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HAMILTONIAN MECHANICS AND DIVERGENCE-FREE FIELDS

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ABSTRACT

The field lines, or integral curves, of a divergence-free field in three dimensions are shown to be topologically equivalent to the trajectories of a Hamiltonian with two degrees of freedom. The consideration of fields that depend on a parameter allow the construction of a canonical perturbation theory which is valid even if the perturbation is large. If the parametric dependence of the magnetic, or the vorticity, field is interpreted as time dependence, evolution equations are obtained which give Kelvin's theorem or the flux conservation theorem for ideal fluids and plasmas. The Hamiltonian methods prove especially useful for study of fields in which the field lines must be known throughout a volume of space.

I) INTRODUCTION

The trajectories of the field lines of a divergence-free field are mathematically equivalent to the trajectories of a particle with a Hamiltonian of two degrees of freedom.[1-5] This property of divergence-free fields, such as the magnetic field $\mathbb{B}(x)$ and the vorticity $\boldsymbol{\omega}(\mathbb{x})$, has important applications to the determination of the trajectories of the field lines[6] and the evolution of the field structure. The Hamiltonian property of the magnetic field also leads to simple methods for determining particle drift motion in complicated magnetic fields[7] and to methods for studying the equilibrium and stability of plasmas.[8]

The Hamiltonian formulation for divergence-free fields consists of two parts: the Hamiltonian as a function of its canonical variables and the position vector $\mathbb{x}$ also as a function of the canonical variables. The Hamiltonian contains all the topological information on the field line trajectories: the formation of surfaces, islands, or stochastic regions by the trajectories and the rotation number, or rotational transform, of the trajectories which form surfaces. The solution of many problems that involve divergence-free fields requires the determination of every field line trajectory in some volume of space. This is in contrast to the traditional problems of mechanics, such as planetary orbits, which require the solution of only a single trajectory. The techniques of Hamiltonian mechanics that are associated with action-angle variables and canonical perturbation theory permit the determination of all the trajectories with an effort which is comparable to that required to find a single trajectory. The procedure is based on finding canonical coordinates in which the Hamiltonian is in action-angle, or near action-angle, form so that the trajectories in canonical coordinate space are simple. The spatial position $\mathbb{x}$ as a function of the canonical coordinates then gives the trajectory in ordinary space. This procedure has long been used in fluid mechanics and plasma physics under special circumstances. The canonical coordinates in these special circumstances are called Clebsch coordinates or magnetic coordinates.

The Hamiltonian formulation separates different parts of the physics associated with the evolution of a magnetic field embedded in a plasma of low resistivity or of the vorticity of a fluid of low viscosity. In an ideal fluid or plasma, one obtains two distinct theorems: Kelvin's circulation theorem and the flux conservation theorem. Kelvin's circulation theorem says the circulation, or the toroidal magnetic flux, is tied to the fluid. The flux conservation theorem says that the poloidal magnetic flux and the toroidal magnetic flux are tied together. The analogous theorem of fluid mechanics is not well-known since it is not generally appreciated that there are two independent circulations in a fluid. The flux conservation theorem says that these two circulations are tied together in an ideal fluid.

The existence of a Hamiltonian for divergence-free fields is demonstrated in section (II). As in other Hamiltonian systems canonical transformations are of central importance. The theory of canonical transformations is given in section (III). The power of canonical methods is greatly extended in section (IV) by considering fields which depend on an arbitrary parameter λ. With λ interpreted as a perturbation parameter, a general canonical perturbation theory is developed which does not require the perturbation to be small, Section (V). The time development of the magnetic, and vorticity, fields is studied in section (VI) by interpreting λ as time t. Kelvin's theorem as well as the flux conservation theorem are derived in that section. The relation of the results to fluid mechanics is briefly discussed in section (VII), and section (VIII) summarizes the paper. The theory of general coordinates and Jacobians that is required to understand the paper is given in the appendix.

II) HAMILTON'S EQUATIONS

A field line, or an integral curve, $\mathbb{X}(u)$ of a field $\mathbb{B}$ is defined as a solution of the differential equation

$$d\mathbb{X}/du = \mathbb{B}(\mathbb{X}). \qquad (1)$$

The parameter u labels points along the field line; du is the differential distance along a field, dl, divided by the field strength B. If the field $\mathbb{B}$ is globally divergence free, then it can always be represented in the form[3]

$$\mathbb{B}(\mathbb{x}) = \nabla\psi \times \nabla\theta + \nabla\varphi \times \nabla\chi \qquad (2)$$

with $\psi(\mathbb{x})$, $\theta(\mathbb{x})$, $\varphi(\mathbb{x})$, and $\chi(\mathbb{x})$ well-behaved functions of position. (It is frequently useful to choose θ and φ as angles so that $\exp(i\theta)$ and $\exp(i\varphi)$ are well-behaved functions of position rather than θ and φ themselves. If θ and φ are angles, then $2\pi\psi$ is the toroidal flux and $2\pi\chi$ is the poloidal flux, Fig. (1).) The quantities ψ, θ, φ, and χ are the canonical variables, and the representation of equation (2) is the canonical representation of the divergence-free field $\mathbb{B}$. The canonical form for an arbitrary vector $\mathbb{A}(\mathbb{x})$, Equation (8), leads to the canonical representation for $\mathbb{B}$ by taking the curl of $\mathbb{A}$.

The simplest Hamiltonian structure for the field lines arises if, for one of the canonical variables, $\mathbb{B}\cdot\nabla\varphi$ is nonzero. When $\mathbb{B}\cdot\nabla\varphi=(\nabla\psi\times\nabla\theta)\cdot\nabla\varphi$ is nonzero, the functions $\psi(\mathbb{x})$, $\theta(\mathbb{x})$, and $\varphi(\mathbb{x})$ can be inverted to obtain $\mathbb{x}(\psi,\theta,\varphi)$, and $\chi(\mathbb{x})$ can be written as $\chi=\chi_H(\psi,\theta,\varphi)$. The field line equations then become

$$d\psi/d\varphi = -\,\partial\chi_H/\partial\theta \quad \text{and} \quad d\theta/d\varphi = \partial\chi_H/\partial\psi. \qquad (3)$$

Since well-behaved transformations, such as $\mathbb{x}(\psi,\theta,\varphi)$, do not alter topological information, all such information is contained in the Hamiltonian $\chi_H(\psi,\theta,\varphi)$.

A general, but more subtle, Hamiltonian structure exists when all four canonical variables are treated symmetrically.[5] The existence of only three spatial coordinates, but four canonical variables, implies the existence of a constraint equation among the canonical variables which will be written as $H(\psi,\theta,\varphi,\chi)=0$. The function H, which is the two-degree-of-freedom field-line Hamiltonian, has a large degree of arbitrariness, but it must satisfy two conditions. First, the Jacobian

$$J \equiv \partial(\mathbf{x},H)/\partial(\psi,\theta,\varphi,\chi) \tag{4}$$

must be nonsingular and well-behaved near the $H=0$ surface. Second, the functions $\psi(\mathbf{x},H)$, $\theta(\mathbf{x},H)$, $\varphi(\mathbf{x},H)$, and $\chi(\mathbf{x},H)$ must equal $\psi(\mathbf{x})$, $\theta(\mathbf{x})$, $\varphi(\mathbf{x})$, and $\chi(\mathbf{x})$ for $H=0$. Any function $H(\psi,\theta,\varphi,\chi)$ that satisfies these two conditions is the field line Hamiltonian.

