-solutions $\psi(x)$ of the Schrödinger equation

$$(-\Delta + V - E)\psi = 0 \quad \text{for} \quad x \in \Omega_R,$$

$$\Omega_R = \{ x \in \mathbb{R}^2 | r < |x| < R \}, \quad R > 0,$$

(where the Sobolev space $W^{2,2}$ is defined as in [2]). In the following it will always be assumed that

$$E < 0 \quad \text{(1.2)}$$

and that

$$V(x) \text{ is real valued and continuous in } \Omega_R$$

$$\text{and } \lim_{r \to \infty} V(x) = 0. \quad \text{(1.3)}$$

Due to these assumptions we can choose $R$ so that

$$\inf_{x \in \Omega_R} (V(x) - \frac{1}{4r^2} - E) > 0 \quad \text{(1.4)}$$

which implies that the Dirichlet problem (1.1) with continuous boundary data is uniquely solvable (see [1]). Note also that $\psi \in \mathcal{C}^1(\Omega_R)$ (see e.g. [2]). In the following we shall use polar coordinates $x_1 = r \cos \omega, x_2 = r \sin \omega$ with $r \geq R$ and $\omega \in [-\pi, \pi)$, and denote $\psi = \psi(r, \omega)$.

Under additional suitable assumptions on $V$ the generally unbounded nodal set of $\psi$, i.e. $\{ x \in \Omega_R, \psi(x) = 0 \}$ will be investigated for $r \to \infty$. Particularly it will be shown (Theorem 2.3) that for large $r$ the nodal set of $\psi$ consists of non-intersecting nodal lines which look roughly speaking asymptotically either like straight lines or like branches of parabolas.
By the following example (compare also [1]) it is illustrated that already for spherically symmetric $V$ it is in general far from trivial to determine the asymptotic behaviour of the zeros of $\phi$. Let $a_k, b_k \in \mathbb{R}$ for $0 \leq \kappa < \infty$ with $\kappa \in \mathbb{N} \cup \{0\}$ and denote by $W_{\kappa, \ell}(r)$ for $0 \leq \ell < \infty$ the Whittaker functions (see [3]). Define

$$\phi(r, \omega) = r^{-1/2} \sum_{\ell=0}^{\infty} (a_{\ell} \sin \ell \omega + b_{\ell} \cos \ell \omega) W_{\kappa, \ell}(r).$$

Then it is easily seen that $(- \Delta + 1/4) \phi = 0$ in $\mathcal{D}'$ and since for all $\ell$

$$W_{\kappa, \ell}(r) = e^{-r/2} (1 + O(r^{-1}))$$

(see [3]). Define

$$A(w) = \lim_{r \to \infty} \frac{\phi(r, \omega)}{r^{-1/2} W_{\kappa, 0}(r)} = \sum_{\ell=0}^{\infty} (a_{\ell} \sin \ell \omega + b_{\ell} \cos \ell \omega)$$

whereby $(- \Delta + 1/4) r^{-1/2} W_{\kappa, 0}(r) = 0$ in $\mathcal{D}'$. Obviously given any $M > 1$, then $m, a_k, b_k$ can be chosen suitably so that $A$ vanishes e.g. in $u = 0$ of order $M$. In Theorem 2.3 it is demonstrated how the order $M$ of the zero of $A$ is connected with the asymptotics of the nodes of $\phi$ in a cone $|w| < \varepsilon$ for $\varepsilon$ small enough.

In the following we suppose (as in [1]) that

$$V(x) = V_1(x) + V_2(x)$$

(1.5)

where $V_1$ and $V_2$ obey (1.3) and (1.4) separately.

The above assumptions imply (see [1] and [4]) that there exists $v \in L^2(\mathbb{R})$, $v > 0$ for $r > R$ such that

$$(- \Delta + V_1 - E)v = 0 \quad \text{for} \quad r > R. \quad (1.6)$$

Now define

$$u(r, \omega) = \phi(r, \omega)/v(r) \quad (1.7)$$

and note that $u$ and $\phi$ have the same zeros. The derivation of our results
on the nodal lines of \( \psi \) will be based on results on the asymptotic behaviour of \( u \) given in \([1]\) (see also \([5]\)). We shall summarize these relevant results in Theorem 1.1. For this and later on we need

**Def. 1.1.** (i) Let \( I', I \subset \mathbb{R} \) denote finite open intervals and let \( f : (\mathbb{R}, \omega) \rightarrow \mathbb{R} \) denoted by \( f = f(r, s) \). \( f \) is called real analytic in \( s \) uniformly with respect to \( r \) if \( \forall r \in \mathbb{R} \) \( f \) is real analytic in the variable \( s \) \( \forall r \geq R \) and if \( \forall I' \subset I \) there exist \( \delta, C > 0 \) (not depending on \( r \)) such that

\[
|f_s(r, s) - f_s(s')| \leq C \delta (s-s')^k
\]

for \( s \in I' \), \( r \in \mathbb{R} \) and for \( k \in \mathbb{N} \cup \{0\} \).

(ii) Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) and define \( \omega \mu \in (0, \pi) \) \( \tilde{\theta}_0^{-1}(\mu) = (\cos(\omega \mu), \sin(\omega \mu)) \in S^1 \) \( \forall \mu \in [-\pi, \pi] \). We say \( g \) is real analytic in \( \mu \) uniformly with respect to \( r \), if for all \( \tilde{\omega} \) \( g(r \tilde{\theta}_0^{-1}(\mu)) \) is real analytic in \( \mu \) uniformly with respect to \( r \) (as defined in (i)) with \( C, \delta \) not depending on \( \tilde{\omega} \). In accordance with the foregoing we denote \( g(r \tilde{\theta}_0^{-1}(\mu)) = g(r, \mu) \).

According to \([1]\) (resp. \([4]\)) we have

**Theorem 1.1.** Let \( V = V_1 + V_2 \) be given according to (1.3), (1.4) and (1.5). Assume that \( V_1 \) is continuously differentiable with

\[
\left| \frac{dV_1}{dr} \right| \leq c r^{-1-\varepsilon} \quad \text{for} \quad r > R \quad (1.8)
\]

for some \( c, \varepsilon > 0 \) and that

\[
\text{for some } \alpha > \frac{1}{2}, r^{1+\alpha} V_2 \quad \text{is real analytic in } \omega \quad \text{uniformly with respect to } r. \quad (1.9)
\]

Let \( \psi \) and \( v \) be given according to (1.1) and (1.6).

(i) Then \( u \) is real analytic in \( \omega \) uniformly with respect to \( r \),

\[
\lim_{r \to \infty} u(r, \omega) = A(\omega)
\]

exists, \( A \) is real analytic in \( \omega \) and for \( k \in \mathbb{N} \cup \{0\} \)

\[
|u(r, \omega) - A(\omega)| \leq C_r (r - A)^k \quad \text{for } \omega \in R \quad r \geq R \quad \text{large enough, with some } C_r < \quad (1.10)
\]

(not depending on \( r \)).
(ii) Let $\theta \in (0, \frac{1}{2})$ and $D_\theta = \{x \in \Omega_\theta : |w| < r^{-\theta}\}$ with $R_\theta$ sufficiently large. Suppose $A(0) = 0$, then for some $M \in \mathbb{N}$ and $|w|$ small

$$A(w) = w^M + O(w^{M+1})$$

and in $D_\theta$ for some $\nu, \delta > 0$

$$u(r,w) = (2b)^{-M} r^{-M/2} H_M(b \sqrt{r} \omega) (1 + O(r^{-\nu})) + O(r^{-M/2-\delta})$$  \hspace{1cm} (1.11)

where $b = (|\xi|/4)^{1/4}$ and $H_M$ denotes the Hermite polynomial of order $M$

$$H_M(z) = \sum_{k=0}^{[M/2]} \frac{(-1)^k M!}{k!(M-2k)!} (2z)^{M-2k}, \quad z \in \mathbb{R}.$$  

($[M/2]$ denoting the integer part of $M/2$).

