Presented at the 1985 U.S. Particle Accelerator School, SLAC, Stanford, California, July 1985

BNL--38824

DE87 002392

INTRODUCTION TO THE THEORY OF FREE ELECTRON LASERS

Samuel Krinsky
National Synchrotron Light Source
Brookhaven National Laboratory, Upton, NY 11973

TABLE OF CONTENTS

ABSTRACT

We present an introduction to some fundamental aspects of the theory of free electron lasers. Spontaneous radiation emitted by electrons traversing a wiggler magnet is briefly reviewed, and stimulated emission in the low-gain regime is discussed using Colson's pendulum equations and Madey's theorems. The high-gain regime is treated by an extention of the work of Bonifacio, Pellegrini, and Narducci. We introduce dynamical variables to describe the radiation field, and a Hamiltonian formulation of Maxwell's equations is employed. A canonical transformation to the interaction representation factors out the fast time variation of the radiation field, and the slow time dependence is determined by linearized equations for the appropriate collective variables. As an application of this technique we consider self-amplified spontaneous radiation, and we comment upon the relationship between our approach and the use of coupled Vlasov-Maxwell equations.

INTRODUCTION TO THE THEORY OF FREE ELECTRON LASERS

Samuel Krinsky
National Synchrotron Light Source
Brookhaven National Laboratory, Upton NY 11973

1. INTRODUCTION

The transverse acceleration experienced by an electron passing through an external magnetic field results in the emission of electromagnetic radiation. Relativistic electrons passing through a uniform magnetic field are deflected along the arc of a circle and one speaks of synchrotron radiation.[1] In the discussion of free electron lasers,[2,3] we shall be concerned with electrons traversing a wiggler magnet,[4,5] i.e. a device that produces a magnetic field alternating in polarity along its axial direction, taken parallel to the average velocity of the electron beam. The wiggler magnet produces no net deflection of the electrons, but it does wiggle them, causing the emission of radiation predominantly in the forward (axial) direction. In the context of free electron lasers, the radiation emitted by individual accelerated electrons is referred to as "spontaneous radiation."

Our discussion will be confined to wiggler fields periodic in the axial direction, and we denote the period length by λ_w and the number of periods N_w. For sufficiently weak magnetic fields, the spontaneous radiation in the forward direction is peaked at frequency ω_o given by[6,7]

$$\omega_o \simeq 2\gamma^2(2\pi c/\lambda_w) \; , \tag{1}$$

where c is the velocity of light and γ is the electron energy in units of its rest mass. The linewidth of the radiation spectrum in the forward direction is

$$\frac{\Delta\omega}{\omega_o} \simeq \frac{1}{N_w} \; , \tag{2}$$

a consequence of the fact that a wave packet emitted by an electron passing through the wiggler magnet contains N_w electromagnetic periods of frequency ω_o. The spontaneous radiation satisfying Eqs. (1) and (2) is often referred to as "undulator" rather than "wiggler" radiation, but we shall continue to use the term "wiggler." In Section 2, we present a brief review of the spontaneous radiation from a helical wiggler.[7,8]

An electron beam passing through a wiggler magnet can serve as a medium for the amplification[9] of an external electromagnetic wave (Fig. 1) with frequency near ω_o, "stimulated emission." In the low-gain regime, corresponding to small electron density, one can ignore the increment of the radiated field during a single pass when computing the energy transfer between the electrons and the wave,

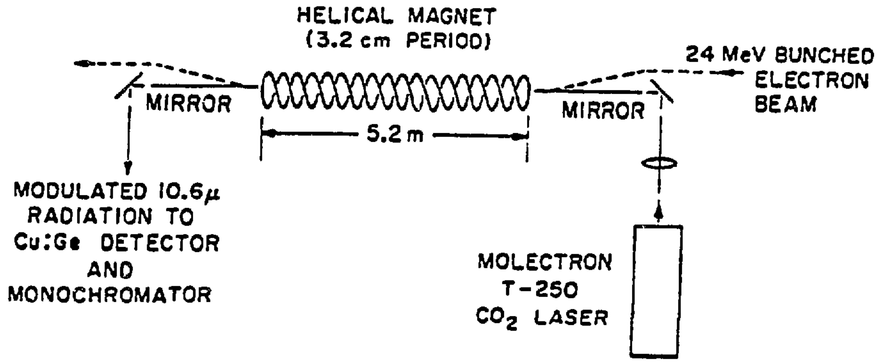

Fig.1. Experimental arrangement for measuring the amplification by an electron beam in a wiggler magnet of infrared radiation from a carbon dioxide laser.[9]

and one can consider each electron to interact with the electromagnetic wave independently of the other electrons. The amplification process can be imagined to proceed in three overlapping stages. When the electron bunch enters the wiggler and has traversed only a small number of periods, the predominant effect is a redistribution in energy among the electrons, some gaining and some losing energy, thus increasing the energy spread with essentially no net energy transfer to the electromagnetic wave. This increase in the energy spread is linear in the number of wiggler periods traversed, $[<(\Delta\gamma)^2>]^{1/2} \sim N_w$, the average being over the initial phase between the electron motion and the electromagnetic wave. Farther along the wiggler, the second-stage action occurs: the electrons begin to shift in phase, and the electron spatial distribution becomes modulated on the scale of the radiation wavelength. This process depends quadratically on the number of wiggler periods traversed, i.e. electron density modulation $\sim N_w^2$. Finally, significantly more than half the electrons become so located in phase as to transfer energy to the photon field, and laser action with positive gain takes place. The net gain $\sim N_w^3$.

Since all electrons move relative to the fixed phase of the external electromagnetic wave, the stimulated emission process is coherent. In principle, the linewidth of the amplified radiation can be as small as

$$\frac{\Delta\omega}{\omega_o} \simeq \frac{\lambda}{\ell_e} , \qquad (3)$$

where λ is the radiation wavelength and ℓ_e the length of the electron bunch. Equation (3) merely states that a wave packet can contain at most ℓ_e/λ electromagnetic periods. In Section 3, we give a brief description of the amplification process in the low-gain regime.