To prove that $H(\psi,\theta,\varphi,\chi)$ is the field line Hamiltonian, consider

$$\frac{\partial H}{\partial \psi} = J\,\frac{\partial(H,\theta,\varphi,\chi)}{\partial(\mathbf{x},H)}$$

$$= -J\,\mathbf{B}\cdot\nabla\theta. \tag{5}$$

The constraint $H=0$ implies that dH/du is zero along the trajectory; so

$$d\theta(\mathbf{x},H)/du = (\nabla\theta)\cdot d\mathbf{x}/du$$

$$= -\frac{1}{J}\frac{\partial H}{\partial\psi}. \tag{6}$$

Define a parameter τ so that $du/d\tau = J(\psi,\theta,\varphi,\chi)$, then one obtains Hamilton's equations

$$\frac{d\psi}{d\tau} = \frac{\partial H}{\partial\theta} \qquad \frac{d\theta}{d\tau} = -\frac{\partial H}{\partial\psi}$$

$$\frac{d\varphi}{d\tau} = \frac{\partial H}{\partial\chi} \qquad \frac{d\chi}{d\tau} = -\frac{\partial H}{\partial\varphi}. \tag{7}$$

In practice one does not care whether points along the trajectory are labeled by τ rather than u, but if one did care, $du/d\tau = J$ could be integrated to obtain the trajectory as a function of u. Actually, one can always make the Jacobian unity in the $H=0$ surface by choosing a new Hamiltonian $H_n = H_0/J_0$ with H_0 and J_0 the old Hamiltonian and Jacobian.

The integration of Hamilton's equations gives ψ, θ, φ, and χ as functions of τ along a field line trajectory. The transformation equations $\mathbf{x}(\psi,\theta,\varphi,\chi)$ then give the trajectory $\mathbf{x}(\tau)$ in ordinary space. Since well-behaved transformation equations do not alter topological properties, all topological information on the trajectories is contained in the Hamiltonian $H(\psi,\theta,\varphi,\chi)$.

In summary, the general Hamiltonian structure involves two sets of four variables. One set is the three spatial coordinates $\mathbf{x}$ plus the Hamiltonian H; the other set is the four canonical variables ψ,θ,φ,χ. The field line trajectories lie in the three-dimensional surface defined by $H(\psi,\theta,\varphi,\chi)=0$ and are the solution of Hamilton's equations. The simple Hamiltonian structure, with Hamiltonian $\chi_H(\psi,\theta,\varphi)$, corresponds in the general theory to $\psi(\mathbf{x},H)=\psi(\mathbf{x})$, $\theta(\mathbf{x},H)=\theta(\mathbf{x})$, $\varphi(\mathbf{x},H)=\varphi(\mathbf{x})$, and $H=\chi-\chi_H(\psi,\theta,\varphi)$.

III) CANONICAL TRANSFORMATIONS

A divergence-free field is uniquely specified by giving the four canonical variables ψ, θ, φ, and χ as functions of the position $\mathbf{x}$. There is not, however, a unique set of four functions. The existence of many such sets, all of which represent the same field $\mathbf{B}(\mathbf{x})$, leads to the freedom of making canonical transformations. In the application of Hamiltonian mechanics to the study of divergence-free fields, it is advantageous to choose canonical variables which simplify the Hamiltonian. The theory of canonical transformation theory is, therefore, of central importance.

There are two peculiarities of canonical transformation theory for divergence-free fields. First, there are four canonical variables; so the natural phase space of canonical transformations is four dimensional even though the natural phase space of a divergence-free vector is ordinary three-dimensional space. Second, canonical transformations are more naturally studied using the vector potential $\mathbb{A}(\mathbb{x})$ rather than the divergence-free field $\mathbb{B}=\nabla\times\mathbb{A}$.

Poincaré proved that any globally divergence-free field has a well-behaved, single-valued vector potential; we will assume that $\mathbb{A}(\mathbb{x})$ has these properties. Any vector potential, like any other vector, can be written in the canonical form

$$\mathbb{A}(\mathbb{x}) = \psi\,\nabla\theta - \chi\,\nabla\varphi + \nabla G. \tag{8}$$

The curl of the canonical form gives the canonical representation of $\mathbb{B}$. The gauge function G is irrelevant to the representation of $\mathbb{B}$, but it does have a significant role in the theory of canonical transformations. In applications of the Hamiltonian formulation to particle drift orbits and to evolving fields, it is advantageous to choose the gauge of the vector potential to be the natural one for a given set of canonical coordinates, which means $G=0$.

The vector potential $\mathbb{A}(\mathbb{x})$ can be specified by giving the four canonical variables as functions of position $\mathbb{x}$ as well as the gauge function G as a function of ψ, θ, φ, and χ. The condition that two sets of canonical variables represent the same vector potential is

$$\nabla(G_2 - G_1) = \psi_1\nabla\theta_1 - \psi_2\nabla\theta_2 - \chi_1\nabla\varphi_1 + \chi_2\nabla\varphi_2. \tag{9}$$

This condition can be satisfied by defining a function $F\equiv G_2-G_1$ with the total derivative of F equal to

$$dF = \psi_1\, d\theta_1 - \psi_2\, d\theta_2 - \chi_1\, d\varphi_1 + \chi_2\, d\varphi_2. \tag{10}$$

This means that F is a function of θ_1, θ_2, φ_1, and φ_2 with $\psi_1=\partial F/\partial\theta_1$, $\psi_2=-\partial F/\partial\theta_2$, $\chi_1=-\partial F/\partial\varphi_1$, and χ_2-$\partial F/\partial\varphi_2$. These are the standard relations of canonical transformation theory with F a generating function. To interpret canonical transformations, let $\psi_1(\mathbb{x})$, $\theta_1(\mathbb{x})$, $\varphi_1(\mathbb{x})$, $\chi_1(\mathbb{x})$, and $G_1(\psi_1,\theta_1,\varphi_1,\chi_1)$ be a set a functions that specify a vector potential $\mathbb{A}(\mathbb{x})$. The same vector potential is specified by $\psi_2(\mathbb{x})$, $\theta_2(\mathbb{x})$, $\varphi_2(\mathbb{x})$, $\chi_2(\mathbb{x})$, and $G_2(\psi_2,\theta_2,\varphi_2,\chi_2)$ provided the functions $\psi_2(\psi_1,\theta_1,\varphi_1,\chi_1)$, $\theta_2(\psi_1,\theta_1,\varphi_1,\chi_1)$, $\varphi_2(\psi_1,\theta_1,\varphi_1,\chi_1)$, $\chi_2(\psi_1,\theta_1,\varphi_1,\chi_1)$, and $G_2(\psi_2,\theta_2,\varphi_2,\chi_2)$ satisfy the constraints of a canonical transformation.

The generating function F not only assures that the two sets of canonical variables represent the same vector potential, but also that the Hamiltonians are equal, $H_1(\psi_1,\theta_1,\varphi_1,\chi_1)=H_2(\psi_2,\theta_2,\varphi_2,\chi_2)$. The proof follows from considering each of ψ_2, θ_2, φ_2, and χ_2 as a function of $\psi_1,\theta_1,\varphi_1,\chi_1$. The Hamiltonian H_1 gives the τ derivatives of ψ_1, θ_1, φ_1, and χ_1 and, by use of the chain rule, the τ derivatives of ψ_2, θ_2, φ_2, and χ_2. One then finds that the correct τ derivatives of ψ_2, θ_2, φ_2, and χ_2 are obtained if, and only if, H_2 is chosen to be equal to H_1 within an additive constant. This additive constant must be zero due to the $H=0$ constraint. One can also easily prove that the Jacobian of the canonical variables $\psi_1,\theta_1,\varphi_1,\chi_1$ with respect to the canonical variables $\psi_2,\theta_2,\varphi_2,\chi_2$ is unity. Both the unit Jacobian and the equality of the Hamiltonian are standard results of canonical transformation theory.