Some immediate consequences of Theorem 1.1 on the nodes of $\phi$ have been already noted in [1]. See Remark 2.3.

**Corollary 1.1.** Choosing $\omega = z/(b \sqrt{r})$, (1.11) implies

$$u(r, \frac{z}{b \sqrt{r}})^{M/2} = (2b)^{-M} H_M(z) \quad \text{for} \quad r = 0, \forall z \in \mathbb{R}$$  \hspace{1cm} (1.12)

and the convergence is uniformly in any compact interval.

In the following we denote

$$U_N^{(k)} = \frac{d^k}{ds^k} H_N,$$

$$U_N(r,s) = u(r, \frac{s}{b \sqrt{r}})^{M/2} \quad \text{and} \quad U_N^{(k)} = \frac{d^k}{ds^k} U_N,$$  \hspace{1cm} (1.13)

Note that $U_N^{(k)}$ exists since $\partial^k u / \partial w^k$ exists for all $k \in \mathbb{N}$ due to Theorem 1.1.

Theorem 2.1 deals with the behaviour of $U_N^{(k)}$ for $r = 0$ for $k \in \mathbb{N}$. In Theorem 2.2 the asymptotics of $\omega u/\omega r$ is characterized. With the help of these two theorems the main result on the nodal lines of $\phi$, stated
in Theorem 2.3 will be obtained. In sections 3, 4, and 5 the theorems
given in section 2 are proven.

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2. Statement of the Results

The first result is concerned with the asymptotic properties of $U_M$ as defined in (1.13).

**Theorem 2.1.** Under the assumptions of Theorem 1.1, $U_M(r,z)$ is real analytic in $z$ uniformly with respect to $r$ (in the sense of Def. 1.1) whereby $z \in \mathbb{R}$, $I$ any finite open interval.

Furthermore, for $k \in \mathbb{N} \cup \{0\}$

$$
\lim_{r \to 0} \frac{3^k}{2^k} U_M(r,z) = (2b)^{-M} \frac{d^k}{dz^k} H_M(z) \quad (2.1)
$$

with $b = (|z|/4)^{1/4}$, for $z \in \mathbb{R}$,

and the convergence is uniformly in any compact interval.

**Remark 2.1.** Clearly (2.1) implies that $U_M^{(k)}(0) = 0$ for $r = 0$, $\forall z \in \mathbb{R}$ for $k > M+1$.

The next result gives detailed information on the asymptotics of $\partial u/\partial r$.

**Theorem 2.2.** Under the assumptions of Theorem 1.1, $r^{1+a} \frac{2u}{\partial r}$ (with $a = \min(1, \alpha)$) is real analytic in $\omega$ uniformly with respect to $r$.

Further let $A(0) = 0$ with

$$
A(\omega) = \omega^M + d\omega^{M+1} + O(\omega^{M+2}) \quad \text{for } \omega \text{ small} \quad (2.2)
$$

for some $d \in \mathbb{R}$ and $M \in \mathbb{N}$.

If $M = 1$, then for some $\epsilon > 0$

$$
r^2 \frac{2u}{\partial r}(r,\omega) = \frac{d}{\sqrt{|E|}} + O(r^{-\epsilon}) \quad (2.3)
$$

for all $\omega$ with $|\sqrt{r} \omega|$ bounded for $r = 0$. 


If \( M = 2m, \ m \geq 1 \), then
\[
\frac{\partial}{\partial r} |_{r=\sqrt{b/r}}^{r=M/2+1} |_{r=\sqrt{b/r}} = (2b)^{-M} M(M-1)H_{M-2}(r) + o(1) \tag{2.4}
\]
for \(|z|\) bounded and \( r \to \infty \).

If \( M = 2m+1, \ m \geq 1 \), then for some \( \varepsilon > 0 \)
\[
\frac{\partial}{\partial r} |_{r=\sqrt{b/r}}^{r=M/2+1} |_{r=\sqrt{b/r}} = (2b)^{-M} M(M-1)H_{M-2}(r) +
\]
\[
+ (\varepsilon M(m+1))^{-1/2} (1 + O(r^{-\varepsilon})) +
\]
\[
+ O(r^{-1/2-\varepsilon}) + |z| o(1) \tag{2.5}
\]
for \(|z|\) bounded and \( r \to \infty \).

**Remark 2.2.** (a) An immediate consequence of Theorem 2.2 is that
\( r\frac{\partial}{\partial r} W_{M}^{r} \to 0 \) for \( r \to \infty \) for \( z \in B \).

(b) Clearly (2.5) implies that (2.4) holds for \( M \) odd. However, for \( M \) odd we shall need the more detailed asymptotics of (2.5) later on.

Theorem 2.1 and 2.2 together will enable us to obtain our main result:

**Theorem 2.3.** Suppose the assumptions of Theorem 1.1 hold. Assume \( A(0) = 0 \) with
\[
A(\omega) = \omega^M + \omega^{M+1} + O(\omega^{M+2}) \text{ for } |\omega| \text{ small} \tag{2.6}
\]
for some \( d \in \mathbb{R} \) and \( M \in \mathbb{N} \). Let \( z_i \in \mathbb{R} \) for \( 1 \leq i \leq M \) denote the zeros of the Hermite polynomial \( H_M \), i.e., \( H_M(z_i) = 0 \) for \( 1 \leq i \leq M \).

Then for \( \varepsilon > 0 \) sufficiently small and \( R \) large the nodal set of \( \varphi \) in \( D_\varepsilon = \{ z \in \mathbb{C} | r > R, |\omega| < \varepsilon \} \) consists of \( M \) nodal lines (corresponding to the \( M \) zeros of \( H_M \)). They admit a representation in cartesian coordinates \((x_1, x_2) \in \mathbb{R}^2\) denoted by \( x_i = C_i(x_1) \) for \( 1 \leq i \leq M \). Therefore denoting \( \varphi = \varphi(x_1, x_2), \delta(x_1, C_i(x_1)) = 0 \) for \( 1 \leq i \leq M \). For all \( i \), \( C_i \) is continuously differentiable and the nodal lines have the following asymptotic behaviour:
For $M \geq 2$ and $z_i \neq 0$

$$\tau_i(x_i) = \left( \frac{x_i}{b} + o(1) \right) \sqrt{x_i} \quad \text{for large } x_i \quad (2.7)$$

with $b = \left( \frac{|z_i|}{4} \right)^{1/4}$. Further if $x_i > 0$ ($< 0$), then $G_{\Gamma}$ is strictly monotonically increasing (decreasing) for large $x_i$.

For $M$ odd, $E_M(0) = 0$ and without loss let $z_j = 0$, then

$$E_j(x_j) = \frac{d}{\sqrt{|x|}} + o(1) \quad \text{for large } x_j \quad (2.8)$$

with $d$ given in (2.6).