A free electron laser oscillator[10] consists of the amplifier just described situated in an optical cavity bounded by mirrors (Fig. 2). The electrons interact with the cavity field rather than

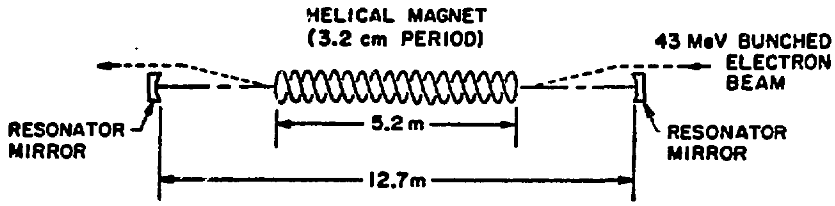

Fig.2. The Stanford free electron laser oscillator.[10]

with an external electromagnetic wave produced by some other laser. Since the cavity field interacts repeatedly with the electron beam, its strength can increase enormously even though the gain per pass is small, assuming it is larger than the cavity losses.

Of particular interest is the high-gain regime[11-17] corresponding to larger electron beam densities. In this case, when considering a free electron laser amplifier or oscillator, it is necessary to take into account the increment of the electromagnetic field during a single pass. The electrons modify the electromagnetic wave, which then acts back upon the electrons. This gives rise to a positive feedback loop between the electron beam distribution and the radiation. The more the electrons radiate, the greater the modification of the distribution; and the greater the modification of the distribution, the more power radiated. This collective instability results in an exponential increase in the radiated power per pass with the number of wiggler periods, i.e. radiated power $\sim \exp(\alpha N_w)$.

The high-gain regime is discussed in Section 4. The work presented is an extension of the work of Bonifacio, Pellegrini, and Narducci.[17] We introduce dynamical variables to describe the variation of the electromagnetic field and use a Hamiltonian formulation of Maxwell's equations. A canonical transformation to the interaction representation allows us to factor out the fast time dependence of the laser field, and the slow time evolution of the envelope of the radiation field is determined by linearized equations for the appropriate collective variables. Self-amplified spontaneous radiation is considered, and the relationship between our approach and the recent work of Kim[18] and of Wang and Yu[19] using the coupled Vlasov-Maxwell equations[11,20] is commented upon.

Let us conclude this section with a few remarks on the development of real free electron laser devices.[21] Since for fixed wiggler parameters the optical gain is proportional to the product of the peak current of the electron beam and the radiation wavelength to the 3/2 power, it is more difficult to build a free electron laser operating at short wavelengths. An infrared free electron laser can be made to operate with the electron beam from a linac or microtron, but for operation in the ultraviolet a current accumulator, such as a storage ring, may be required. Recently, experimental studies of a low-gain free electron laser in a storage ring have been reported in the visible region of the spectrum.[22,23] At present new storage ring designs are being considered at Stanford University[24] and

Lawrence Berkeley Laboratory[25] to obtain higher gain in the ultraviolet and soft x-ray regions. Since good mirrors for an optical cavity are difficult to achieve at short wavelengths, the highest possible gain is essential. In fact, there is much interest in eliminating the need for mirrors by using a single-pass free electron laser[26] in the high-gain regime for the production of self-amplified spontaneous radiation. The shot noise power present in the electron beam at the entrance of the wiggler grows exponentially along the device. A successful experiment on the high-gain free electron laser in the microwave range using a linear accelerator has been carried out,[27] and the goal of generating shorter-wavelength radiation from a single-pass device is of great interest.

2. SPONTANEOUS RADIATION

Consider a helical wiggler[7,8] with its axis parallel to the unperturbed electron motion (z-direction). The magnet produces a helical magnetic field $\vec{B}_w$, which has a period length λ_w in the z-direction, and is well approximated by

$$\vec{B}_w = B_w(\hat{x}\cos k_w z + \hat{y}\sin k_w z) , \tag{4}$$

where $k_w = 2\pi/\lambda_w$. The magnetic field causes an electron to be deflected. Measured in units of its rest mass mc^2, the electron has energy γ, and the electron velocity $\vec{\beta} = \vec{v}/c$ measured relative to the speed of light c is

$$\vec{\beta} = \beta_o\hat{z} + \vec{\beta}_\perp , \tag{5}$$

where

$$\beta_o = 1 - \frac{1 + K^2}{2\gamma^2} , \tag{6}$$

$$\vec{\beta}_\perp = \frac{K}{\gamma}(\hat{x}\cos k_w z + \hat{y}\sin k_w z) , \tag{7}$$

and the magnetic field strength parameter K is defined by

$$K = \frac{eB_w\lambda_w}{2\pi mc^2} . \tag{8}$$

Since the electron is being accelerated, it radiates, and the energy radiated per unit solid angle per unit frequency interval in the forward direction is given by[1,7,8]

$$\left.\frac{dI(\omega)}{d\Omega}\right|_{\theta=0} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_0^t dt'\vec{\beta}_\perp(t')e^{i\omega(1-\beta_o)t'}\right|^2$$

$$= \frac{e^2\omega^2K^2}{4\pi^2c\gamma^2}\,\frac{2\sin^2\left(\omega_w(1-\frac{\omega}{\omega_o})\,\frac{t}{2}\right)}{\omega_w^2(1-\frac{\omega}{\omega_o})^2} \tag{9}$$

where the peak frequency is

$$\omega_o = \omega_w/(1-\beta_o) = 2\gamma^2\omega_w/(1+K^2)\ , \tag{10}$$

and $\omega_w = k_w c$.

The frequency ω_o (Eq. 10) at which the spectral peak is centered can be understood on the basis of a simple intuitive argument. Consider a wavefront radiated in the z-direction by an electron passing through the periodic magnetic field. At time $\lambda_w/\beta_o c$ later, the electron has passed through one period of the magnet, and a second wavefront emitted at this time will follow the first by a time interval

$$\tau = \lambda_w/\beta_o c - \lambda_w/c\ . \tag{11}$$

An observer downstream of the magnet looking in the forward direction sees a radiation spectrum peaked about

$$\omega_o = 2\pi/\tau\ , \tag{12}$$

which agrees with Eq. (10).

For a magnet with N_w periods, the radiated pulse from one electron passing through the device contains N_w radiation periods, consequently the linewidth of the radiation in the forward direction is

$$\Delta\omega/\omega_o \sim 1/N_w\ . \tag{13}$$

This is also evident from Eq. (9) with $\omega_w t = 2\pi N_w$.