The independent variables of the canonical transformation equations can be changed by a Legendre transformation. For example, if $S\equiv F+\theta_2\psi_2-\varphi_2\chi_2$, then the total derivative of S is

$$dS = \psi_1\, d\theta_1 + \theta_2\, d\psi_2 - \chi_1\, d\varphi_1 - \varphi_2\, d\chi_2. \tag{11}$$

The generating function $S(\psi_2,\theta_1,\varphi_1,\chi_2)$ is particularly important since it defines the identity transformation simply, $S=\theta_1\psi_2-\varphi_1\chi_2$.

The most important canonical transformations in the Hamiltonian theory of divergence-free fields are the infinitesimal canonical transformations. Suppose the canonical variables and the gauge function depend on a parameter λ, for example $\psi(\mathbb{x},\lambda)$. Two sets of canonical variables can be defined by giving two values, λ_1 and λ_2, of the parameter, so $\psi_1=\psi(\mathbf{x},\lambda_1)$, etc. A canonical transformation between the two sets is defined by

$$S = \theta_1\psi_2 - \varphi_1\chi_2 + (\lambda_2-\lambda_1)\, s(\psi_2,\theta_1,\varphi_1,\chi_2,\lambda_1), \tag{12}$$

which is the identity transformation for $\lambda_1=\lambda_2$. The equation for the total derivative of S implies $\psi_1=\partial S/\partial\theta_1$, which implies that

$$\psi(\mathbf{x},\lambda_1) = \psi(\mathbf{x},\lambda_2) + (\lambda_2-\lambda_1)\, \partial s/\partial\theta_1. \tag{13}$$

If the difference between λ_1 and λ_2 is allowed to approach zero, the equations of infinitesimal canonical transformations are obtained:

$$\begin{aligned} \partial\psi(\mathbf{x},\lambda)/\partial\lambda &= -\,\partial s/\partial\theta \qquad & \partial\theta(\mathbf{x},\lambda)/\partial\lambda &= +\,\partial s/\partial\psi \\ \partial\varphi(\mathbf{x},\lambda)/\partial\lambda &= -\,\partial s/\partial\chi \qquad & \partial\chi(\mathbf{x},\lambda)/\partial\lambda &= -\,\partial s/\partial\varphi \end{aligned} \tag{14}$$

$$\partial G(\mathbf{x},\lambda)/\partial\lambda = -\,\psi\,(\partial\theta/\partial\lambda) + \chi\,(\partial\varphi/\partial\lambda) + s(\psi,\theta,\varphi,\chi,\lambda) \tag{15}$$

$$\partial H(\psi,\theta,\varphi,\chi,\lambda)/\partial\lambda = [H,s] \tag{16}$$

$$[H,s] \equiv \frac{\partial H}{\partial\psi}\frac{\partial s}{\partial\theta} - \frac{\partial H}{\partial\theta}\frac{\partial s}{\partial\psi} + \frac{\partial H}{\partial\varphi}\frac{\partial s}{\partial\chi} - \frac{\partial H}{\partial\chi}\frac{\partial s}{\partial\varphi}. \tag{17}$$

The object [H,s] is the Poisson bracket of H and s. The generating function for infinitesimal canonical transformations $s(\psi,\theta,\varphi,\chi,\lambda)$, like any other generating function, can be chosen freely.

IV) FIELDS WITH PARAMETRIC DEPENDENCE

In this section we will study the Hamiltonian properties of divergence-free fields that depend on an arbitrary parameter λ. The equations that will be derived can be applied to the evolution of fields with λ interpreted as time (Sec. VI) or to canonical perturbation theory with λ the perturbation parameter (Sec. V).

The vector potential of a divergence-free field which depends on a parameter λ is $\mathbb{A}(\mathbb{x},\lambda)$. The derivative of $\mathbb{A}$ with respect to λ is

$$\partial\mathbb{A}/\partial\lambda = -(\partial\theta/\partial\lambda)\,\nabla\psi + (\partial\psi/\partial\lambda)\,\nabla\theta - (\partial\chi/\partial\lambda)\,\nabla\varphi + (\partial\varphi/\partial\lambda)\,\nabla\chi + \nabla s. \tag{18}$$

The function $s(\mathbb{x},\lambda)$ is defined as

$$s \equiv \partial G/\partial\lambda + \psi\,(\partial\theta/\partial\lambda) - \chi\,(\partial\varphi/\partial\lambda) \tag{19}$$

with G and the canonical variables interpreted as functions of $\mathbb{x}$ and λ. The function s, taken as a function of $\psi,\theta,\varphi,\chi,\lambda$, satisfies Eq. (15) for the generating function of infinitesimal canonical transformations. The function $s(\psi,\theta,\varphi,\chi,\lambda)$ represents the freedom in the choice of the canonical variables as λ evolves for a given $\mathbb{A}(\mathbb{x},\lambda)$. If the vector potential is independent of λ, this is precisely the freedom of infinitesimal canonical transformations.

The canonical variables ψ,θ,φ,χ and s can be considered to depend on both $\mathbb{x}$, the spatial position, and H, the Hamiltonian, but equation (18) for $\partial\mathbb{A}/\partial\lambda$ places conditions only on the three spatial derivatives. The natural additional condition, which is needed to constrain the H derivatives, is

$$-\frac{\partial\theta}{\partial\lambda}\frac{\partial\psi}{\partial H} + \frac{\partial\psi}{\partial\lambda}\frac{\partial\theta}{\partial H} - \frac{\partial\chi}{\partial\lambda}\frac{\partial\varphi}{\partial H} + \frac{\partial\varphi}{\partial\lambda}\frac{\partial\chi}{\partial H} + \frac{\partial s}{\partial H} = 0. \tag{20}$$

In the simplest Hamiltonian structure, the spatial position has the form $\mathbb{x}(\psi,\theta,\varphi,\lambda)$ and $H=\chi-\chi_H(\psi,\theta,\varphi,\lambda)$; so the H derivatives of ψ, θ, and φ at fixed position $\mathbb{x}$ are zero. The constraint on the H derivatives then becomes

$$\partial\varphi/\partial\lambda = -\,\partial s(\psi,\theta,\varphi,\chi,\lambda)/\partial\chi. \tag{21}$$

Functions that depend on H have physical relevance only in an infinitesimal neighborhood of the H=0 plane. Therefore, without loss of generality, the function s can be assumed to have the form

$$s(\psi,\theta,\varphi,\chi,\lambda) = s_0(\psi,\theta,\varphi,\lambda) + (\chi-\chi_H)\, s_1(\psi,\theta,\varphi,\lambda); \tag{22}$$

the quantities s_0 and s_1 can also be considered functions of just $\mathbb{X}$ and λ. In the physical, or H=0, plane, equations (18) and (21) become:

$$\partial\mathbb{A}/\partial\lambda = -(\partial\theta/\partial\lambda)\,\nabla\psi + (\partial\psi/\partial\lambda)\,\nabla\theta - (\partial\chi/\partial\lambda)\,\nabla\varphi + \nabla s_0; \tag{23}$$

$$\partial\varphi/\partial\lambda = -\, s_1. \tag{24}$$

The orthogonality relations of general coordinates (See the Appendix) imply that

$$\partial\psi/\partial\lambda = -\,\partial s_0/\partial\theta + (\partial\mathbb{A}/\partial\lambda)\cdot(\partial\mathbb{X}/\partial\theta) \tag{25}$$

$$\partial\theta/\partial\lambda = \partial s_0/\partial\psi - (\partial\mathbb{A}/\partial\lambda)\cdot(\partial\mathbb{X}/\partial\psi) \tag{26}$$

$$\partial\chi/\partial\lambda = \partial s_0/\partial\varphi - (\partial\mathbb{A}/\partial\lambda)\cdot(\partial\mathbb{X}/\partial\varphi). \tag{27}$$

If $\partial\mathbb{A}/\partial\lambda=0$, equations (24) to (27) are the equations of infinitesimal canonical transformations. As λ varies, the canonical variables change due to the effect of $\partial\mathbb{A}/\partial\lambda$ and due to the canonical transformations produced by $s_0(\psi,\theta,\varphi,\lambda)$.

