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UCID--20938

DE87 003609

F. A. Howes

December 4, 1986

Lawrence Livermore National Laboratory

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Work performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore National Laboratory under Contract W-7405-Eng-48.

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by

F. A. Howes
Lawrence Livermore National Laboratory
P.O. Box 808, L-321
Livermore, CA 94550

ABSTRACT

We study the response of a turbulent boundary layer to an outer-flow disturbance in the form of a small-amplitude wave travelling along the bottom of a smooth channel. In a previous paper[2] we proposed a model for the viscous attenuation of a wave propagating along the interface between two superposed fluids inside a laminar boundary layer attached to the bottom wall. We obtained precise estimates on the amount of attenuation suffered by the oscillatory component of the motion as a result of viscous dissipation. This was accomplished by means of a representation of the solution as the asymptotic sum of a Blasius boundary layer profile and a modified Stokes layer profile. The present paper contains a similar asymptotic decomposition of the solution of the appropriate turbulent Prandtl equations when the outer flow is a small-amplitude travelling wave, and so it may be considered an extension of our previous work to the more realistic case of turbulent flow.

I. REVIEW OF THE LAMINAR CASE

In this section we recall briefly our past results[2] on the interaction between a small-amplitude travelling wave and a laminar boundary layer. The outer-flow disturbance is of the form

$$U_e(x,y,t) = U + i\varepsilon f(y)e^{i(\omega t-kx)}, \qquad y > -h, \tag{1.1}$$

where U is the constant mean flow, $f(y) = k \cosh[k(y+h)]$ with k the wavenumber and ω the frequency, and $y=-h$ is the location of the bottom of the channel. In addition, the positive parameter ε represents the ratio of the oscillatory part of the flow to the steady part. We make the basic assumption that ε and k are small, that is, we assume the outer flow is a uniform flow on which is superimposed a small oscillatory component in such a way that the wavelength of the disturbance, which is proportional to k^{-1}, is large compared to the boundary layer thickness.

In view of the form (1.1) of this disturbance we make the ansatz that the response of the boundary layer is of a similar form; namely (cf.[4; Chap. 5]),

$$u(x,y,t) = u_o(x,y) + \varepsilon u_1(x,y)e^{i(\omega t-kx)} + \ldots$$
$$v(x,y,t) = v_o(x,y) + \varepsilon v_1(x,y)e^{i(\omega t-kx)} + \ldots , \tag{1.2}$$

where u and v are the tangential and normal components of the velocity in the boundary layer, respectively. Now these components are solutions of the Prandtl boundary layer problem

$$u_t + uu_x + vu_y = U_{e,t} + U_eU_{e,x} + \nu u_{yy}$$
$$u_x + v_y = 0 \tag{1.3}$$
$$u = v = 0 \text{ at } y = -h;\ u \to U_e \text{ as } y \to \infty ,$$

and so inserting (1.2) into (1.3) and rearranging in powers of ε yield the following two problems for (u_o,v_o) and (u_1,v_1):

$$u_ou_{o,x} + v_ou_{o,y} = \nu u_{o,yy}$$
$$u_{o,x} + v_{o,y} = 0 \tag{1.4}$$
$$u_o = v_o = 0 \text{ at } y = -h;\ u_o \to U \text{ as } y \to \infty ,$$

$$(i\omega - iku_o + u_{o,x})\, u_1 + u_ou_{1,x} + u_{o,y}v_1 + u_{1,y}v_o = -(\omega-Uk)f + \nu u_{1,yy}$$
$$u_{1,x} + v_{1,y} - iku_1 = 0 \tag{1.5}$$
$$u_1 = v_1 = 0 \text{ at } y = -h;\ u_1 \to if(y) \text{ as } y \to \infty .$$