3. STIMULATED RADIATION: LOW-GAIN REGIME

Now consider[28,29] an electron moving both in the magnetic field of the helical wiggler, Eq. (4), and in the radiation field:

$$\vec{E} = E_o\left(\hat{x}\sin(kz-\omega t+\psi_o) + \hat{y}\cos(kz-\omega t+\psi_o)\right)\ ,$$

$$\vec{B} = \hat{z}\times\vec{E}\ . \tag{14}$$

The rate of change of the electron energy is

$$\dot{\gamma} = \frac{e}{mc} \vec{\beta}_{\perp} \cdot \vec{E} . \quad (15)$$

Assuming the EM-field is sufficiently weak compared to the static wiggler magnetic field, we ignore the change in the transverse velocity due to the electromagnetic wave. Then the transverse velocity $\vec{\beta}_{\perp}$ is given by Eq. (7), and the rate of change of the electron energy is

$$\dot{\gamma} = - \frac{eE_o K}{mc\gamma} \sin\psi , \quad (16)$$

the phase ψ being defined by

$$\psi = (k_w + k)z - \omega t + \psi_o . \quad (17)$$

Approximating $\dot{z} \simeq \beta_o c$, with β_o given by Eq. (6), the rate of change of the phase is seen to be

$$\dot{\psi} = \omega_w (1 - \frac{k}{k_w} \frac{1 + K^2}{2\gamma^2}) . \quad (18)$$

It is apparent that there can be a net transfer of energy between the electron and the EM-wave, provided that $\vec{\beta}_{\perp}$ and $\vec{E}$ remain in phase over the length of the wiggler magnet. Equation (16) shows that this is the case when $\dot{\psi}$ is zero or is sufficiently small. For a given electron energy γ, a resonant frequency can, therefore, be defined by the condition $\dot{\psi} = 0$. From Eq. (18) we see that the resonant frequency is given by

$$\omega_o = \omega_w/(1 - \beta_o) = 2\gamma^2\omega_w/(1 + K^2) , \quad (19)$$

which is, in fact, the peak frequency of the spontaneous radiation given earlier in Eq. (10). The physical meaning of the resonance condition (19) can be understood by noting that an electron takes time $\lambda_w/\beta_o c$ to traverse one period of the wiggler, while the EM-wave makes the transit in time λ_w/c. At resonance the difference between these transit times equals one period $\tau = \lambda/c$ of the EM-wave.

Colson's pendulum equations[28] can be derived if we assume that the electron energy change in traversing the wiggler is small. Just as we defined a resonant frequency ω_o, in Eq. (19), corresponding to a given electron energy, we can define a resonant electron energy γ_R by

$$\gamma_R^2 = \frac{\omega}{2\omega_w} (1 + K^2) , \quad (20)$$

corresponding to a given frequency ω of the external EM-wave. Then, assuming the initial electron energy is close to the resonant value, we define

$$\eta = \frac{\gamma - \gamma_R}{\gamma_R} << 1 , \tag{21}$$

and Eqs. (16) and (18) become

$$\dot{\eta} = -\frac{\Omega^2}{2\omega_w} \sin\psi , \tag{22}$$

$$\dot{\psi} = 2\omega_w \eta , \tag{23}$$

where the "pendulum frequency" is

$$\Omega^2 = \frac{2\omega_w e E_o K}{mc\gamma_R^2} . \tag{24}$$

Combining Eqs. (22) and (23) yields the pendulum equation

$$\ddot{\psi} + \Omega^2 \sin\psi = 0 . \tag{25}$$

The change in energy of an electron due to its interaction with the EM-wave is determined from Eq. (22) to be

$$\begin{aligned} \frac{\Delta\gamma}{\gamma_R} &= -\frac{\Omega^2}{2\omega_w} \int_o^t dt' \, \sin\left(\omega_w (1 - \frac{\omega}{\omega_o}) t + \psi_o\right) \\ &= -\frac{\Omega^2}{2\omega_w} \frac{\cos\psi_o - \cos\left(\omega_w (1 - \frac{\omega}{\omega_o}) t + \psi_o\right)}{\omega_w (1 - \frac{\omega}{\omega_o})} . \end{aligned} \tag{26}$$

Let us introduce the average over the phase ψ_o,

$$<F> \equiv \int_o^{2\pi} \frac{d\psi_o}{2\pi} F \cdot \tag{27}$$

Then, assuming the electron beam to be initially uniformly distributed in ψ_o, the energy spread introduced in the electron beam is

$$<(\frac{\Delta\gamma}{\gamma_R})^2> = (\frac{\Omega^2}{2\omega_w})^2 \frac{2 \sin^2\left(\omega_w (1 - \frac{\omega}{\omega_o}) \frac{t}{2}\right)}{\omega_w^2 (1 - \frac{\omega}{\omega_o})^2} , \tag{28}$$

as is easily shown using Eq. (26). Now recalling Eq. (9) for the spontaneous energy radiated per unit solid angle per unit frequency interval in the forward direction, we see that

$$<(\Delta\gamma)^2> = \frac{4\pi^2 E_o^2}{m^2\omega^2 c} \frac{dI(\omega)}{d\Omega}\bigg|_{\theta=0} . \qquad (29)$$

This reciprocity relation,[30] known as Madey's first theorem,[31] determines the energy spread induced in the electron beam from the spectral distribution of the spontaneous radiation.

Madey's second theorem[30-33] (gain-spread theorem) is the fluctuation-dissipation[34] relation applied to the free electron laser,

$$<\Delta\gamma> = \frac{1}{2} \frac{\partial}{\partial\gamma} <(\Delta\gamma)^2> , \qquad (30)$$

where $<\Delta\gamma>$ is the net energy change of the electron beam due to its interaction with the EM-wave. We rewrite Eq. (28) as

$$<(\frac{\Delta\gamma}{\gamma_R})^2> = (\frac{\Omega}{2\omega_w})^4 (\omega_w t)^2 \frac{2\sin^2\xi}{\xi^2} , \qquad (31)$$

having defined

$$\xi = \omega_w(1 - \frac{\gamma_R^2}{\gamma^2}) \frac{t}{2} , \qquad (32)$$

and used

$$\omega_o/\omega = \gamma^2/\gamma_R^2 . \qquad (33)$$

Since $d\xi/d\gamma \simeq \omega_w t/\gamma_R$, Eqs. (30) and (31) show that

$$\frac{<\Delta\gamma>}{\gamma_R} = (\frac{\Omega}{2\omega_w})^4 (\omega_w t)^3 \frac{d}{d\xi} (\frac{\sin^2\xi}{\xi^2}) , \qquad (34)$$

which can also be written as

$$\frac{<\Delta\gamma>}{\gamma_R} = -(\frac{\Omega}{2\omega_w})^4 (2\omega_w t)^3 \frac{1}{(2\xi)^3} \left(1 - \frac{2\xi}{2} \sin 2\xi - \cos 2\xi\right) . \qquad (35)$$