The general equations for the evolution of the canonical variables follows in a similar manner if one uses the othogonality relations for four vectors (See the Appendix). The answer is

$$\partial\psi/\partial\lambda = -\,\partial s/\partial\theta + (\partial\mathbb{A}/\partial\lambda)\cdot(\partial\mathbb{X}/\partial\theta) \tag{28}$$

$$\partial\theta/\partial\lambda = \partial s/\partial\psi - (\partial\mathbb{A}/\partial\lambda)\cdot(\partial\mathbb{X}/\partial\psi) \tag{29}$$

$$\partial\varphi/\partial\lambda = -\,\partial s/\partial\chi - (\partial\mathbf{A}/\partial\lambda)\cdot(\partial\mathbf{x}/\partial\chi) \tag{30}$$

$$\partial\chi/\partial\lambda = \partial s/\partial\varphi - (\partial\mathbf{A}/\partial\lambda)\cdot(\partial\mathbf{x}/\partial\varphi). \tag{31}$$

The equations that we have just derived determine the evolution of the canonical variables with respect to λ at fixed position $\mathbf{x}$ and H. It is, in general, more useful to obtain the evolution of the position and the Hamiltonian with respect to λ for fixed values of the canonical variables, that is, the functions $\mathbf{x}(\psi,\theta,\varphi,\chi,\lambda)$ and $H(\psi,\theta,\varphi,\chi,\lambda)$. The triviality $(\partial\mathbf{x}/\partial\lambda)_{x,H}=0$ implies

$$\begin{aligned}\partial\mathbf{x}(\psi,\theta,\varphi,\chi,\lambda)/\partial\lambda = &-(\partial\psi/\partial\lambda)(\partial\mathbf{x}/\partial\psi) - (\partial\theta/\partial\lambda)(\partial\mathbf{x}/\partial\theta)\\ &-(\partial\varphi/\partial\lambda)(\partial\mathbf{x}/\partial\varphi) - (\partial\chi/\partial\lambda)(\partial\mathbf{x}/\partial\chi).\end{aligned} \tag{32}$$

If the expressions for $\partial\psi/\partial t$, $\cdots$, $\partial\chi/\partial t$ are substituted from equations (28) to (31), the right-hand side of the $\partial\mathbf{x}/\partial\lambda$ equation is a known function of $\psi,\theta,\varphi,\chi,\lambda$ provided $\mathbf{A}(\mathbf{x},\lambda)$, $s(\psi,\theta,\varphi,\chi,\lambda)$, and $\mathbf{x}(\psi,\theta,\varphi,\chi,\lambda)$ are given. The generating function $s(\psi,\theta,\varphi,\chi,\lambda)$ is arbitrary; it can be chosen in any convenient way. Given initial transformation equations $\mathbf{x}(\psi,\theta,\varphi,\chi,\lambda=0)$, the $\partial\mathbf{x}/\partial\lambda$ equation can be integrated to obtain the transformation equations at any other value of λ, $\mathbf{x}(\psi,\theta,\varphi,\chi,\lambda)$.

The evolution equation for the Hamiltonian $H(\psi,\theta,\varphi,\chi,\lambda)$ is slightly more difficult to obtain in the most useful form. A somewhat messy identity is proven in the Appendix

$$(\partial\mathbf{x}/\partial\psi)\times\mathbf{B} = -\,[\,\nabla\theta + (\partial H/\partial\psi)\,\boldsymbol{\beta}], \tag{33}$$

$$\boldsymbol{\beta} \equiv (\partial\theta/\partial H)\nabla\psi - (\partial\psi/\partial H)\nabla\theta + (\partial\chi/\partial H)\nabla\varphi - (\partial\varphi/\partial H)\nabla\chi. \tag{34}$$

The important feature of $\boldsymbol{\beta}$ is that

$$\boldsymbol{\beta}\cdot\mathbf{B}=1/J. \tag{35}$$

The triviality $(\partial H/\partial\lambda)_{x,H}=0$ implies, using the chain rule, that

$$\frac{\partial H}{\partial\lambda} = -\frac{\partial H}{\partial\psi}\frac{\partial\psi}{\partial\lambda} - \frac{\partial H}{\partial\theta}\frac{\partial\theta}{\partial\lambda} - \frac{\partial H}{\partial\varphi}\frac{\partial\varphi}{\partial\lambda} - \frac{\partial H}{\partial\chi}\frac{\partial\chi}{\partial\lambda}. \quad (36)$$

Equation (32) for $\partial\mathbb{x}/\partial t$ can then be used to show that

$$\partial\mathbb{A}/\partial\lambda = (\partial H/\partial\lambda)\,\mathbb{B} + (\partial\mathbb{x}/\partial\lambda) \times \mathbb{B} + \nabla s. \quad (37)$$

In Section (VI) this equation for $\partial\mathbb{A}/\partial\lambda$ will be used to study the time development of the magnetic field and the vorticity. The component of equation (37) parallel to $\mathbb{B}$ gives the desired equation for $\partial H/\partial\lambda$,

$$\begin{aligned} \partial H/\partial\lambda &= [(\mathbb{B}\cdot\partial\mathbb{A}/\partial\lambda) - (\mathbb{B}\cdot\nabla s)]\, J \\ &= (\mathbb{B}\cdot\partial\mathbb{A}/\partial\lambda)\, J + [H,s]. \end{aligned} \quad (38)$$

The last line is derived using equation (5) for the relation between the derivatives of H and $\mathbb{B}\cdot\nabla$ operating on the canonical variables and equation (17) for the definition of the Poisson bracket.

V) HAMILTONIAN PERTURBATION THEORY

The equations for fields with parametric dependence define a canonical perturbation theory. Conceptually, a field with a vector potential $\mathbb{A}_1(\mathbb{x})$ is evolved into the field with a vector potential $\mathbb{A}_2(\mathbb{x})$ by letting $\mathbb{A}(\mathbb{x},\lambda)=(1-\lambda)\mathbb{A}_1+\lambda\mathbb{A}_2$. The transformation equations and Hamiltonian of a field determined by $\mathbb{A}_1(\mathbb{x})$ can be evolved into the transformation equations and Hamiltonian of any other field $\mathbb{A}_2(\mathbb{x})$ by integrating the evolution equations (32) and (38) through the range $0\leq\lambda\leq1$. This technique can be very powerful if clever choices are made for $\mathbb{A}_1$ and $\mathbb{A}_2$.

The most important type of perturbation analysis is the simplification of the Hamiltonian by canonical transformations. During the explanation of this procedure, we will assume $\mathbb{B}\cdot\nabla\varphi$ is nonzero and use the Hamiltonian structure $\mathbb{X}(\psi,\theta,\varphi,\lambda)$ and $\chi_H(\psi,\theta,\varphi,\lambda)$. The objective of the procedure is to transform canonically the Hamiltonian into a form that is as close as possible to the action-angle, or magnetic coordinate, form. Suppose the Hamiltonian can be transformed into the action-angle, or magnetic coordinate, form, $\chi_H(\psi)$. The field line trajectories are then trivial to integrate: $\psi=\psi_0$ and $\theta=\theta_0+\iota\varphi$ with ψ_0 and θ_0 the values of ψ and θ at $\varphi=0$ and $\iota\equiv d\chi_H(\psi)/d\psi$. If the field line trajectories remain in a bounded region of space, the existence of a Hamiltonian $\chi_H(\psi)$ implies the field lines wind about a torus in nested surfaces. The coordinates θ and φ are angles; so the determination[10] of a set of Fourier coefficients $\mathbb{X}_{nm}(\psi)$, with

$$\mathbb{X}(\psi,\theta,\varphi) = \Sigma\ \mathbb{X}_{nm} \exp[i(n\varphi-m\theta)], \tag{39}$$

and the rotational transform, or the rotation number, $\iota(\psi)$ effective $\jmath$ give all the field line trajectories in a volume of space.