Thus the zeroth-order problem (1.4) is simply the classical Blasius problem for a flat plate (cf. [3;Chap.7]) with a known solution (u_B,v_B), and so the problem (1.5) for the first-order correction is a nonhomogeneous, linear problem with known coefficients. In order to solve this problem we make the high-frequency assumption that $\omega \gg 1$ in (1.5) and arrive thereby at the simpler system

$$i(\omega - kU)u_1 = -(\omega - kU)f + \nu u_{1,yy}$$
$$u_{1,x} + v_{1,y} - iku_1 = 0 . \qquad (1.6)$$

(Note also that we have replaced u_0 in (1.5) with its asymptotic value U at the outer edge of the boundary layer. This permits us to find a solution of (1.6) satisfying the matching condition $u_1 \to if(y)$ as $y \to \infty$, as is readily apparent from (1.6) if we neglect the viscous term $\nu u_{1,yy}$. Indeed, $\nu u_{1,yy} \to 0$ as $y \to \infty$!) We seek a particular solution of the nonhomogeneous equation (1.6) in the form

$$u_1(x,y) = C \cosh [k(y + h)] ,$$

and a short calculation reveals that

$$C = \frac{ik(\omega - kU)}{(\omega - kU) + i\nu k^2} \sim ik \text{ as } \nu \to 0$$

is the desired constant. The corresponding homogeneous equation has the general solution $u_1(x,y) = D_1 \exp[\lambda(y + h)] + D_2 \exp[-\lambda(y + h)]$, for $\lambda = [i(\omega - kU)/\nu]^{1/2}$, and so in view of the boundary conditions on u_1, we set $D_1 = 0$, $D_2 = -C$ and arrive at the following asymptotic solution of (1.6):

$$\begin{aligned} u_{st}(x,y) &= \left[\frac{ik(\omega - kU)}{(\omega - kU) + i\nu k^2}\right] \{\cosh[k(y + h)] - \exp[-\lambda(y + h)]\} \\ &\sim ik\{\cosh[k(y + h)] - \exp[-[i(\omega - kU)/\nu]^{1/2}(y + h)]\} . \end{aligned} \qquad (1.7)$$

To summarize, then, we have shown that the tangential component of the velocity in a laminar boundary layer subjected to the outer disturbance (1.1) is the sum of a Blasius boundary layer and a modified Stokes boundary layer, that is,

$$u(x,y,t) \sim u_B(x,y) + \varepsilon u_{st}(x,y) e^{i(\omega t - kx)} . \qquad (1.8)$$

Using (1.8) we can proceed to calculate the attentuation suffered by the wave through the action of viscous dissipation and other quantities of interest; cf. [2] for the details.

2. THE TURBULENT CASE

We now wish to study the response of a turbulent boundary layer to the outer-flow disturbance (1.1) assuming, as before, that the wavelength of the disturbance is much larger than the boundary layer thickness. Let us proceed as in (1.2) by making the ansatz that the response is of the form

$$u(x,y,t) = u_o(x,y) + \varepsilon u_1(x,y)e^{i(\omega t - kx)} + \ldots$$

$$v(x,y,t) = v_o(x,y) + \varepsilon v_1(x,y)e^{i(\omega t - kx)} + \ldots ,$$

where now the quantities u_o, v_o, u_1 and v_1 are random complex-valued functions. In other words, we assume that the boundary layer responds to the outer flow with a *deterministic fluctuation* of the same frequency and an amplitude that contains random turbulent fluctuations; cf. [4; Chap. 6]. We further assume that the organized fluctuations, measured by the size of ε, are *small compared to the mean flow*. The difference between this analysis and the analysis in Sec. 1 consists in the fact that here all terms contain a random part. In particular, the random fluctuations are of two types: the fluctuations of the steady part of the flow and the fluctuations of the organized motion.