**Remark 2.3.** As a consequence of the results summarized in Theorem 1.1 it was noted in [1] that in $D_{\Gamma}$ for each $\tau$ there exist $w_i(\tau), 1 \leq i \leq M$ with $w(\tau, w_i(\tau)) = 0$. In Theorem 2.3 the case $A(0) = 0$ is considered without loss of generality, since by rotation of the coordinate system corresponding results to (2.7) and (2.8) are immediately obtained if, for instance,

$$A(\omega) = (\omega - \omega_0)^M + d(\omega - \omega_0)^{M+1} + o((\omega - \omega_0)^{M+1}) \text{ for } |\omega - \omega_0| \text{ small.}$$

Note that since $A$ is real analytic it has only a finite number of zeros. Hence the zero set of $\Phi$ consists of non-intersecting nodal lines characterized by the results given in Theorem 2.3.

**Remark 2.4.** In some sense our asymptotic results on nodes might be considered as analogs of the local results on nodes of L. Bers [6], S.Y. Cheng [7] and recently L.A. Caffarelli and A. Friedman [8].

There are some results on generic properties of eigenfunctions of elliptic operators on compact manifolds by J. Albert [9] and K. Uhlenbeck [10]. In the appropriate setting the generic case for the nodal lines of $\Phi$ for $\tau = \omega$ should be straight lines as given in (2.8). We hope to investigate this problem in future work.
Remark 2.5. The results given in Theorem 1.1 have been generalized to the n-dimensional case in [5]. Naturally the structure of the nodal set near infinity of such a solution can show a much more complicated pattern than in two dimensions. Partial results will be given in [11].

3. Proof of Theorem 2.1

To verify the uniform real analyticity of $J_N$ it suffices to show that given $I$, then for some $c$, $\delta > 0$,

$$|U_N^{(k)}(r,z)| \leq c \frac{k!}{\delta^k} \quad \forall z \in I \quad \text{and} \quad \forall r > \bar{R} \quad (3.1)$$

for some $\bar{R} > R$ large.

To derive (3.1) we first show that given any compact interval $J \subset \mathbb{R}$, then the family of functions

$$F_k = \{U_N^{(k)}(r, \cdot) : J \rightarrow \mathbb{R}, r > \bar{R}\} \quad \text{(with some } \bar{R} > R)$$

is uniformly bounded for $0 < k < M$.

This can be verified by making use of the following inequality:

If $f$ is an $n$-times differentiable function on a closed interval $J \subset \mathbb{R}$ of length $|J|$ and if $|f(x)| < M_0$ and $|f^{(n)}(x)| < M_n$, where $M_j = \sup_{x \in J} |f^{(j)}(x)|$, $0 \leq j \leq n$, then for $x \in J$ and for $0 < k < n$

$$|f^{(k)}(x)| \leq c_{n,k} M_0^{1-k/n} M_n^{k/n} \quad (3.2)$$

where $M_0 = \max_{n, M_0 \cdot |J|^{n}}$ and $c_{n,k}$ is a constant depending only on $n$ and $k$ (See e.g. [12]).

Since for every arbitrary fixed $r > R$, $U_N(r, z)$ fulfills the above conditions (due to the known properties of $u$) on any compact interval $J \subset \mathbb{R}$, inequality (3.2) can be applied and it remains to show that $\sup_{z \in J} |U_N(r, z)|$ and $\sup_{z \in J} |U_N^{(M)}(r, z)|$ are bounded for $r \rightarrow \infty$, which will

(Prove the rest of the theorem...)
become clear from the following: That

$$\sup_{z \in J} |u^0_n(r, z)| \leq C(J) \text{ for } r > \tilde{a}$$

is an immediate consequence of Corollary 1.1. On the other hand, since

$$u^0_M = b^{-M} \frac{\partial^M}{\partial r^M} u(r, \frac{z}{b^r})$$

we conclude by (1.10) (with \(k = M\)) that for all \(z \in J\)

$$|U^M(r, z) - b^{-M} \frac{\partial^M}{\partial r^M} u(r, \frac{z}{b^r})| \leq C(J) r^{-d} \quad \text{for large } r$$

(3.4)

with some \(C(J) < \infty\). Particularly (3.4) implies that

$$U^M(r, z) = (\frac{1}{2b})^M U^M(s) \quad \text{for } r = s \quad \text{uniformly in } J.$$ (3.5)

Hence it follows via inequality (3.2) that \(F_k\) is uniformly bounded for \(0 \leq k \leq M\). For \(k > M+1\) the uniform boundedness of \(F_k\) is easily seen from

$$U^{M+j}(r, z) = b^{-M-j} r^{-j/2} \frac{\partial^{M+j}}{\partial r^{M+j}} u(r, \frac{z}{b^r}) \quad \forall j \in \mathbb{N}$$ (3.6)

and the fact that due to Theorem 1.1 \(u\) is real analytic in \(w\) uniformly with respect to \(r\). But this implies further that given \(I \subseteq \mathbb{R}\), then for some \(c, \delta > 0\),

$$|U^{M+j}(r, z)| \leq c \ r^{-j/2} (\frac{M+j+1}{M+j})^\delta \quad \forall z \in I \text{ and large } r.$$ (3.7)

(3.7) together with the uniform boundedness of \(F_k\) for \(0 \leq k \leq M\) verifies (3.1). Furthermore (3.7) implies (2.1) for \(k > M+1\).

So finally it remains to verify (2.1) for \(0 \leq k \leq M\). Note first that \(F_k\) is for all \(k \in \mathbb{N}\) an equicontinuous family of functions since for \(z_1, z_2 \in J \text{ and } r > \tilde{a}\)

$$|U^k(r, z_1) - U^k(r, z_2)| \leq \int_{z_1}^{z_2} |U^k(r, z)|dz \leq c_{k+1} |z_2 - z_1|.$$
for some $c_{k+1} < \infty$ (not depending on $r$) due to the uniform boundedness of $F_k$ for all $k$.

To simplify notation let $g(z) = (2b)^{-N} H_n(z)$ and $g_n(z) = U_n(r_n,z)$ with $z \in J$, where $(r_n)$ is an arbitrary but fixed sequence with $r_n \to \infty$ for $n \to \infty$.

From Theorem 1.1 we know that $g_n \to g$ uniformly in $J$. Now let $k \in \{1,2,\ldots,N-1\}$, $z \in J$ arbitrary but fixed and let $a_k$ denote an accumulation point of the sequence $(g_n(k)(z))$.

Then a subsequence $(g_{n(i)}(k))$ of $(g_n(k))$ exists such that $g_{n(i)}(k)(z) \to a_k$ for $i \to \infty$. But $g_{n(i)}(k) \to g(z)$ uniformly on $J$ and $F_k$ is uniformly bounded and equicontinuous. Hence by Arzela-Ascoli's theorem (see e.g. [13]) it follows that a subsequence $(g_{n(j)}(k))$ of $(g_{n(i)}(k))$ exists with $g_{n(j)}(k) \to g(k)$ for $j \to \infty$ uniformly on $J$ for $j = 0,1,2,\ldots,M$. Therefore $g(k)(z) = a_k$ and further $g_{n}(k)(z) \to g(k)(z)$. Since $z \in J$ was arbitrary we obtain $g_n(k) \to g(k)$ in $J$ for $n \to \infty$, and the convergence is uniformly since $g_n \to g$ uniformly on $J$ due to (3.5).

This completes the proof of Theorem 2.1.