If magnetic coordinate, or action-angle, variables exist, then the use of these canonical variables allows the solution for the trajectories to be written down throughout a volume of space. Unfortunately, magnetic coordinates do not generally exist except in configurations of high symmetry. A magnetic field that confines an isotropic plasma to a bounded region of space must be closely approximated by a field that does have magnetic coordinates. That is, the Hamiltonian for such a field can always be transformed into the form

$$\chi_H(\psi,\theta,\varphi) = \overline{\chi}(\psi) + \hat{\chi}(\psi,\theta,\varphi) \tag{40}$$

with $\hat{\chi} << \overline{\chi}$. The canonical coordinates that place the Hamiltonian in this form are called near-magnetic coordinates. The property of $\hat{\chi}$ that prevents its removal by a canonical transformation is resonance with the Hamiltonian $\overline{\chi}$. This means that if $\hat{\chi}$ is written as a Fourier series

$$\hat{\chi} = \Sigma\, \chi_{nm}(\psi) \exp[i(n\varphi - m\theta)], \tag{41}$$

then the resonant Fourier terms satisfy $m\iota(\psi) = n$ for some value of ψ with $\iota \equiv d\overline{\chi}/d\psi$. Nonresonant Fourier terms can be removed by a canonical transformation so that $\hat{\chi}$ should contain only resonant Fourier terms. The resonant Fourier terms represent the magnetic islands and stochastic regions and provide a relatively simple description of the topological features of the field lines.

Perturbation analysis takes a Hamiltonian $\chi_H(\psi,\theta,\varphi)$ and transformation equations $\mathbb{X}(\psi,\theta,\varphi)$ and finds canonical coordinates that are in near-magnetic coordinate form. That is, perturbation analysis finds a Hamiltonian of the form of equation (40), with $\hat{\chi}$ having only resonant Fourier terms, and the associated transformation equations. This concept of perturbation analysis is more general than that usually given. The traditional view of canonical perturbation theory is that a given Hamiltonian $\overline{\chi}(\psi)$ is perturbed by a small, additive term $\chi_p(\psi,\theta,\varphi)$. The perturbation χ_p is generally assumed nonresonant so that the perturbed Hamiltonian can be written in action-angle, or magnetic coordinate, form by a canonical transformation. Since χ_p is small, this need only be done to some low order in $\chi_p/\overline{\chi}$. The new Hamiltonian and the expression for the old canonical variables in terms of the new are the goals of the analysis.

To develop the perturbation analysis method, equations are required for the λ derivatives of the Hamiltonian and the transformation equations. Equation (38) can be written as

$$\partial\chi_H(\psi,\theta,\varphi,\lambda)/\partial\lambda = \partial s_0/\partial\varphi + [\chi_H, s_0] - (\mathbb{B}\cdot\partial\mathbb{A}/\partial\lambda)/\mathbb{B}\cdot\nabla\varphi. \tag{42}$$

Since the canonical phase space has only two coordinates ψ,θ the Poisson bracket is

$$[\chi_H, s_0] = (\partial\chi_H/\partial\psi)(\partial s_0/\partial\theta) - (\partial\chi_H/\partial\theta)(\partial s_0/\partial\psi). \tag{43}$$

We will choose to hold φ fixed so that $s_1=0$ and $s=s_0$. That is, the subscript can be dropped from the generating function s. The equation for the advancement of the transformation equations follows from equations (32) and (25) to (27).

$$\partial\mathbb{X}/\partial\lambda = [\mathbb{X}, s]. \tag{44}$$

Suppose that a set of canonical variables are determined for a given field $\mathbb{B}(\mathbb{X})$. These original canonical variables are denoted by $\psi_0(\mathbb{X})$, $\theta_0(\mathbb{X})$, $\varphi(\mathbb{X})$, and $\chi_*(\psi_0,\theta_0,\varphi)$. The original Hamiltonian can be written as

$$\chi_* = \overline{\chi_0}(\psi_0) + \hat{\chi}_0(\psi_0,\theta_0,\varphi) + \tilde{\chi}_0(\psi_0,\theta_0,\varphi) \tag{45}$$

with $\overline{\chi_0}$ the n=0, m=0 Fourier term, $\hat{\chi}_0$ the sum of the resonant Fourier terms, and $\tilde{\chi}_0$ the sum of the remaining Fourier terms. The λ dependent vector potential is chosen as

$$\mathbb{A}(\mathbb{X},\lambda) = \psi_0\,\nabla\theta_0 - \chi_0\,\nabla\varphi; \tag{46}$$

$$\chi_0 \equiv \overline{\chi_0}(\psi_0) + \hat{\chi}(\psi_0,\theta_0,\varphi,\lambda) + \lambda\,\tilde{\chi}(\psi_0,\theta_0,\varphi,\lambda) \tag{47}$$

with $\chi_*=\chi_0(\psi_0,\theta_0,\varphi,\lambda=1)$ the original Hamiltonian. The term $\hat{\chi}$ can be chosen as an arbitrary function of λ. We choose $\hat{\chi}$ so that it equals $\hat{\chi}_0$ at $\lambda=0$ and $\partial(\hat{\chi}+\tilde{\chi})/\partial\lambda=0$. This choice implies that $\tilde{\chi}$ equals $\tilde{\chi}_0$ at $\lambda=0$. The remaining freedom in the choice of $\hat{\chi}$ and $\tilde{\chi}$ as functions of λ can be represented by the arbitrary function

$$\sigma \equiv \partial\hat{\chi}(\psi_0,\theta_0,\varphi,\lambda)/\partial\lambda. \tag{48}$$

Let ψ and θ be functions of ψ_0, θ_0, φ, and λ such that ψ and θ equal ψ_0 and θ_0 at $\lambda=0$. At λ equal to zero, the Hamiltonian $\chi_H=\chi_0$ has the desired form $\chi_H=\overline{\chi}_0+\hat{\chi}_0$. We will try to choose s so that χ_H has the desired form

$$\chi_H = \overline{\chi}(\psi,\lambda) + \hat{\chi} \tag{49}$$

for all λ. Equation (42) implies

$$\partial\chi_H(\psi,\theta,\varphi,\lambda)/\partial\lambda = \partial s/\partial\varphi + [\chi_H,s] + \sigma + \tilde{\chi} - \lambda\sigma \tag{50}$$

One can easily show that

$$\partial\hat{\chi}(\psi,\theta,\varphi)/\partial\lambda = [\hat{\chi},s] + \sigma \tag{51}$$

$$\partial\tilde{\chi}(\psi,\theta,\varphi)/\partial\lambda = [\tilde{\chi},s] - \sigma, \tag{52}$$

Equations (49) and (50) imply with

$$\partial\overline{\chi}(\psi,\lambda)/\partial\lambda = \partial s/\partial\varphi + \iota\partial s/\partial\theta + \tilde{\chi} - \lambda\sigma \tag{53}$$

with $\iota\equiv\partial\overline{\chi}/\partial\psi$. The equations (51) to (53) form a complete set of equations for $\overline{\chi}$, s, σ, $\hat{\chi}$, and $\tilde{\chi}$ as functions of λ. Equation (44) can be integrated to obtain $\mathbb{X}$ as a function of λ. At $\lambda=1$ one obtains the full Hamiltonian and transformation equations in the desired near-magnetic-coordinate form.