Let us then decompose the velocity components into the sum of a time-average and a random fluctuation; to wit, for $j = 0,1$

$$u_j = \bar{u}_j + u'_j;\ v_j = \bar{v}_j + v'_j . \tag{2.1}$$

(By definition, the time average of a fluctuation is zero, that is, $\overline{u'_j} = \overline{v'_j} = 0$.) It is not difficult to show that the *mean and fluctuating components* satisfy the continuity equations

$$\bar{u}_{o,x} + \bar{v}_{o,y} = 0;\ \ u'_{o,x} + v'_{o,y} = 0$$

$$\bar{u}_{1,x} + \bar{v}_{1,y} - ik\bar{u}_1 = 0;\ \ u'_{1,x} + v'_{1,y} - iku'_1 = 0;$$

in addition, the fluctuations u'_j, v'_j satisfy the boundary conditions

$$u'_j = v'_j = 0 \text{ at } y = -h;\ u'_j, v'_j \to 0 \text{ as } y \to \infty .$$

Substituting the expressions (2.1) into the boundary layer equation (1.3) and averaging yield boundary value problems for $(\bar{u}_j, \bar{v}_j)$:

$$\begin{aligned} &\bar{u}_o\bar{u}_{o,x} + \bar{v}_o\bar{u}_{o,y} = \nu\bar{u}_{o,yy} - (\overline{u'_o v'_o})_y \\ &\bar{u}_o = \bar{v}_o = 0 \text{ at } y = -h;\ \bar{u}_o \to U \text{ as } y \to \infty \end{aligned} \tag{2.2}$$

and

$$(i\omega - ik\bar{u}_0 + \bar{u}_{0,x})\bar{u}_1 + \bar{u}_0\bar{u}_{1,x} + \bar{u}_{0,y}\bar{v}_1 + \bar{u}_{1,y}\bar{v}_0$$

$$= -(\omega - kU)f + \nu\bar{u}_{1,yy} - (\overline{u_0'v_1'} + \overline{u_1'v_0'})_y - 2ik\overline{u_0'u_1'} \tag{2.3}$$

$$\bar{u}_1 = \bar{v}_1 = 0 \text{ at } y = -h;\ \bar{u}_1 \to if(y) \text{ as } y \to \infty .$$

Problem (2.2) is the classical problem for a steady turbulent boundary layer along a flat plate (cf.[1]) with a known solution (u_T, v_T) that depends on the particular closure law chosen for the estimation of the Reynolds stress term $\overline{u_0'v_0'}$. In order to proceed with the solution of problem (2.3) we again make the high-frequency assumption that $\omega \gg 1$ and so obtain the approximate problem

$$i(\omega - kU)\bar{u}_1 = -(\omega - kU)f + \nu\bar{u}_{1,yy} - 2ik\overline{u_0'u_1'} \tag{2.4}$$

$$\bar{u}_1 = \bar{v}_1 = 0 \text{ at } y = -h;\ \bar{u}_1 \to if(y) \text{ as } y \to \infty ,$$

in which $\bar{u}_0$ has been replaced by its asymptotic value U, as in Sec. 1. We now assume that an approximate particular solution of (2.4) is given by

$$\bar{u}_1 = C \cosh [k(y + h)]$$

for $C = \dfrac{ik(\omega - kU)}{(\omega - kU) + i\nu k^2} \sim ik ,$

as in the laminar case. Note that in deriving this result we have neglected the last term in the righthand side of (2.4), the one due to the turbulent fluctuations u_0' and u_1'. The justification for this resides in the observation that the particular solution describes the behavior of the solution of (2.4) at the outer edge of the turbulent boundary layer, and so the term $\overline{u_0'u_1'}$ is small there, owing to the fact that $u_0', u_1' \to 0$ as $y \to \infty$. Concerning the corresponding homogeneous problem, we make the further assumption that the turbulent forcing term in (2.4) is of the form

$$-2ik\,\overline{u_0'u_1'} = ik\gamma\bar{u}_{1,y} ,$$

where γ is an adjustable phenomenological constant whose magnitude is of the order of $\bar{u}_{0,y}$; cf. [3; Chap. 19]. Granted the validity of this ansatz we now look for solutions of the homogeneous equation