4. Proof of Theorem 2.2

For the proof we shall need the following

**Lemma 4.1.** Let $V_1$ and $\nu$ be given according to Theorem 1.1 so that

$$-\nu^n + (V_1 - 1/4\nu^2 - E)\nu = 0 \quad \text{for } r > R, \quad \text{where } \nu = \sqrt[2]{r} \nu. \text{ Then for large } r$$

$$\int \frac{v^2(x)dx}{r} = \frac{1}{2V_1 - 1/4\nu^2 - E} (1 + O(\nu^{-1})) \nu^2(r), \quad (4.1)$$

for $\nu > 0$

$$\nu^{-2}(r) \int \frac{v^2(x)x^{-1-\gamma} dx}{r} = \frac{1}{2/|\nu|} \nu^{-1-\gamma}(1 + O(\nu^{-\epsilon})) \quad (4.2)$$

for some $\epsilon > 0$. 

Let \( y_i = 0 \) for \( 1 \leq i \leq k, k \in \mathbb{N} \), denote
\[
Q_i = \frac{\nabla^2 (y_i)}{\nabla^2 (x_i)}
\]
and
\[
\sum_{i=1}^{k} Q_i^{-1-y_i} = \int \int \int \cdots \int Q_i^{-1-y_i} y_k \text{d}y_k \text{d}x_k \cdots \text{d}y_1 \text{d}x_1
\]
then for large \( r \)
\[
\sum_{i=1}^{k} Q_i^{-1-y_i} = O(r^{-\gamma}) \quad \text{where} \quad \gamma = \frac{k}{q}, \quad \text{specifically for}
\]
some \( \varepsilon > 0 \)
\[
\sum_{i=1}^{k} Q_i^{-1-y_i} = \left( \frac{1}{2 \sqrt{|B|}} \right)^{2-k} \left( 1 + O(r^{-\varepsilon}) \right).
\]

**Proof of Lemma 4.1.** For a proof of (4.1) see Lemma 2.5 in [1]. Applying (4.1) we obtain immediately that for some \( \varepsilon > 0 \)
\[
\nabla^2(r) \int \frac{\nabla^2 (x)}{r^{1-\gamma}} \text{d}x \leq r^{-1-\gamma} \frac{1}{2 \sqrt{|B|}} (1 + O(r^{-\varepsilon})).
\]

To derive the lower bound we use partial integration, apply (4.1) and obtain for some \( \varepsilon > 0 \)
\[
\int \frac{\nabla^2 (x)}{r^{1-\gamma}} \text{d}x = r^{-1-\gamma} \int \frac{\nabla^2 (x) \text{d}x - (1 + \gamma) \int x^{-2-\gamma} \int \nabla^2 (y) \text{d}y \text{d}x}{r^x} \geq
\]
\[
\geq r^{-1-\gamma} \frac{1 - \varepsilon}{2 \sqrt{|B|}} \nabla^2 (r)
\]
implying (4.2). Using induction (4.3) follows easily by application of (4.2). \( \square \)

Now we investigate the properties of \( w/3 \) for \( r = \infty \). Noting that \( \psi \) obeys (1.1) and \( v \) obeys (1.6) it follows that
\[
-\frac{\partial^2 u}{\partial r^2} - \frac{2u}{r} + \frac{\partial u}{\partial r} - \frac{2 \partial u}{r^2} + (\partial v^2) u = 0 \quad \text{in} \quad \mathbb{R}. \quad (4.4)
\]
Having in mind that \( \lim_{r \to \infty} u = A \) it is easily seen that \( u \) obeys the following integrodifferential equation

\[
u(r, \omega) = A(\omega) + \int \frac{\nu^2(x)}{r} \int \nu^2(y)(-y^2 \frac{\partial^2}{\partial \omega^2} + \nu_2(y, \omega))u(y, \omega) \, dy \, dx \quad (4.5)
\]

(see Equ. (4.2) in [1]). Therefrom

\[
\frac{3\nu}{3r}(r, \omega) = -\nu^2(r) \int \frac{\nu^2(y)(-y^2 \frac{\partial^2}{\partial \omega^2} + \nu_2(y, \omega))u(y, \omega) \, dy}{r} \quad (4.6)
\]

follows. Since \( u, r^\omega \nu_2 \) are real analytic in \( \omega \) uniformly with respect to \( r \), \( \forall \varepsilon > 0 \) small, there exist \( C_0, C_1, \delta > 0 \) such that for \( |\omega| < r_0 - \varepsilon \) and large \( r \)

\[
|\frac{\partial_j}{\partial \omega^j} \nu_2 \, r^\omega| < C_0 \frac{\partial}{\partial \omega^j} \quad \text{and} \quad |\frac{\partial_j}{\partial \omega^j} u| < C_1 \frac{\partial}{\partial \omega^j} \quad \text{for} \quad j \in \mathbb{N} \cup \{0\}
\]

Therefore for some \( C < \) (not depending on \( k \) and \( \gamma \))

\[
|\frac{\partial_k}{\partial \omega^k}(-y^2 \frac{\partial^2}{\partial \omega^2} u(y, \omega) + \nu_2(y, \omega) u(y, \omega))| \leq \\
\leq |y^2 \frac{\partial^{2+k}}{\partial \omega^{2+k}} u| + \sum_{j=0}^{k} \left( \frac{\partial^j}{\partial \omega^j} \nu_2 \right) \left( \frac{\partial^j}{\partial \omega^j} u \right) \leq C \frac{k!}{\delta^k} y^{-1-\varepsilon} \quad \text{for} \quad k \in \mathbb{N} \cup \{0\}
\]

with \( \alpha = \min(1, \alpha) \). Applying (4.2) in Lemma 4.1 we obtain for large \( r \) and some \( C < \)

\[
\frac{\nu^2(r)}{r} \int \nu^2(y)(-y^2 \frac{\partial^2}{\partial \omega^2} + \nu_2(y, \omega)) \, dy \leq C \frac{k!}{\delta^k} r^{-\alpha} \quad \text{for} \quad k \in \mathbb{N} \cup \{0\}. \quad (4.7)
\]

Hence we conclude from (4.6) and (4.7) that for \( k \in \mathbb{N}, \forall |\omega| < r_0 - \varepsilon \) and large \( r \)

\[
\frac{\partial^{k+1}}{\partial r \partial \omega^k} u(r, \omega) = -\nu^2(r) \int \frac{\nu^2(y)(-y^2 \frac{\partial^2}{\partial \omega^2} + \nu_2(y, \omega)) \, dy}{r} \quad (4.8)
\]

and

\[
|\frac{\partial^{k+1}}{\partial r \partial \omega^k} u(r, \omega)| \leq C \frac{k!}{\delta^k} r^{-\alpha} \quad (4.9)
\]

Clearly an analogous estimate to (4.9) is obtained after rotation of the coordinate system by proceeding in the above manner. But this
implies that \( r^1v2u/\partial v \) is real analytic in \( w \) uniformly with respect to \( r \), verifying the first part of Theorem 2.2.