Although the perturbation equations are straightforward to integrate, it is instructive to consider the first λ step. In order to satisfy equation (53) at $\lambda=0$, one must choose the initial value of s, s_0, so that

$$\partial s_0/\partial\varphi + \iota\partial s_0/\partial\theta + \tilde{\chi}_0 = 0. \tag{54}$$

This can always be done since $\tilde{\chi}_0$ contains no resonant Fourier terms. Indeed the initial Fourier series is $s_{nm}=i\tilde{\chi}_{nm}/(n-\iota m)$. The nonresonant Fourier terms in σ are chosen so the $\hat{\chi}$ has only resonant terms for all λ,

$$[\hat{\chi},s]_n + \sigma_n = 0 \qquad (55)$$

with the subscript implying nonresonant. The resonant Fourier terms in σ are chosen so the $\tilde{\chi}-\lambda\sigma$ has no resonant terms for any λ. Using the subscript r to mean resonant, equation (52) implies that

$$2\sigma_r = [\tilde{\chi}_0, s_0] \qquad (56)$$

for $\lambda=0$. Equations (51) and (52) can now be integrated to obtain $\tilde{\chi}$ and $\hat{\chi}$ after the first step in λ since $\overline{\chi}_0$, $\hat{\chi}_0$, and $\tilde{\chi}_0$ are the parts of the given initial Hamiltonian. Equation (53) implies that $\partial\overline{\chi}/\partial\lambda$ is equal to the n=0, m=0 Fourier component of $\tilde{\chi}-\lambda\sigma$; so $\overline{\chi}$ can also be integrated forward.

The standard canonical perturbation analysis is obtained from equations (51) to (53) by setting $\hat{\chi}=0$ and $\sigma=0$ and then solving the equations using a power series in λ. The Lie methods are also, in general, based on $\hat{\chi}$ and σ being zero but solve the differential equations (52) and (53) by more subtle series methods.[11] The perturbation method given here is amenable to solution to arbitrary accuracy using numerical methods. The perturbation need not be small.

VI) TIME-DEPENDENT FIELDS

The λ evolution equations give equations for the time development of fields if λ is interpreted as time t. In this section the language of magnetic fields will be used, but in the next section we will find that a similar analysis holds for the vorticity. Faraday's law says that the electric field satisfies

$$\mathbb{E} = -\partial\mathbb{A}/\partial t - \nabla\Phi_0 \qquad (57)$$

with Φ_0 the electric potential, which is a well-behaved, single-valued function of position if $\mathbb{A}$ is. Equation (37) implies that

$$\mathbb{E} + (\partial \mathbb{x}/\partial t)\times\mathbb{B} = -(\partial H/\partial t)\,\mathbb{B} - \nabla\Phi; \tag{58}$$

$$\Phi \equiv \Phi_0 + s. \tag{59}$$

The potential Φ can be viewed as the electric potential in the frame of the canonical coordinates, and $\mathbb{E} + (\partial \mathbb{x}/\partial t)\times\mathbb{B}$ can be viewed as the electric field in that frame. The velocity through space of a canonical coordinate point is $\partial \mathbb{x}/\partial t$. The component of equation (58) parallel to $\mathbb{B}$ gives

$$\partial H/\partial t = -\mathbb{B}\cdot(\mathbb{E} + \nabla\Phi)J. \tag{60}$$

This equation implies that s can be chosen so that $\partial H/\partial t=0$ if a function Φ exists such that $\mathbb{B}\cdot(\mathbb{E}+\nabla\Phi)=0$. This condition is equivalent to $\oint\mathbb{E}\cdot d\mathbb{l}=0$ around each closed magnetic field line. If the Hamiltonian is time independent, then the topology of the field lines does not change.

The generalized Ohm's law of plasma physics is

$$\mathbb{E} + \mathbb{v}\times\mathbb{B} = \eta\,\mathbb{j} \tag{61}$$

with $\mathbb{v}$ the velocity of the plasma. In an ideal plasma the resistivity η is zero; so $\mathbb{E}\cdot\mathbb{B}=0$. The choice $\Phi=0$ then makes $\partial H/\partial t=0$; so the topology of the field lines does not change in an ideal plasma. The conservation of H by an ideal plasma is the flux conservation law. The choice $\Phi=0$ in an ideal plasma also implies $\partial \mathbb{x}/\partial t=\mathbb{v}$ using equations (58) and (61) with $\partial H/\partial t=0$. This is Kelvin's theorem, which says the canonical coordinates can be viewed as being tied to an ideal plasma. As is usual, these equations are easier to interpret in the Hamiltonian structure $\chi_H(\psi,\theta,\varphi,t)$ and $\mathbb{x}(\psi,\theta,\varphi,t)$. In an ideal plasma the toroidal flux $2\pi\psi$ at a given plasma element can be viewed as fixed as the plasma moves (Kelvin's theorem) and the poloidal flux $2\pi\chi_H$ has a fixed relation to the toroidal flux, $\chi_H(\psi,\theta,\varphi)$ (the flux conservation theorem).

VII) RELATION TO FLUID MECHANICS

There are two divergence-free fields that are of importance to fluid mechanics: the flow velocity of an incompressible fluid and the vorticity. Both fields have an associated Hamiltonian structure. The Hamiltonian structure of the vorticity field is fundamentally identical to that of the magnetic field, even the time evolution equations for the two fields are identical. The Hamiltonian structure of the flow velocity is simple only in the case of a time-independent, divergence-free flow.

The vorticity $\mathbb{\omega}=\nabla\times\mathbb{v}$ obeys the same time-dependent equations as the magnetic field. The Navier-Stokes equation,

$$\partial\mathbb{v}/\partial t + \mathbb{v}\cdot\nabla\mathbb{v} = - \nabla w - \nu\nabla\times(\nabla\times\mathbb{v}) \qquad (62)$$

with w the enthalpy and ν the viscosity, is equivalent to the Ohm's law for the magnetic field. To show this let

$$\mathbb{E}_f \equiv - \partial\mathbb{v}/\partial t - \nabla(v^2/2 + w), \qquad (63)$$

which is the equivalent of Faraday's law. The use of a standard vector identity implies that the Navier-Stokes equation can be written as

$$\mathbb{E}_f + \mathbb{v} \times \mathbb{B} = \nu \nabla\times\mathbb{\omega}, \qquad (64)$$

which is of the same form as Ohm's law, Eq. (61), supplemented by Ampere's law, $\nabla\times\mathbb{B}=\mu_0\mathbb{J}$.

There is a poloidal, and a toroidal, magnetic flux and similarly there is a poloidal, and a toroidal, flux of vorticity. The traditional name for the toroidal flux of vorticity is the circulation, but one should really speak of the toroidal, and the poloidal, circulation. In fluid mechanics it has not been customary to consider vortex lines that circle back into the same region of space. Vortex lines that never circle back on themselves can be described by a single circulation or equivalently by globally well-behaved Clebsch coordinates

$$\boldsymbol{\omega} = \nabla\psi \times \nabla\theta. \tag{65}$$

The vorticity field lines in Clebsch coordinates are given by $\psi=\psi_0$ and $\theta=\theta_0$ with ψ_0 and θ_0 constants, the initial values of ψ and θ. Assuming the existence of a coordinate such that $\boldsymbol{\omega}\cdot\nabla\varphi$ is nonzero, the existence of Clebsch coordinates implies the existence of canonical coordinates in which the Hamiltonian χ_H is zero. The Hamilton-Jacobi equation is the partial differential equation for the generating function for the transformation from arbitrary canonical coordinates to canonical coordinates with zero Hamiltonian. This transformation is always locally well-behaved, but it is globally well-behaved only under restrictive conditions such as the trajectories never circling back.