The small signal gain G is defined to be

G = electron energy loss/energy in EM-field . (36)

Assume that the electron beam and the EM-wave occupy the same volume V and that there are N electrons. Then,

$$G = \frac{-<\Delta\gamma>mc^2 N}{\frac{2E_o^2}{8\pi} V} . \qquad (37)$$

We now define[17] a dimensionless parameter ρ, called the Pierce parameter, which will be shown in the next section to be of fundamental importance in the high-gain regime:

$$\rho^3 = \frac{4\pi e^2 \frac{N}{V} K^2}{16\gamma_R^3 m\omega_w^2} . \tag{38}$$

Also introducing dimensionless parameters τ and δ_k by

$$\tau = 2\rho\omega_w t$$

and

$$2\xi = \delta_k\tau = \omega_w t(1 - \frac{k}{k_o}) , \tag{39}$$

where $k = \omega/c$ and $k_o = \omega_o/c$, we obtain

$$G = \frac{4}{\delta_k^3} \left(1 - \frac{\delta_k\tau}{2} \sin \delta_k\tau - \cos \delta_k\tau\right) . \tag{40}$$

When the radiation field is not too strong, one can choose a value of the initial electron energy $\gamma_o > \gamma_R$ such that there is a net gain of energy by the electromagnetic wave from the electron beam. This results from the dynamical effects which increase the number of electrons having phase proper to do work on the radiation field and decrease the number having phase proper to absorb energy. Basic to this behavior are two facts: first, electrons initially at energy $\gamma > \gamma_R$ which gain energy move away from the resonant energy γ_R, while those losing energy move closer; and second, the rate of phase change is more rapid further from resonance. Therefore, while traversing the wiggler, those electrons initially in a phase region to absorb energy from the radiation, migrate relatively rapidly toward the phase region corresponding to emission while the initially emitting electrons leave their phase region comparatively slowly for phase corresponding to absorption. Hence, the initial, uniform phase distribution develops peaks in regions of phase corresponding to emission and valleys in regions corresponding to absorption. This is to say that the rate of emission is greater than that of absorption, and there results a positive amplification of the radiated intensity. Together, Madey's theorems [Eqs. (29) and (30)] determine the gain in terms of the spontaneous power spectrum, as illustrated in Fig. 3.

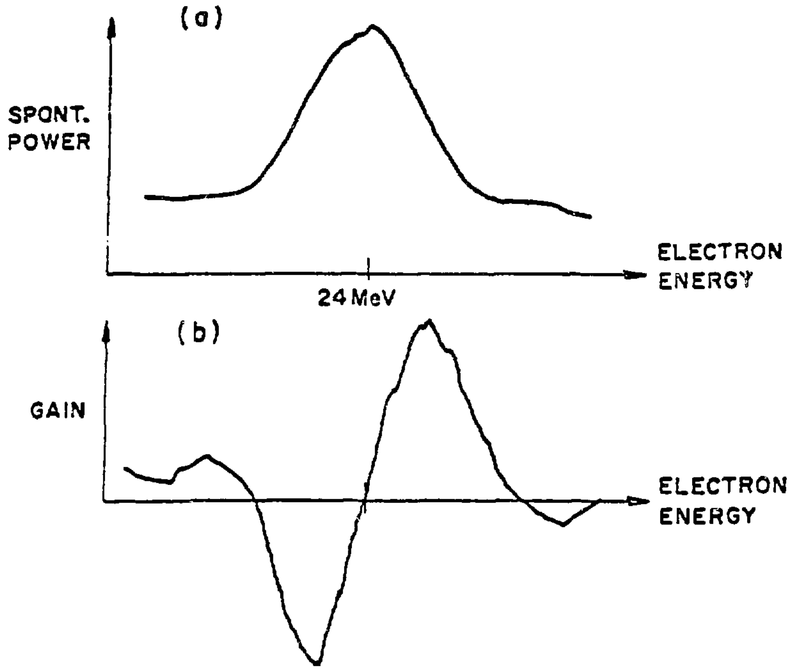

Fig. 3. The lower curve shows the gain in a free electron laser, as measured by Elias et al.[9]; this is proportional to the slope of the spontaneous emision line shape in the upper curve.

It is worthwhile now to give an estimate of the magnitude of the Pierce parameter ρ which might be achievable. Let us introduce the electron beam density

$$n_o = N/V \ ,$$

the classical radius of the electron,

$$r_o = e^2/mc^2 = 2.8 \times 10^{-15} \text{ m} \ ,$$

and the amplitude of the wiggle,

$$a = \frac{\lambda_w}{2\pi} \frac{K}{\gamma} \ .$$

Then Eq. (38) can be rewritten as

$$\rho^3 = \frac{n_o r_o \pi a^2}{4\gamma} \ . \qquad (41)$$

Let us suppose that

$$N = 10^{11} \ , \quad V = 10^{-3} \times 10^{-4} \times 10^{-2} \text{ m}^3 \ , \quad n_o = 10^{20} \text{ m}^{-3} \ .$$

Then, taking

$$K = 1 , \quad \gamma = 1000 , \quad \lambda_w = 0.06 \text{ m} ,$$

we find $a = 10^{-5}$ m and $\rho \sim 10^{-3}$. Hence we see that achievable values of ρ are small, on the order of 10^{-3}. The gain can now be estimated from Eq. (40). Using $\tau = 2\rho\omega_w t = 4\pi\rho N_w$, the gain is

$$G \simeq 4(4\pi\rho N_w)^3 \, f(4\pi N_w \frac{\Delta\gamma}{\gamma_R}) , \tag{42}$$

where

$$f(x) = \frac{1}{x^3} (1 - \frac{x}{2} \sin x - \cos x) , \tag{43}$$

is plotted in Fig. 4. The maximum of $f(x)$ is 0.07 for $x = 2.6$ ($\Delta\gamma/\gamma_R = 0.2/N_w$), so

$$G_{max} \simeq 500 \, (N_w\rho)^3 . \tag{44}$$

For $N_w = 100$ and $\rho = 10^{-3}$, $G_{max} = 0.5$.