To prove the second part of Theorem 2.2 we start with the case \( M = 1 \):

Rewriting (4.6) we have

\[
\frac{3u}{\partial r}(r,w) = -\frac{r^2}{r} \int 2(\gamma)(y)\frac{-y^2}{dr^2} + v_2A)dy - \frac{r^2}{r} \int 2(\gamma)(y)\frac{2r}{dr^2} + v_2(u-A)dy .
\]

Having in mind (1.10) of Theorem 1.1 application of Lemma 4.1 leads to

\[
\frac{3u}{\partial r}(r,w) = \frac{d^2A(w)/dw^2}{2/|E|} r^{-1}\left(1 + O(r^{-6})\right) + A(w)O(r^{-1/2}) + O(r^{-1/2}) .
\]

Since for \( w = O(r^{-1/2}) \), \( d^2A/\partial r^2 = 2d + O(r^{-1/2}) \) and \( A = O(r^{-1/2}) \), (2.2) follows immediately from the above.

Now we have to investigate the case \( M > 1 \):

Due to the real analyticity of \( u/\partial r \) we have

\[
\frac{3u}{\partial r}(r,w) = \sum_{k=0}^{\infty} \frac{1}{k!} u(r,0) \frac{d^k}{dw^k} w^k \text{ for small } |w| \text{ and large } r .
\]

To derive the asymptotics of the r.h.s. of (4.10) we shall use (4.8) with \( w = 0 \) and the fact that

\[
\frac{d^k}{dw^k} u(r, w) = \frac{k}{b} r^{(-M+k)/2} u_M(k, r) \text{ for } z \in \mathbb{R} \text{ and } k \in \mathbb{N} \cup \{0\} .
\]

Therefore we obtain

\[
\sum_{k=1}^{\infty} \frac{1}{k!} w^k \frac{d^k}{dw^k} u(r,0) = \sum_{k=1}^{\infty} \frac{1}{k!} w^k \int 2(\gamma)(y)\frac{-y^{1+k-M)/2}{dr^2} b^{k+2} u_M(k+2)(r,0) - \frac{k}{b} \sum_{j=0}^{k} \frac{1}{j!} \frac{1}{b^j} v_2(y,0) u_M^{(j)}(y,0)b^j y^{(j-M)/2}dy .
\]

Since due to Theorem 2.1 \( u_M^{(j)}(r,0) - (2b)^{-M} w_M^{(j)}(0) = 0 \) for \( r = \) the r.h.s. of the above equation
Further since due to our assumptions \( \frac{\partial^j}{\partial r^j} V_2 \) is uniformly bounded for \( r \rightarrow \) for all \( j \in \mathbb{N} \cup \{0\} \), application of Lemma 4.1 implies that the above
\[
\begin{align*}
\sum_{k=1}^{N-1} \frac{b^k}{b^{k+1}} \frac{\theta^k}{\theta^{k+1}} \int_{\mathbb{R}^d} \frac{1}{r^{1-(k-\eta)/2}} (2b)^{-\eta} H^M_{\eta}(k+2)(0) + o(1)) (1 + O(r^{-\xi})) + \pi = r^{-1-\eta/2} (2b)^{-\eta} H^M_{\eta}(k+2)(0) + o(1)) (1 + O(r^{-\xi})) + \pi = r^{-1-\eta/2} (2b)^{-\eta} H^M_{\eta}(k+2)(0) + o(1)) (1 + O(r^{-\xi})) + \pi \end{align*}
\]
Therefore for \( u = z/(b^2) \) and \( z \in J \) (J an arbitrary but fixed compact interval)
\[
\sum_{k=1}^{N-1} \frac{b^k}{b^{k+1}} u(r,0) \frac{\theta^k}{\theta^{k+1}} r^{-k/2} = \sum_{k=1}^{N-1} \frac{b^k}{b^{k+1}} u(r,0) \frac{\theta^k}{\theta^{k+1}} r^{-k/2} = \sum_{k=1}^{N-1} \frac{b^k}{b^{k+1}} u(r,0) \frac{\theta^k}{\theta^{k+1}} r^{-k/2} = \sum_{k=1}^{N-1} \frac{b^k}{b^{k+1}} u(r,0) \frac{\theta^k}{\theta^{k+1}} r^{-k/2}
\]
On the other hand by applying (4.9) it is straightforward to see that for \( z \in J \)
\[
\left| \sum_{k=1}^{N-1} \frac{b^k}{b^{k+1}} u(r,0) \frac{\theta^k}{\theta^{k+1}} \right| \leq C |z| r^{-M/2-1-\xi} \left| \sum_{k=1}^{N-1} \frac{b^k}{b^{k+1}} u(r,0) \frac{\theta^k}{\theta^{k+1}} \right| \leq C |z| r^{-M/2-1-\xi}
\]
for large \( r \) with some \( C < \infty \). Combining (4.13) and (4.14) we arrive at
\[
\begin{align*}
\frac{\partial^2}{\partial r^2} (r,0) \left| \sum_{k=1}^{N-1} \frac{b^k}{b^{k+1}} u(r,0) \frac{\theta^k}{\theta^{k+1}} \right| \leq C |z| r^{-M/2-1-\xi} \left| \sum_{k=1}^{N-1} \frac{b^k}{b^{k+1}} u(r,0) \frac{\theta^k}{\theta^{k+1}} \right| \leq C |z| r^{-M/2-1-\xi}
\end{align*}
\]
Due to (4.6) and (4.11) and Theorem 2.1 it follows that
\[
\frac{\partial u}{\partial \tau}(r,0) = \gamma^2(r) \int \gamma^2(y) y^{-1-M/2} b^2 y^2 u_{M}^{(1)}(y,0) + v_2(y,0) y^{-M/2} u_M(y,0) dy = \\
\frac{1}{4}(2b)^{-M} H_M^{(2)}(0) + o(1) r^{-1-M/2} (1 + O(r^{-5})) + O(r^{-1-M/2}) .
\] (4.16)

Further note that
\[
\sum_{k=0}^{M-1} z^k u_M^{(k+2)}(0) = H_M^{(2)}(z) = 4H(M-1) H_{M-2}(z) .
\] (4.17)

Now if \( M = 2m, m \in \mathbb{N} \), then (4.16), (4.17) together with (4.15) obviously verify (2.4).

Finally let \( M = 2m+1, m \in \mathbb{N} \): Since \( H_M(0) = 0 \) (4.17) together with (4.15) yield
\[
\frac{\partial u}{\partial \tau}(r,0) = r^{1+M/2} + O(r^{-\infty})
\] for large \( r \) and \( z \in J \). However (4.16) only implies that
\[
\frac{\partial u}{\partial \tau}(r,0) \rightarrow 1+M/2 + O(r^{-\infty}) \quad \text{for } r \rightarrow 0 .
\]

Suppose we have shown

**Lemma 4.2.** For \( M = 2m+1, m \in \mathbb{N} \),
\[
r^{m+2} \frac{\partial u}{\partial \tau}(r,0) = \frac{(-1)^m (M+1)!}{(2b)^{M+1}} d(1 + O(r^{-6})) + O(r^{-8}) .
\] (4.19)

Then (4.19) together with (4.18) verify (2.5), finishing the proof of Theorem 2.2.