$$i(\omega - kU)\,\bar{u}_1 = \nu\bar{u}_{1,yy} + ik\gamma\bar{u}_{1,y}$$

of the form

$$\bar{u}_1 = \exp[\lambda(y + h)]\,,$$

where λ is a root of the quadratic $\nu\lambda^2 + ik\gamma\lambda - i(\omega - kU)$. A little algebra reveals that

$$\lambda = \frac{-ik\gamma \pm [-k^2\gamma^2 + 4i\nu(\omega - kU)]^{1/2}}{2\nu}\,, \tag{2.5}$$

and so depending on the size of γ we can write down the complete asymptotic solution of problem (2.4).

First of all, if γ is much smaller than $\nu^{1/2}$ then we may, to lowest order, neglect the quadratic term in γ in the formula (2.5) and obtain the asymptotic formula

$$\lambda_{\pm} \sim \pm[i(\omega - kU)/\nu]^{1/2}.$$

In view of the far-field matching condition that $\bar{u}_1 \sim if(y)$ as $y \to \infty$, we reject the root λ_+ and find that [cf.(1.7)]

$$\bar{u}_1(x,y) = \left[\frac{ik(\omega - kU)}{(\omega - kU) + i\nu k^2}\right] \{\cosh[k(y + h)] - \exp[\lambda_-(y + h)]\}$$

$$\sim ik\{\cosh[k(y + h)] - \exp[-[i(\omega - kU)/\nu]^{1/2}(y + h)]\}$$

is the desired approximate solution of problem (2.4). It is valid in the viscous sublayer of the turbulent boundary layer where the viscous stresses dominate the Reynolds stresses. Secondly, if γ is much larger than $\nu^{1/2}$, as is the case in the remainder of the turbulent layer outside the viscous sublayer, then formula (2.5) reduces to

$$\lambda \sim \frac{-ik\gamma \pm [ik\gamma + 2\nu(\omega - kU)/(k\gamma)]}{2\nu}\,.$$

This follows from the asymptotic result that for $|\mu| \ll 1$ $(\Gamma^2 + \mu)^{1/2} \sim \Gamma + \frac{1}{2}\mu/\Gamma$, which in turn follows from the Mean Value Theorem applied to the function $(\Gamma^2 + \mu)^{1/2}$. Therefore, the two roots are approximately

$$\lambda_+ \sim (\omega - kU)/(k\gamma)$$

and

$$\lambda_- \sim -ik\gamma/\nu - (\omega - kU)/(k\gamma)\ .$$

We reject the root λ_+ because of the far-field matching condition and use λ_- to obtain the approximate solution

$$\bar{u}_1(x,y) = [\frac{ik(\omega - kU)}{(\omega - kU) + i\nu k^2}]\ \{\cosh[k(y+h)] - \exp[\lambda_-(y+h)]\}$$

$$\sim ik\{\cosh[k(y+h)] - \exp[-(ik\gamma/\nu)(y+h)]\exp[-[(\omega - kU)/(k\gamma)](y+h)]\}$$

of the problem (2.4). Thus the travelling-wave disturbance induces in the tangential component $\varepsilon\bar{u}_1(x,y)e^{i(\omega t - kx)}$ of the first-order correction to the mean turbulent flow a rapidly oscillatory motion with amplitude ε and frequency $k\gamma/\nu \gg 1/\nu^{1/2}$ described by the term $\exp[-(ik\gamma/\nu)(y+h)]$.

ACKNOWLEDGMENTS

This work was performed under the auspices of the U.S. Department of Energy at the Lawrence Livermore National Laboratory under contract number W-7405-ENG-48 and Army Research Laboratories DOE-DOD Joint Memorandum of Understanding. The author thanks Al Buckingham for his generous hospitality and Joan Balaris for her secretarial assistance.

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