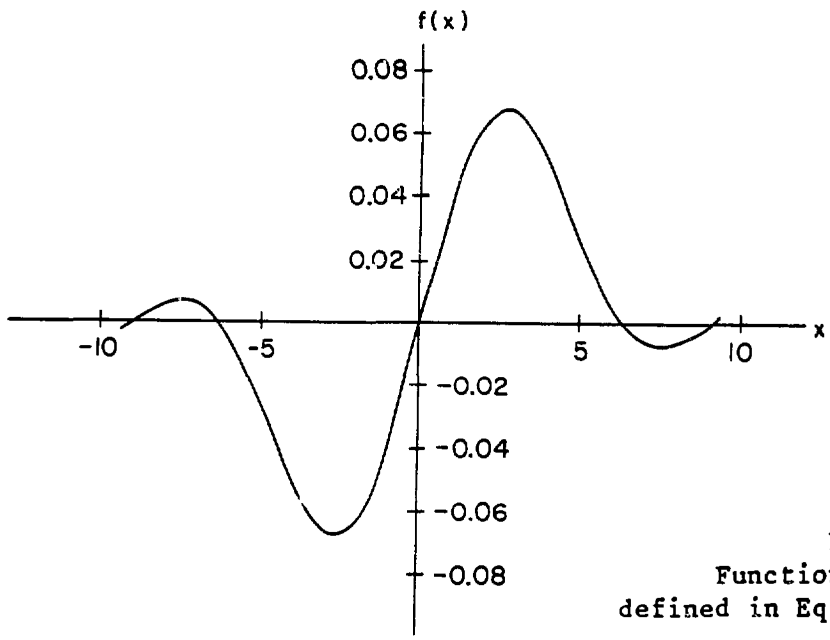

Fig. 4.
Function $f(x)$,
defined in Eq. (43).

4. HIGH-GAIN REGIME

Let us now take into account the change of the EM-field during a single traversal of the wiggler by an electron. We shall introduce dynamical variables P_k and Q_k to describe the EM-field and use a Hamiltonian formulation of Maxwell's equations as described, e.g. by Landau and Lifshitz.[35] In the high-gain regime,[11-19] the collective instability produced because the electrons radiating causes the electron beam to bunch and the bunching enhances the radiation, leads to exponential growth in the radiated field until saturation is reached.

We consider the free electron laser to be described by the Hamiltonian

$$H = \sum_{j=1}^{N} c\sqrt{(\vec{\pi}_j - \frac{e}{c}\vec{A}(z_j,t))^2 + m^2c^2} + \frac{1}{2}\sum_k(P_k^2 + \omega_k^2 Q_k^2) \ , \qquad (45)$$

where the canonical electron momentum $\vec{\pi}_j$ is given by

$$\vec{\pi}_j = \vec{p}_j + \frac{e}{c}\vec{A}(z_j,t) \ . \qquad (46)$$

Here $j = 1, \ldots, N$ labels the different electrons in the beam, $\vec{p}_j = \gamma m\vec{v}_j$, and $\omega_k = kc$. We take

$$A_z = 0 \quad \text{so} \quad \pi_{jz} = p_{jz} \qquad (47)$$

and

$$\vec{\pi}_{j_\perp} = 0 \quad \text{so} \quad \vec{p}_{j_\perp} - \frac{e}{c}\vec{A}_\perp = 0 \ , \qquad (48)$$

hence the Hamiltonian simplifies to

$$H = \sum_{j=1}^{N} c\sqrt{p_j^2 + \frac{e^2}{c^2}\vec{A}^2(z_j,t) + m^2c^2} + \frac{1}{2}\sum_k(P_k^2 + \omega_k^2 Q_k^2) \ , \qquad (49)$$

using the notation $p_j = p_{jz}$.

The vector potential can be written

$$\vec{A}(z,t) = \vec{A}_w(z) + \vec{A}_L(z,t) \ , \qquad (50)$$

with the vector potential of the static wiggler field

$$\vec{A}_w(z) = A_w(\hat{x}\cos k_w z + \hat{y}\sin k_w z) \qquad (51)$$

and that of the laser field

$$\vec{A}_L(z,t) = \sum_k G_k\{(\frac{\hat{x} + i\hat{y}}{\sqrt{2}})(\frac{\omega_k Q_k + iP_k}{\sqrt{2\omega_k}})\, e^{ikz} + \text{c.c.}\} \ . \qquad (52)$$

The EM-field is considered to be confined to a volume V, and the normalization factor G_k is given by

$$G_k = (2\pi\omega_k/k^2V)^{1/2} \ . \qquad (53)$$

Our discussion is one-dimensional since we do not consider the tranverse variation of the vector potential, and we consider only the electromagnetic field radiated in the forward direction.

It is now convenient to make a canonical transformation[36] to the interaction picture, i.e. we transform from the original dynamical variables describing the laser field to new action-angle variables I_k, ϕ_k using the generating function

$$F = -\sum_k \frac{1}{2}\omega_k Q_k^2 \tan(\omega_k t + \phi_k) . \tag{54}$$

Then

$$P_k = \frac{\partial F}{\partial Q_k} , \tag{55}$$

$$I_k = -\frac{\partial F}{\partial \phi_k} , \tag{56}$$

$$\hat{H} = H - \sum_k \omega_k I_k . \tag{57}$$

It follows that

$$G_k \frac{\omega_k Q_k + iP_k}{\sqrt{2\omega_k}} = G_k \sqrt{I_k}\, e^{-i(\omega_k t+\phi_k)} \equiv a_k\, e^{-i\omega_k t} . \tag{58}$$

The key point is that the fast time variation, $\exp(-i\omega_k t)$, has been factored out, and a_k has only the slow time variation.