**Proof of Lemma 4.2.** For the proof we shall proceed in an analogous way as in [11 resp. [5] for working out the asymptotics of \( u \). We shall use the following notation (compare Lemma 4.1):
\[
\tau_1 = - y^2 \frac{\partial^2}{\partial \omega^2} v_2(y_1,\omega) , \quad \tau = - y^2 \frac{\partial^2}{\partial \omega^2} v_2(y,\omega) .
\]
where \( f \) is identified with \( u \) or \( A \). When necessary the dependence of \( \ldots \) on the variable \( r \) will be denoted by \( \ldots \). Using the above notation, equation (4.5) resp. (4.6) read

\[
 u = A + \langle Q_i T_i, u \rangle 
\]

Iterating equation (4.20) gives

\[
 u = A + \sum_{i=1}^{M} \langle Q_i T_i, A \rangle + \sum_{i=1}^{M} \langle Q_i T_i, (u - A) \rangle 
\]

Combining (4.22) with (4.21) leads to

\[
 \frac{3u}{\sqrt{r}}(r, w) = - \sqrt{2}(r) \int \sqrt{2}(y) T(A + \sum_{i=1}^{M} \langle Q_i T_i, A \rangle + \sum_{i=1}^{M} \langle Q_i T_i, (u - A) \rangle)dy
\]

Now we investigate the asymptotics of the terms on the r.h.s. of (4.23):

Note that due to assumption (1.9) \( r^{1+a} \frac{2^k}{2^{\omega_k}} v_k \) is uniformly bounded in \( r \sparse r = \infty \) for all \( k \in \mathbb{N} \cup \{0\} \) and that \( r^{a} \frac{j^k}{2^\omega_k} (u - A) \) is in the same sense bounded because of (1.10). Taking this into account and using Lemma 4.1 it is straightforward to show that for all \( w \) and large enough \( r \)

\[
 T = \Pi_{i=1}^{M} Q_i T_i (u - A) = \Pi_{i=1}^{M} Q_i T_i (u(y_i M, w) - A) = \mathcal{O}(y^{-aM - 2a - 1}) = \mathcal{O}(y^{-\infty})
\]

and therefore
Next we observe in an analogous way that for \( 1 \leq \ell \leq m \)

\[
\begin{align*}
\sum_{\ell=1}^{m} Q_{\ell}^{2}\sum_{i=1}^{l} \left( y^{2}(y) T_{i}^{2} (u - A) y_{i} \right) dy &= O(r^{-2-2m}). \quad (4.24)
\end{align*}
\]

where \( R \) is a sum of terms, each of them depending on \( \frac{d^{k}}{du} A(0) \) for some \( k \) with \( 0 \leq k \leq 2\ell \). Since \( \frac{d^{k}}{du} A(0) = 0 \) for \( 0 \leq k \leq 2m \) and \( 1 \leq \ell \leq m \) the above implies that

\[
\begin{align*}
\sum_{\ell=1}^{m} Q_{\ell}^{2}\sum_{i=1}^{l} \left( y^{2}(y) T_{i}^{2} (u - A) y_{i} \right) dy |_{u=0} &= \left\{ \begin{array}{ll}
0 & \text{for } 1 \leq \ell \leq m-1 \\
\sum_{i=1}^{m} Q_{i}^{2} y_{i}^{2} (-1)^{m} y^{-2m} \frac{d^{2m+2}}{du^{2m+2}} A(0) & \text{for } \ell = m
\end{array} \right.
\end{align*}
\]

and we conclude from the above via Lemma 4.1 that for some \( \epsilon > 0 \)

\[
-\sum_{\ell=1}^{m} Q_{\ell}^{2}\sum_{i=1}^{l} \left( y^{2}(y) T_{i}^{2} (u - A) y_{i} \right) dy |_{u=0} = \frac{(-1)^{m}}{m!} \frac{1}{(4\ell^{2})^{m}} \left( 1 + O(r^{-\epsilon}) \right) \sum_{\ell=1}^{m} Q_{\ell}^{2} \sum_{i=1}^{l} \left( y^{2}(y) T_{i}^{2} (u - A) y_{i} \right) dy |_{u=0} \]

\[
= \frac{(-1)^{m}}{m!} (N+1)! \frac{d}{(4\ell^{2})^{m+1}} \left( 1 + O(r^{-\epsilon}) \right) r^{-m-2}, \quad (4.25)
\]

where we used \( \frac{d^{2m+2}}{du^{2m+2}} A(0) = d(N+1)! \).

It remains to investigate the asymptotics of \( \sum_{\ell=1}^{m} Q_{\ell}^{2}\sum_{i=1}^{l} \left( y^{2}(y) T_{i}^{2} (u - A) y_{i} \right) dy |_{u=0} \) for \( m+1 \leq \ell \leq 2m+1 \): Thereby it is not difficult to see that

\[
\begin{align*}
\sum_{\ell=1}^{m} Q_{\ell}^{2}\sum_{i=1}^{l} \left( y^{2}(y) T_{i}^{2} (u - A) y_{i} \right) dy |_{u=0} &= \sum_{i=1}^{m} Q_{i}^{2} y_{i}^{2} (-1)^{m} y^{-2m} \frac{d^{2m+2}}{du^{2m+2}} A(0) \times y_{2}(y, 0) (-1)^{t} \frac{d^{2t+2}}{du^{2t+2}} A(0) + R \quad (4.26)
\end{align*}
\]
where the rest $R$ is a finite sum of terms of two types denoted by $J_{k,l}$ and $I_{k,l}$ with $1 \leq k+l \leq t$ characterized by the following:

Let $(i_1, \ldots, i_t)$ denote a permutation of $(1,2,\ldots,t)$ and let $a \in \mathbb{N}$, $a_j \in \mathbb{N} \cup \{0\}$ for $k+l \leq j \leq t$, then

$$J_{k,l} = \frac{\prod_{k=1}^{l} q_i^k}{(q_i^k - 1)^2} \sum_{j=k+1}^{l} a_j \frac{d^a}{d\omega^a} \Delta(0)$$

with $2m+1 \leq a \leq 2t$ and $\sum_{j=k+1}^{l} a_j + a = 2k + 2$.

and

$$I_{k,l} = \frac{\prod_{k=1}^{l} q_i^k}{(q_i^k - 1)^2} \sum_{j=k+1}^{l} a_j \frac{d^a}{d\omega^a} \Delta(0)$$

with $2m+1 \leq a \leq 2t-2$, $\sum_{j=k+1}^{l} a_j + a = 2k$.

Analogously to the foregoing considerations it follows that

$$J_{k,l} = 0(\prod_{k=1}^{l} q_i^k (y^{-2} \prod_{l \geq 0} y^{-l-a} y^{-1-a} y^{-2})) = 0(y^{-2-k-s(t-k)}) = 0(y^{-2-m-a})$$

using $k > m$ in the last step (according to (4.27)) and

$$I_{k,l} = 0(\prod_{k=1}^{l} q_i^k (y^{-2} \prod_{l \geq 0} y^{-l-a} y^{-1-a} y^{-2})) = 0(y^{-k-s(t-k)-1-a}) = 0(y^{-2-m-2a})$$

using $k > m+1$ in the last step (according to (4.28)). (4.29) and (4.30) imply that

$$R = O(y^{-2-m-a})$$

Therefore and again with the help of Lemma 4.1, (4.26) yields
Application of (4.31), (4.25) and (4.29) to (4.23) verifies Lemma 4.2.0.

Remark 4.1. (4.19) holds also for \( m = 0 \) as can be seen easily by proceeding as above, but we shall not need it for the following.