The trajectories of fluid elements in a steady incompressible flow are also mathematically equivalent to the trajectories of a time-independent Hamiltonian of two degrees of freedom. The equation of motion for the fluid elements is

$$d\mathbf{x}/dt = \mathbf{v}(\mathbf{x}). \tag{66}$$

There is a special case in which one can find a Hamiltonian structure with the velocity time dependent. In this case, well-behaved, time-independent transformation equations $\mathbf{x}(\psi,\theta,\varphi)$ exist, but $\chi_H(\psi,\theta,\varphi,t)$ is time dependent. The special Hamiltonian is

$$H_s(\psi,\theta,\varphi,p_\varphi,t) = (p_\varphi + \chi_H)^2/2 \tag{67}$$

with $p_\varphi=\mathbf{v}\cdot\nabla\varphi-\chi_H$. Hamilton's equations predict nontrivial behavior when the trajectories circle back on themselves. The effect of time dependence on the Hamiltonian properties of the flow velocity is fundamentally different from the effect of time dependence on the Hamiltonian properties of the vorticity or the magnetic field.

VIII) DISCUSSION

The application of Hamiltonian mechanics to the study of continuum fields, such as the vorticity and the magnetic field, is not only aesthetically pleasing but also useful. In this paper, the formal structure has been emphasized, but a number of applications to plasma problems have been published. These include the determination of the structure of a given magnetic field,[6] the drift orbits of charged particles in magnetic and electric fields[7] including the application to the "fishbone" instability,[12] and the conditions for the opening magnetic islands in toroidal equilibria,[5] the so-called tearing modes.

The canonical description of field line trajectories consists of two parts: an autonomous Hamiltonian of two degrees of freedom and the transformation equations from the canonical variables of the Hamiltonian to the ordinary spatial variables. The canonical variables for a magnetic field can be chosen so that the toroidal flux and the poloidal angle are one pair of canonically conjugate coordinates, and the toroidal angle and the poloidal flux are the other pair. All topological information on the field line trajectories is contained in the Hamiltonian. Frequently, the two degree of freedom, autonomous, Hamiltonian can be reduced to a one degree of freedom, time-dependent, Hamiltonian with the toroidal angle the timelike variable. This reduction leads to a considerable simplification of the theory.

Canonical transformations are an important part of the Hamiltonian theory. The most important canonical coordinates are those close to the action-angle coordinates, which are known as magnetic coordinates to plasma theorists. The most important canonical transformations are the infinitesimal canonical transformations. If the transformation equations for two sets of canonical coordinates can be smoothly distorted into each other, the equations for infinitesimal canonical transformations can be integrated to produce the transformation between the two sets of canonical variables.

If a divergence-free field depends on a parameter λ, then equations can be derived for variation in the Hamiltonian and the transformation equations between infinitesimally separated values of λ. These equations reduce to the equations for infinitesimal canonical transformations in the trivial case in which the field does not change as λ changes. One can use these equations to specify a canonical perturbation theory that is valid even if the perturbation is large.

The time dependence of the vorticity or the magnetic field appears in the Hamiltonian formulation as a parametric dependence of the Hamiltonian and the transformation equations. The evolution equations for the vorticity and the magnetic field are mathematically identical.

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APPENDIX: GENERAL COORDINATES

In this appendix we will give the basic properties of coordinate transformations. Although the results given in this appendix are taught in courses on differential forms, the simple properties that are required to understand this paper can be learned quickly and are not well-known to many plasma, or fluid dynamics, theorists.

First, consider three-dimensional space. A position $\mathbf{x}$ in the space can be specified by three quantities or coordinates ξ^1, ξ^2, ξ^3 which can be written as $\mathbf{x}(\xi^1,\xi^2,\xi^3)$. The fundamental relation in the theory of general coordinates is the orthogonality relation. To prove this relation, consider Cartesian coordinates

$$\mathbf{x} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}. \qquad \text{(A1)}$$

The position associated with each value of ξ^1,ξ^2,ξ^3 can be specified by giving x, y, and z as functions of ξ^1, ξ^2, and ξ^3. One can also think of ξ^1,ξ^2,ξ^3 as functions of the position $\mathbf{x}$ or, equivalently, as functions of x, y, and z. The orthogonality relation relates the derivative of one of the coordinates with respect to position, $\nabla\xi^i$, and the derivative of position with respect to one of the coordinates, $\partial\mathbf{x}/\partial\xi^j$. The triviality that $\partial\xi^i/\partial\xi^j$ is zero unless i=j and the chain rule imply the orthogonality relation

$$\begin{aligned}\delta^i{}_j &= \partial\xi^i/\partial\xi^j \\ &= (\partial\xi^i/\partial x)(\partial x/\partial\xi^j) + (\partial\xi^i/\partial y)(\partial y/\partial\xi^j) + (\partial\xi^i/\partial z)(\partial z/\partial\xi^j) \\ &= \nabla\xi^i\cdot(\partial\mathbf{x}/\partial\xi^j).\end{aligned} \tag{A2}$$

The orthogonality relation holds under very general assumptions and has an obvious extension to spaces of arbitrarily high dimension. The metric tensor is not used in the paper, but

$$g^{ij} = \nabla\xi^i\cdot\nabla\xi^j \quad\text{and}\quad g_{ij} = \partial\mathbf{x}/\partial\xi^i\cdot\partial\mathbf{x}/\partial\xi^j. \tag{A3}$$

One can show that g^{ij} and g_{ij} are matrix inverses of each other. An orthogonal coordinate system implies g^{ij} and g_{ij} are diagonal matrices. The condition that the metric tensor be a diagonal matrix is rarely satisfied by general, or nonorthogonal, coordinates, but the orthogonality relation, Equation (A2), always holds, even for nonorthogonal coordinates.

There are two obvious representations of a vector $\mathbf{B}$: the covariant representation $\mathbf{B}=\Sigma B_i\nabla\xi^i$ and the contravariant representation $\mathbf{B}=\Sigma B^i\partial\mathbf{x}/\partial\xi^i$. The orthogonality relations imply that $B_i=\mathbf{B}\cdot\partial\mathbf{x}/\partial\xi^i$ and $B^i=\mathbf{B}\cdot\nabla\xi^i$. These representations exist provided the Jacobian J is neither zero nor infinity,

$$J \equiv \frac{\partial \mathbf{x}}{\partial \xi^1} \cdot \left(\frac{\partial \mathbf{x}}{\partial \xi^2} \times \frac{\partial \mathbf{x}}{\partial \xi^3}\right); \tag{A4}$$

$$1/J = \nabla \xi^1 \cdot (\nabla \xi^2 \times \nabla \xi^3). \tag{A5}$$

The relation between J and 1/J can be proved using the important dual relations,

$$\partial \mathbf{x}/\partial \xi^1 = J\, \nabla \xi^2 \times \nabla \xi^3; \tag{A6}$$

$$\nabla \xi^1 = (\partial \mathbf{x}/\partial \xi^2) \times (\partial \mathbf{x}/\partial \xi^3)/J. \tag{A7}$$

The dual relations are proved by expanding $\nabla \xi^2 \times \nabla \xi^3$ as a contravariant vector and $(\partial \mathbf{x}/\partial \xi^2) \times (\partial \mathbf{x}/\partial \xi^3)$ as a covariant vector and evaluating the components. These relations plus the facts that the divergence of crossed gradients is zero, $\nabla \cdot (\nabla \xi^2 \times \nabla \xi^3) = 0$, and the curl of a gradient is zero, $\nabla \times (\nabla \xi^1) = 0$ allow one to carry out the required calculations in three-dimensional space.