The vector potential of the laser field can now be written

$$\vec{A}_L(z,t) = \sum_k G_k \sqrt{2I_k}\left(\hat{x}\cos(kz - \omega_k t - \phi_k) - \hat{y}\sin(kz - \omega_k t - \phi_k)\right)$$

$$= \sum_k \left[\left(\frac{\hat{x} + i\hat{y}}{\sqrt{2}}\right) a_k\, e^{i(kz-\omega_k t)} + \text{c.c.}\right] \tag{59}$$

and the transformed Hamiltonian becomes

$$\hat{H} = \sum_{j=1}^{N} c\sqrt{p_j^2 + \frac{e^2}{c^2}(\vec{A}_w^2 + 2\vec{A}_w\cdot\vec{A}_L + \vec{A}_L^2) + m^2c^2} . \tag{60}$$

For the purpose of our discussion we shall neglect the term $\vec{A}_L^2$. From Eqs. (51) and (59) we see that

$$2\vec{A}_w \cdot \vec{A}_L = 2A_w \sum_k G_k \sqrt{2I_k}\cos(\psi_{kj} - \phi_k)$$

$$= \sqrt{2}\, A_w \sum_k (a_k\, e^{i\psi_{kj}} + a_k^*\, e^{-i\psi_{kj}}) \tag{61}$$

with

$$\psi_{kj} \equiv (k + k_w)z_j - \omega_k t . \tag{62}$$

Now introducing the effective mass

$$M^2c^2 = m^2c^2 + \frac{e^2}{c^2} A_w^2 , \tag{63}$$

the Hamiltonian in the interaction representation is

$$\tilde{H} = c \sum_{j=1}^{N} \sqrt{p_j^2 + M^2c^2 + \sqrt{2}\,\frac{e^2}{c^2} A_w \sum_k (a_k e^{i\psi_{kj}} + a_k^* e^{-i\psi_{kj}})} \ . \qquad (64)$$

The equations of motion are given by

$$\dot{p}_j = -\frac{\partial \tilde{H}}{\partial z_j} , \qquad (\dot{\gamma}_j \simeq p_j/mc) \qquad (65)$$

$$\dot{z}_j = \frac{\partial \tilde{H}}{\partial p_j} , \qquad (66)$$

$$\dot{a}_k = -iG_k^2 \frac{\partial \tilde{H}}{\partial a_k^*} . \qquad (67)$$

The equations of Bonifacio, Pellegrini, and Narducci[17] (keeping all k-values) follow if one neglects[37] the effect of the radiation field on the electron trajectory. Introducing the Fourier coefficient of the electric field

$$c_k = ik\, a_k , \qquad (68)$$

the magnetic field strength parameter

$$K = eA_w/mc^2 , \qquad (69)$$

and the average density of the electron beam

$$n_o = N/V , \qquad (70)$$

it is straightforward to derive

$$\dot{\psi}_{kj} = \omega_w \left(1 - \frac{k}{k_w} \frac{1 + K^2}{2\gamma_j^2}\right) , \qquad (71)$$

$$\dot{\gamma}_j = -\frac{ecK}{\sqrt{2}mc^2\gamma_j} \sum_k (c_k e^{i\psi_{kj}} + c_k^* e^{-i\psi_{kj}}) , \qquad (72)$$

$$\dot{c}_k = \frac{2\pi ecKn_o}{\sqrt{2}} \frac{1}{N} \sum_{j=1}^{N} \frac{e^{-i\psi_{kj}}}{\gamma_j} . \qquad (73)$$

Let us consider solving these equations with the initial conditions

$$\gamma_j = \gamma_o \qquad (j = 1, \ldots, N) , \qquad (74)$$

$$\psi_{kj} = \psi_{kj}^{(0)} = (k + k_w)z_j^{(0)} \quad (j = 1, \ldots, N) \ . \tag{75}$$

We shall linearize the equations by introducing θ_{kj}, η_j, and A_k, which will be considered small quantities, according to the definitions

$$\psi_{kj} = \psi_{kj}^{(0)} + \omega_w(1 - \frac{k}{k_o})t + \theta_{kj} \ , \tag{76}$$

$$\gamma_j = \gamma_o(1 + \eta_j) \quad , \tag{77}$$

$$c_k = (2\pi\gamma_o mc^2 n_o \rho)^{1/2} \, e^{-i\omega_w(1-\frac{k}{k_o})t} A_k \ , \tag{78}$$

where

$$k_o = \frac{2\gamma_o^2 k_w}{1 + K^2} \tag{79}$$

is the resonant wave number corresponding to the initial energy γ_o and

$$\rho^3 = \frac{4\pi e^2 n_o K^2}{16\gamma_o^3 m\omega_w^2} \tag{80}$$

is the Pierce parameter.

The quantities θ_{kj}, η_j, and A_k are taken to be initially small, and we keep only lowest-order terms in them. We introduce collective variables[17]

$$x_k = \frac{1}{N}\sum_{j=1}^{N} \theta_{kj} \, e^{-i\psi_{kj}^{(0)}} \tag{81}$$

and

$$y_k = \frac{1}{N}\sum_{j=1}^{N} \frac{\eta_j}{\rho} \, e^{-i\psi_{kj}^{(0)}} \ . \tag{82}$$

The linearized version of Eqs. (71) to (73) becomes

$$\frac{dx_k}{d\tau} = (1 - 2\rho\delta_k)y_k \ , \tag{83}$$

$$\frac{dy_k}{d\tau} = -\sum_p \left(A_p f_{k-p} + A_p^* f_{k+p+2k_w}\right) \ , \tag{84}$$

$$\frac{dA_k}{d\tau} = i\delta_k A_k - ix_k - \rho y_k + f_{k+k_w} \ , \tag{85}$$

where

$$\tau = 2\rho\omega_w t \ , \tag{86}$$

$$\delta_k = (1 - \frac{k}{k_o})/2\rho \ , \tag{87}$$

and

$$f_k = \frac{1}{N} \sum_{j=1}^{N} e^{ikz_j^{(0)}} \tag{88}$$

is the Fourier transform of the initial spatial distribution of the electron beam.

Note that the electron beam distribution enters into the linearized equations in two ways. In Eq. (85), the term f_{k+k_w} acts as a driving term for the laser field A_k. For the very short radiation wavelengths in which we are interested, the Fourier transform of the bunch density is dominated by the high frequency shot noise due to the discrete nature of the individual electrons. However, the terms f_{k-p} and f_{k+p+2k_w} appearing in Eq. (84) play a different role. They introduce a coupling between different Fourier components of the laser field due to the finite bunch length of the electron beam. The magnitude of f_k is largest for small k, less than the inverse bunch length, so coupling between A_k and $A_{k'}$ will be negligible if $|k - k'| \gg 1/\sigma$ where σ is the bunch length. Let us denote the smooth distribution with the finite bunch length σ by D_k, so

$$f_k = D_k + \text{shot noise} \ . \tag{89}$$

Then Eq. (84) becomes to good approximation

$$\frac{dy_k}{d\tau} \simeq - \sum_p A_p D_{k-p} \ . \tag{90}$$

For a Gaussian bunch $D_k = \exp(-k^2\sigma^2/2)$ and for a uniform coasting beam