5. Proof of Theorem 2.3

We first show the existence of exactly \( N \) nodal lines in \( D_\varepsilon \): Let

\[
H_N(\tilde{s}) = 0
\]

and choose \( \delta_0 > 0 \) such that (without loss)

\[
\frac{d}{ds} H_N(s) > 0 \quad \text{for } s \in I_{\delta_0}(\tilde{s}), \quad \text{where } I_{\delta_0}(\tilde{s}) = \{s | |s - \tilde{s}| < \delta_0\} \tag{5.1}
\]

This is possible since \( H_N \) has only nondegenerate zeros. Further choose \( R_0 \) so large that

\[
U_M^{(1)}(r,s) > 0 \quad \text{for } r > R_0 \quad \text{and } s \in I_{\delta_0}(\tilde{s}) \tag{5.2}
\]

which is possible due to Theorem 2.1. Further by Theorem 2.1 the above implies that \( \forall \delta \in (0,\delta_0) \) there is some \( R_\delta > R_0 \) such that for \( r > R_\delta \)

\[
\text{sgn } U_M(r,s + \delta) = \text{sgn } H_M(\tilde{s} + \delta).
\]

Hence for all \( r > R_\delta \) there exists \( g(r) \in I_{\delta}(\tilde{s}) \) with \( U_M(r,g(r)) = 0 \) and due to (5.2) it is unique. Having in mind that \( u \in C^1(\Omega_\varepsilon) \) the implicit function theorem implies that \( g \) is continuously differentiable. Furthermore it follows that \( g(r) = \tilde{s} \) for \( r = \infty \).

The foregoing considerations imply that for each zero \( s_i \) \( (1 \leq i \leq N) \) of \( H_N \) there is at least one nodal line of \( u \) in \( D_\varepsilon \) given by \( u_i = g_i(r) \phi\sqrt{r} \),

\[
u(r,s_i) = 0 \quad \text{where } g_i(r) = s_i \quad \text{for } r = \infty.
\]
Now suppose there exists \( r_n \to \infty \) for \( n \to \infty \) and \( \tilde{w}(r_n) \), such that \( \forall m \)

\[
|\tilde{w}(r_n)| < \varepsilon, \quad \tilde{w}(r_n) \neq w_i(r_n), \quad \text{for } 1 \leq i \leq M \text{ and } u(r_n, \tilde{w}(r_n)) = 0. \]

Then since \( u + A \) for \( r \to \infty \) uniformly and \( A(\omega) \neq 0 \) for \( 0 < |\omega| < \varepsilon \) for \( \varepsilon \), small enough, \( \tilde{w}(r_n) \to 0 \) for \( n \to \infty \) follows. Together with the foregoing consideration we obtain that for some \( \omega(r_n) \to 0 \) for \( n \to \infty \), \( \frac{\partial^2 R_0}{\partial X^2} (r_n, \omega(r_n)) = 0 \) \( \forall n \). Since \( \frac{\partial^2 R_0}{\partial X^2} '> 0 \) uniformly we obtain \( \frac{\partial A(\omega)/\partial \omega} = 0 \)

which is a contradiction to the assumption on \( A \). Hence there are exactly \( N \) nodal lines of \( \omega \) in \( D \) for \( \varepsilon \) small enough.

Let \( g = \tilde{g} \) for \( r = - \) be given as before and denote \( f(r) = g(r)/(b/\sqrt{r}) \).

Then for large \( r \)

\[
u(r,f(r)) = 0
\]

and

\[
\frac{3u}{3r}(r,u)|u=f(r) + \frac{2u}{3u}(r,f(r)) f'(r) = 0. \quad (3.3)
\]

This implies further

\[
f'(r) = - \frac{2u}{3u}(r,u)|u=f(r) r^{(N-1)/2} (b U_n^{(1)}(r,g(r)))^{-1}. \quad (5.4)
\]

Since due to Theorem 2.1

\[
\lim_{r \to \infty} U_n^{(1)}(r,g(r)) = (2b)^{-N} 2N U_n^{(1)}(\varepsilon) \neq 0 \quad (5.5)
\]

and since due to Theorem 2.2 \( r^{|1+M/2}} \frac{3u}{3r}(r,u)|u=f(r) \) is bounded for \( r \to \infty \) we obtain from (5.4) that for some \( C < \infty \)

\[
|f'| \leq C r^{-3/2} \quad \text{for large } r. \quad (5.6)
\]

Denoting \( \gamma_1(r) = r \cos f(r), \gamma_2(r) = r \sin f(r) \) we conclude from (5.6) that for large \( r \) for some \( c > 0 \)

\[
\gamma_1'(r) = \cos f(1-r f \tan f) \geq \cos f(1+0(r^{-1})) > c > 0. \quad (5.7)
\]

Therefore the inverse \( \gamma_1^{-1} \) exists, implying the representation of the
modal line in cartesian coordinates $(x_1, x_2) \in \mathbb{R}^2$ by $x_2 = G(x_1)$ with $G = r_2 \circ v_1^{-1}$.

Next we verify the asymptotics of the nodal lines of $\psi$. We start with the simplest case:

$M = 1$

We use the asymptotics of $\partial u/\partial r$ given in (2.3) of Theorem 2.2, take into account (5.5) and apply these findings to (5.4). This gives

$$f'(r) = \left(-\frac{d}{\sqrt{|x|}} + o(1)\right) r^{-2}$$

and integrating from $r$ to $\infty$ gives

$$f(r) = \left(-\frac{d}{\sqrt{|x|}} + o(1)\right) r^{-1}.$$ 

Therefore

$$\gamma_2(r) = r \sin f(r) = rf + O(r^{-1/2}) = \frac{d}{\sqrt{|x|}} + o(1)$$

and $\gamma_1(r)/r = \cos f(r) + 1$ for $r \to \infty$, implying $G(x_1) = d/\sqrt{|x|} + o(1)$ and verifying (2.8) for $M = 1$.

Next we consider the case

$M \geq 2$ and $\bar{x} \neq 0$

Since $g(r) = \bar{x}$ for $r \to \infty$ we obviously have

$$\gamma_2(r) = \frac{\bar{x}}{b/\ell} = \frac{\bar{x}}{\ell} \sqrt{\ell}(1 + O(\ell^{-1})) = \left(\frac{\ell}{\bar{x}} + o(1)\right) \sqrt{\ell}$$

and $\gamma_1(r)/r + 1$ for $r \to \infty$ and therefore $G(x_1) = (\bar{x}/b + o(1))/\sqrt{\ell}$ for large $x_1$ verifying (2.7).

To prove the monotonicity of $G(x_1)$ it suffices (because of (5.7)) to show the monotonicity of $\gamma_2(r)$; since

$$\gamma_2'(r) = \cos f(r) + e^{g}(f) = \cos \left(\frac{d}{\sqrt{\ell}}(r^{3/2} + \sqrt{\ell}(1 + O(r^{-1})))\right)$$

we have to investigate the asymptotics of $f'$: Taking into account (2.4)
and (2.5) of Theorem 2.2 we conclude
\[ r^{1+H/2} \frac{du}{dr}(r,\omega)|_{\omega=f(r)} = H(N-1)(2b)^{-H} H_{H-2}(z) + o(1) \]  
(5.9)

and note that \( H_{H-2}(z) \neq 0 \). Applying (5.9) and (5.5) to (5.4) we arrive at
\[ f''(r) = -r^{-3/2} \frac{1}{2b} H_{H-2}(z) + o(1) \]  
(5.10)

Noting that \( H_{H}(z) = 0 = 2z H_{H-1}(z) - 2(N-1) H_{H-2}(z) \), (5.10) leads to
\[ f''(r) = (-\frac{z}{2b} + o(1))r^{-3/2} \]  
(5.11)

Combining (5.11) with (5.8) and taking into account that \( \sqrt{r} f = \bar{z}/b \) for \( r \to \infty \) we obtain
\[ \text{sgn} y'_2(r) = \text{sgn} (\bar{z}) \quad \text{for large } r . \]  
(5.12)

(5.12) together with (5.7) shows that \( y_2 \circ y_1^{-1} \) is strictly monotonously increasing for \( \bar{z} > 0 \) respectively decreasing for \( \bar{z} < 0 \).