The Jacobian J is actually the determinant of a matrix, the Jacobian matrix, which implies that the Jacobian has all the properties of a determinant. Let the three Cartesian coordinates x,y,z be denoted by x^1, x^2, x^3, then the Jacobian matrix between Cartesian coordinates and general coordinates ξ^1, ξ^2, ξ^3 is $J^i{}_j \equiv \partial x^i/\partial \xi^j$. A standard notation for the Jacobian is $J = \partial(x^1, x^2, x^3)/\partial(\xi^1, \xi^2, \xi^3)$. If η^1, η^2, η^3 is any other set of coordinates, the properties of determinants imply that

$$\frac{\partial(x^1, x^2, x^3)}{\partial(\eta^1, \eta^2, \eta^3)} = \frac{\partial(x^1, x^2, x^3)}{\partial(\xi^1, \xi^2, \xi^3)} \frac{\partial(\xi^1, \xi^2, \xi^3)}{\partial(\eta^1, \eta^2, \eta^3)}. \tag{A8}$$

That is the Jacobian of the η coordinate system is the product of the Jacobian of the ξ coordinate system and the Jacobian between the ξ and η coordinates. The partial derivative of a quantity f with respect to ξ^1, holding ξ^2 and ξ^3 constant, can be written in Jacobian notation as $\partial f/\partial \xi^1 = \partial(f, \xi^2, \xi^3)/\partial(\xi^1, \xi^2, \xi^3)$.

The general coordinate relations, except for the dual relations, have obvious extensions to higher dimensional spaces. Actually the dual relations exist in higher dimensional spaces; they are just messier. For example, if $\mathbb{X}$ is considered to be a function of ψ,θ,φ,χ and each of the canonical coordinates is considered to be a function of $\mathbb{X}$ and H, then

$$\partial\mathbb{X}/\partial\psi = J[(\partial\chi/\partial H)(\nabla\theta\times\nabla\varphi) + (\partial\varphi/\partial H)(\nabla\chi\times\nabla\theta) + (\partial\theta/\partial H)(\nabla\varphi\times\nabla\chi)] \qquad \text{(A9)}$$

with $J \equiv \partial(\mathbb{X},H)/\partial(\psi,\theta,\varphi,\chi)$ the four-space Jacobian. If one uses the canonical form for $\mathbb{B}$, then a straightforward calculation gives

$$\begin{aligned}(\partial\mathbb{X}/\partial\psi)\times\mathbb{B} &= J\{\nabla\theta\,[-(\nabla\theta\times\nabla\varphi)\cdot\nabla\psi(\partial\chi/\partial H) + (\nabla\psi\times\nabla\theta)\cdot\nabla\chi(\partial\varphi/\partial H) - \\ &\quad (\nabla\chi\times\nabla\psi)\cdot\nabla\varphi(\partial\theta/\partial H)] + \nabla\psi\,(\mathbb{B}\cdot\nabla\theta)(\partial\theta/\partial H) + \\ &\quad \nabla\varphi\,(\mathbb{B}\cdot\nabla\theta)(\partial\chi/\partial H) + \nabla\chi\,(-\mathbb{B}\cdot\nabla\theta)(\partial\varphi/\partial H)\} \\ &= -[\nabla\theta + (\partial H/\partial\psi)\mathscr{B}] \qquad \text{(A10)}\end{aligned}$$

with

$$\mathscr{B} \equiv (\partial\theta/\partial H)\nabla\psi - (\partial\psi/\partial H)\nabla\theta + (\partial\chi/\partial H)\nabla\varphi - (\partial\varphi/\partial H)\nabla\chi. \qquad \text{(A11)}$$

This result was used in section (IV).

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FIGURE CAPTION

Fig. 1. Canonical Coordinates: If the canonical variables θ and φ are interpreted as a poloidal, and a toroidal, angle, then $2\pi\chi$ and $2\pi\psi$ are fluxes. The toroidal flux inside a constant ψ surface is $2\pi\psi$, and the poloidal flux outside a constant χ surface is $2\pi\chi$. The ψ and χ surfaces need not coincide, although they do in magnetic coordinates. The usual name for the toroidal flux of vorticity is the circulation, but since there are two fluxes, one should speak of a toroidal, and a poloidal, circulation.

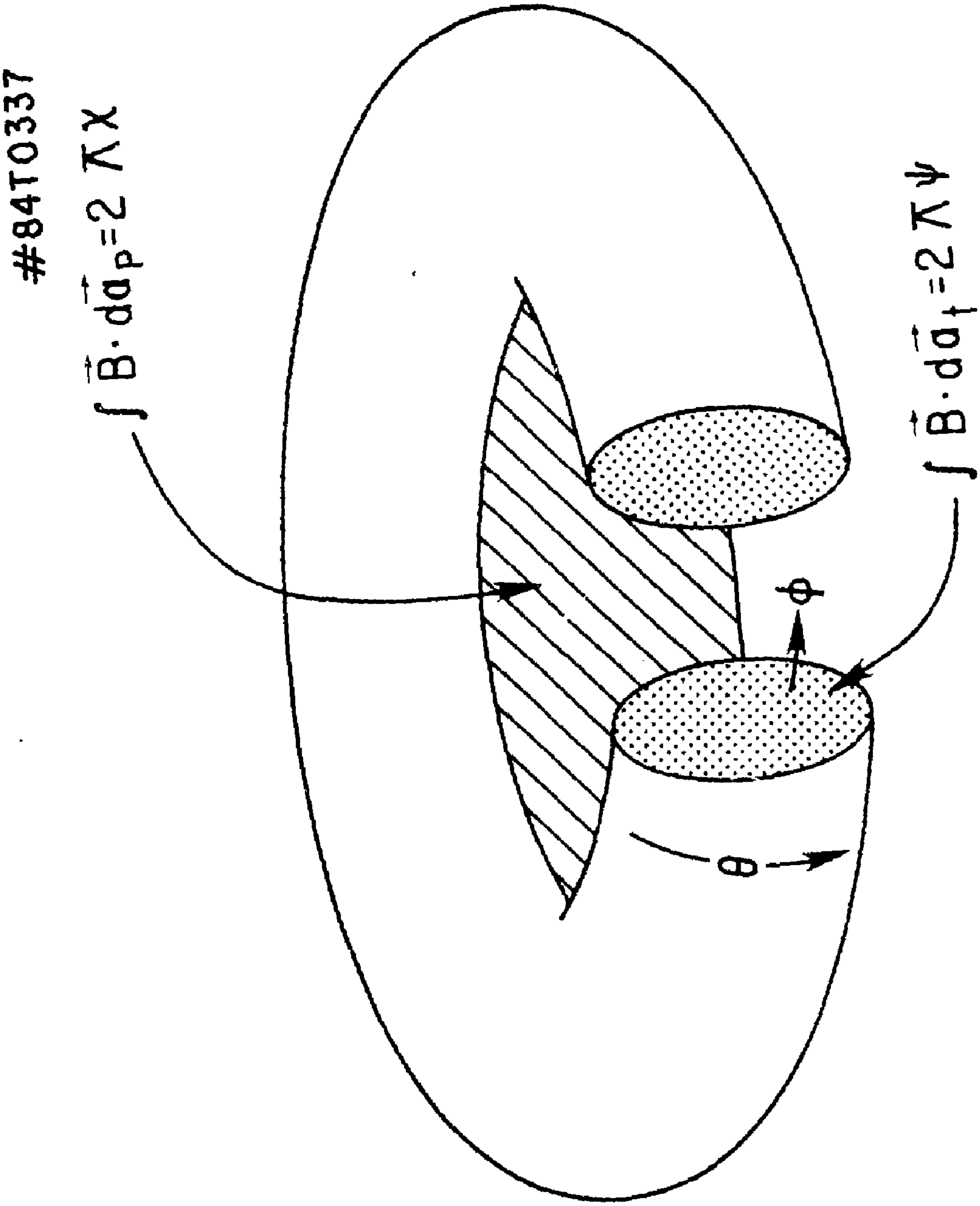
#84T0337
$\int \vec{B} \cdot d\vec{a}_p = 2\pi\chi$
$\int \vec{B} \cdot d\vec{a}_t = 2\pi\psi$
ϕ
θ

Fig. 1

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