$$D_k = \delta(k) \ . \tag{91}$$

Restricting ourselves now to a uniform coasting beam the linearized equations decouple:

$$x_k' = (1 - 2\rho\delta_k)y_k \qquad (x_k' = dx_k/d\tau) \ , \tag{92}$$

$$y_k' = -A_k \ , \tag{93}$$

$$A_k' = i\delta_k A_k - ix_k - \rho y_k + f_{k+k_w} \ . \tag{94}$$

The radiated field is determined by

$$A_k''' = i\delta_k A_k'' + \rho A_k' + i(1 - 2\rho\delta_k)A_k \ , \tag{95}$$

which follows from twice differentiating Eq. (94) and using Eqs. (92) and (93). The term $2\rho\delta_k$ can be neglected. Solving Eq. (95) by Laplace transform, we obtain

$$A_k(t) = e^{i\mu_1(2\rho\omega_w t)} \frac{\mu_1^2 A_k(0) - i\mu_1 f_{k+k_w}}{(\mu_1 - \mu_2)(\mu_1 - \mu_3)}$$

$$+ e^{i\mu_2(2\rho\omega_w t)} \frac{\mu_2^2 A_k(0) - i\mu_2 f_{k+k_w}}{(\mu_2 - \mu_1)(\mu_2 - \mu_3)}$$

$$+ e^{i\mu_3(2\rho\omega_w t)} \frac{\mu_3^2 A_k(0) - i\mu_3 f_{k+k_w}}{(\mu_3 - \mu_1)(\mu_3 - \mu_2)} , \qquad (96)$$

where μ_1, μ_2, μ_3 are the three solutions of the dispersion relation

$$\mu^3 - \delta_k \mu^2 + \rho\mu + 1 = 0 . \qquad (97)$$

The terms in Eq. (96) proportional to $A_k(0)$ describe the amplification of an external EM-wave, while those proportional to f_{k+kw} correspond to self-amplified spontaneous radiation. In what follows we shall carry out the analysis of the self-amplified spontaneous radiation and leave it as an exercise to the reader to work out the amplification of an external wave.[38] However, in Appendix B, we recover the result of Eq. (40) for the gain G in the low-gain regime, from the general expression given in Eq. (96).

The energy in the radiation field is determined by

$$\varepsilon = \frac{V}{2\pi} \sum_{\vec{k}} k^2 a^*_{\vec{k}} a_{\vec{k}} = \frac{V}{2\pi} \int \frac{V d^3k}{(2\pi)^3} k^2 a^*_{\vec{k}} a_{\vec{k}}$$

$$= \frac{V^2}{(2\pi)^4} \int dk d\Omega k^4 a^*_{\vec{k}} a_{\vec{k}} = \int d\omega d\Omega \frac{dI(\omega)}{d\Omega} .$$

Hence, the energy radiated per unit solid angle per unit frequency interval in the forward direction is given by

$$\left.\frac{dI(\omega)}{d\Omega}\right|_{\theta=0} = \frac{V^2}{(2\pi)^4 c} k^4 a^*_k a_k . \qquad (98)$$

Employing $c_k = ika_k$ and the relation (78) between c_k and A_k, we find

$$\left.\frac{dI(\omega)}{d\Omega}\right|_{\theta=0} = \frac{NV}{(2\pi)^3 c} (\gamma_o mc^2\rho) k^2 |A_k|^2 . \qquad (99)$$

Let us now recover the spectral distribution for spontaneous radiation discussed earlier in Section 2 and given in Eq. (9). Consider $\rho \to 0$ and $\delta_k = (1 - k/k_o)/2\rho \to \infty$, and assume that at time

$t = 0$ the radiation field vanishes, $A_k(0) = 0$. Then the solutions of the dispersion relation (97) are to good approximation given by

$$\mu_1 \simeq \delta_k , \qquad \mu_2 \simeq 1/\sqrt{\delta_k} , \qquad \text{and} \qquad \mu_3 \simeq -1/\sqrt{\delta_k} . \tag{100}$$

It then follows from Eq. (96) that

$$A_k(t) \simeq \frac{e^{2i\delta_k \rho\omega_w t} - 1}{i\delta_k} f_{k+k_w} , \tag{101}$$

hence from Eq. (99) we get

$$\left.\frac{dI(\omega)}{d\Omega}\right|_{\theta=0} = \frac{e^2 K^2 \omega^2}{8\pi^2 c\gamma_o^2} \frac{4\sin^2\left((1 - \frac{k}{k_o}) \frac{\omega_w t}{2}\right)}{(1 - \frac{k}{k_o})^2 \omega_w^2} \left| \sum_{j=1}^{N} e^{-i(k+k_w)z_j^{(0)}} \right|^2 , \tag{102}$$

the correct result.

To study the self-amplification of the spontaneous radiation, we consider $\rho \to 0$, $\delta_k \to 0$. For $\rho = \delta_k = 0$, the dispersion relation (97) becomes

$$(\mu^{(0)})^3 + 1 = 0 , \tag{103}$$

hence

$$\mu^{(0)} = e^{-i\pi/3}, \ e^{i\pi/3}, \ -1 . \tag{104}$$

Using the expansion

$$\mu = \mu^{(0)} + \mu^{(1)}\delta_k + \mu^{(2)}\delta_k^2 + \dots , \tag{105}$$

one finds

$$\mu^{(1)} = 1/3 , \qquad \mu^{(2)} = 1/(9\,\mu^{(0)}) . \tag{106}$$

The fastest growing mode $\left(\exp(i\mu_1\tau)\right)$ corresponds to

$$\mu_1 = \frac{1}{2} + \frac{\delta_k}{3} + \frac{\delta_k^2}{18} - i\,\frac{\sqrt{3}}{2}\,(1 - \frac{\delta_k^2}{9}) , \tag{107}$$

and for sufficiently long times this exponentially growing mode will dominate. It follows from Eq. (96) that

$$|A_k|^2 = \frac{1}{9}|f_{k+k_w}|^2 \, e^{\sqrt{3}\tau} \, e^{-\frac{\sqrt{3}}{9}\tau \left(\frac{k-k_o}{2\rho k_o}\right)^2} , \tag{108}$$

where k_o was defined in Eq. (79). From Eq. (88) it is seen that

$$|f_{k+k_w}|^2 = \frac{1}{N^2} \left| \sum_{j=1}^{N} e^{-i(k+k_w)z_j^{(0)}} \right|^2 \sim \frac{1}{N} , \tag{109}$$

where the approximate equality to 1/N holds assuming the initial electron positions $z_j^{(0)}$ (j = 1, ..., N) are uncorrelated and randomly distributed. Using Eq. (99) for $dI(\omega)/d\Omega$, together with Eqs. (108) and (109), we find the energy radiated per unit solid angle about the forward direction:

$$\left.\frac{dW}{d\Omega}\right|_{\theta=0} = \int_0^\infty d\omega \left.\frac{dI(\omega)}{d\Omega}\right|_{\theta=0}$$

$$= \frac{V}{(2\pi)^3}(\gamma_o mc^2\rho)\,\frac{1}{9}\,e^{\sqrt{3}\tau}\,k_o^2\int_{-\infty}^{\infty}dk\;e^{-(k-k_o)^2/2\sigma_k^2}\,. \tag{110}$$