Finally we have to consider the case
\[ M = 2m+1, \ m \in \mathbb{N} \text{ and } \bar{z} = 0; \]
Due to (2.5) of Theorem 2.2 we have
\[ r^{1+H/2} \frac{du}{dr}(r,\omega)|_{\omega=g(r)/(b/r)} = \frac{d_1}{r^3} (1 + O(r^{-c})) + O(r^{-1/2-a}) + \]
\[ \frac{N(N-1)}{(2b)^H} H_{H-2}(g(z)) + o(g(r)) \]  
(5.13)

with \( d_1 = (-1)^m \frac{(N+1)!}{m!} d (2b)^{-N-1} \),

and via Theorem 2.1
\[ u_N^{(1)}(r,\bar{z}) = (2b)^{-H} 2M H_{H-1}(g) + o(1) \]  
(5.14)
Applying (5.13) and (5.14) to (5.6) and taking into account that

\[ f' = \frac{1}{b\sqrt{r}} (g' - \frac{A}{2r}) \]

we obtain

\[
g' = \frac{A}{2r} - r^{N/2} \frac{e^2}{b\sqrt{r}} \left( \frac{1}{g(r)} \left( \frac{1}{k} \right) \right) \left( g(r) \right)^{-1} =
\]

\[
= \frac{1}{2r} \left( \frac{b}{\sqrt{r}} \left( 1 + o\left( r^{-2} \right) \right) - o(1) \right) - (N-1) \frac{g}{r} \frac{g''}{g' - o(1)} \left( g(r) \right)^{-1} \quad (5.15)\]

Since for \( k \in \mathbb{N} \), \( H_{2k}(0) = (-1)^k (2k)! / k! \) and \( H_{N-2}(0) = 2(N-2)H_{N-3}(0) \),

\[
\frac{H_{N-2}(g)}{H_{N-1}(g)} = g \left( \frac{H_{N-2}(0) + o(|g|^2)}{H_{N-1}(0) + o(|g|^2)} \right) = -g(1 + o(1))
\]

we obtain from (5.15) for some \( \delta_1(r), \delta_2(r) > 0 \) for \( r \to +\infty \) that for large \( r \)

\[
g' = \frac{1}{r} \left[ \frac{N}{2} g(1 + \delta_1(r)) - \frac{(N+1)}{4b} + \delta_2(r) \right] r^{-1/2} \quad (5.16)
\]

In the following we show with the help of (5.16) that

\[
\lim_{r \to +\infty} \sqrt{r} g = \frac{d}{2b} \quad (5.17)
\]

Let us first consider the case \( d > 0 \):

Suppose that for some \( \bar{r} \) large \( g(\bar{r}) < 0 \), then because of (5.16), for \( r \geq \bar{r} \)

\( g < 0 \) and \( g \) strictly monotonously increasing follows, contradicting \( g \to 0 \)

for \( r \to +\infty \). Therefore \( g > 0 \) for large \( r \). Let

\[
\tilde{c}_1 = \frac{N}{2} (1 + \tilde{\delta}_1) \quad \text{and} \quad \tilde{c}_2 = \frac{d(N+1)}{4b} - \tilde{\delta}_2
\]

for some \( \tilde{\delta}_1, \tilde{\delta}_2 > 0 \) arbitrarily small, then due to (5.16)

\[
g' \leq \frac{1}{r} \left[ g \tilde{c}_1 - \tilde{c}_2 \right] r^{-3/2} \quad \text{for large enough} \ r.
\]
Further let

\[ h(r) = \frac{S_2}{c_1 + \frac{1}{r}} r^{-1/2}, \]

then

\[ h' = \frac{1}{r} c_1 h - S_2 r^{-3/2} \]

and hence

\[ (h - g)' \geq c_1 \frac{1}{r} (h - g) \text{ for large } r. \]

Suppose there exists \( \tilde{r} \) (arbitrarily large) with \( (h - g)(\tilde{r}) > 0 \). Then the above inequality implies that \( 0 < h - g \) and \( h - g \) strictly monotonously increasing for \( r \geq \tilde{r} \), which contradicts \( h - g = 0 \) for \( r = \infty \). Hence for large \( r \), \( g \geq h \) and therefore with some \( \overline{\epsilon} > 0 \) arbitrarily small

\[ g(r) \geq \left( \frac{d}{2b} - \overline{\epsilon} \right) r^{-1/2} \text{ for large } r. \quad (5.18) \]

Combining (5.16) with (5.18) we obtain for some \( \hat{\epsilon}_1 \), \( \hat{\epsilon}_2 \), \( \delta > 0 \) arbitrarily small

\[ g' \geq \frac{d}{2b}(1 - \hat{\epsilon}_1) \left( \frac{d}{2b} - \overline{\epsilon} \right) - \left( \frac{d(H+1)}{4b} + \hat{\epsilon}_2 \right) r^{-3/2} \geq - \left( \frac{d}{4b} + 4 \right) r^{-3/2} \]

for large \( r \). Integrating the above inequality leads to

\[ g \leq \left( \frac{d}{2b} + \overline{\epsilon} \right) r^{-1/2} \quad (5.19) \]

with some \( \overline{\epsilon} > 0 \) arbitrarily small for \( r = \infty \). (5.18) and (5.19) imply (5.17) for \( d > 0 \).

The case \( d < 0 \) follows in the same way.

For \( d = 0 \) \( \lim_{r \to \infty} \sqrt{r} g = 0 \) can be seen by the following: Suppose there exists \( r = a \) such that \( \sqrt{r} g(r) = k \) for \( r = a \) with \( 0 < k < \infty \). Then because of (5.16)
\[ g' = \left( \sqrt{r}g \frac{N}{2}(1 + \delta_1(r)) - \delta_2(r) \right)r^{-3/2} \]

and therefore \( g > 0 \) and \( g \) is strictly monotonously increasing for large \( r \), contradicting \( g = 0 \) for \( r \to \infty \). For \( 0 < k < \infty \) the conclusion is analogously. This proves (3.17).

By (3.17) we finally obtain

\[ v_2(r) = r \sin \frac{\delta}{b/r} = \frac{\sqrt{r}}{b} = O(g^2/r^3) \frac{r^{-2}}{2b^2} + o(1) \]

and since \( v_1(r)/r \to 1 \) for \( r \to \infty \) this verifies (2.8) and finishes the proof of Theorem 2.3.
References

[1] H. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and J. Swetina,
Continuity and nodal properties near infinity for solutions
of 1-dimensional Schrödinger equations, Duke Math. J. 53
(1946) 271 - 306.

equations of second order, Springer Verlag, Berlin, Heidel-

[3] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions,
Dover, New York, 1968.

Pointwise bounds on the asymptotics of spherically averaged

Asymptotics and continuity properties near infinity of solu-
H. Poincaré, in press.

[6] L. Bers, Local behaviour of solutions of general elliptic equations,

51 (1976) 43 - 55.

of solutions of linear and super-linear elliptic equations,

differential operators, Trans. Am. Math. Soc. 238 (1978) 341 -
354.


of nodes of $L^2$-solutions of Schrödinger equations,
in preparation.