The radiation bandwidth σ_k is given by[18,19]

$$2\sigma_k^2 = \frac{9(2\rho k_o)^2}{\sqrt{3}\tau}\,. \tag{111}$$

An estimate of the total energy radiated in the self-amplified spontaneous emission process has been given by Kim.[18] He assumes the electron beam to have cross-sectional area A, and the solid angle corresponding to diffraction is

$$\Omega_D = \lambda_o^2/A\,, \tag{112}$$

where $\lambda_o = 2\pi/k_o$ is the radiated wavelength. The radiated energy ε is approximated by

$$\varepsilon = \Omega_D \left.\frac{dW}{d\Omega}\right|_{\theta=0}\,, \tag{113}$$

with $(dW/d\Omega)_{\theta=0}$ given by Eq. (110). Defining

$$\int_{-\infty}^{\infty}dk\;e^{-(k-k_o)^2/2\sigma_k^2} = \sqrt{2\pi}\,\sigma_k = \Delta k = 2\pi/\ell_c\,, \tag{114}$$

with ℓ_c called the coherence length, Eqs. (110) to (114) yield[18]

$$\varepsilon = \frac{1}{9N_c}(\rho N\gamma_o mc^2)\,e^{\sqrt{3}\tau}\,, \tag{115}$$

where $N_c = A\ell_c N/V$ is the number of electrons in a coherence length.

This estimate of the total radiated energy is the result of a one-dimensional calculation. Recent work has taken into account some three-dimensional effects.[39]

The radiation bandwidth σ_k (Eq. (111)) has recently been discussed by Kim[18] and by Wang and Yu.[19] Using $\tau = 4\pi\rho N_w$ in Eq. (111) one obtains

$$\frac{\sigma_k}{k_o} = \left(\frac{3\sqrt{3}\rho}{2\pi N_w}\right)^{1/2}$$

where N_w is the number of wiggler periods. The radiation bandwith is observed to decrease as $1/N_w^{1/2}$ for self-amplified spontaneous emission, rather than as $1/N_w$ for the spontaneous radiation itself.

We shall close this section with the derivation of a relation between the total energy radiated in self-amplified spontaneous radiation and the energy spread induced in the electron beam. Recall the definition of the collective variable y_k in Eq. (82),

$$y_k = \frac{1}{N} \sum_{j=1}^{N} \frac{\eta_j}{\rho} e^{-i(k+k_w)z_j^{(0)}} ,$$

where $\eta_j = (\gamma_j - \gamma_o)/\gamma_o$. Summing over k we find

$$\sum_k |y_k|^2 = \frac{1}{N} \sum_{j=1}^{N} \frac{\eta_j^2}{\rho^2} = \langle(\frac{\Delta\gamma}{\rho\gamma})^2\rangle . \tag{116}$$

Now use Eq. (93),

$$y_k' = -A_k ,$$

and note that in the exponential regime where one mode dominates,

$$y_k = A_k/i\mu_1 . \tag{117}$$

Using Eqs. (116), (117), and $|\mu_1| \sim 1$, it follows that

$$\begin{aligned} \langle(\frac{\Delta\gamma}{\rho\gamma_o})^2\rangle &= \sum_k |A_k|^2 = \int \frac{Vk^2 dk d\Omega}{(2\pi)^3} |A_k|^2 \\ &= \Omega_D \int \frac{Vk^2 dk}{(2\pi)^3} |A_k|^2 . \end{aligned} \tag{118}$$

The total radiated energy is

$$\varepsilon = \Omega_D(\rho N\gamma_o mc^2) \int \frac{Vk^2 dk}{(2\pi)^3} |A_k|^2 , \tag{119}$$

therefore,

$$\varepsilon = \rho(N\gamma_o mc^2) \langle(\frac{\Delta\gamma}{\rho\gamma_o})^2\rangle . \tag{120}$$

Recently Eq. (120) has been independently derived from the Vlasov-Maxwell equations by Kim.[18]

From Eq. (120) we see that energy ε can be radiated only with an associated increase in the energy spread $\langle(\Delta\gamma)^2\rangle$ of the electron beam. Exponential growth cannot continue indefinitely, and saturation is expected when

$$\langle(\frac{\Delta\gamma}{\gamma_o})^2\rangle \simeq \rho^2 , \tag{121}$$

hence from Eq. (120) it is seen that the limiting value of the radiated energy at saturation is expected to be

$$\varepsilon \simeq \rho(N\gamma_0 mc^2) \ . \qquad (122)$$

Therefore, ρ is the efficiency of energy extraction from the electron beam whose initial total energy is $N\gamma_0 mc^2$.

5. CONCLUDING REMARKS

At this point we conclude our brief introduction to the theory of free electron lasers. Experimentally, these devices offer exciting promise as sources of coherent radiation with wavelengths from millimeters to soft x-rays. Theoretically, the study of free electron lasers provides a very clean, but challenging, set of problems basic to the collective interaction of electrons and the electromagnetic field. Of great importance is the further development of three-dimensional treatments, in particular the elucidation of the guiding[40,41] of the generated radiation by the electron beam. Also, theories of the saturation[42] of free electron lasers must be carried forward. Clearly, important correspondences exist between the theory of free electron lasers and the description of such important accelerator physics problems as coherent instabilities and new acceleration schemes. We may hope that techniques developed in one area will find application in the other.

ACKNOWLEDGMENTS

I wish to thank J. Murphy and C. Pellegrini for introducing me to the physics of the high-gain regime. I am grateful to K.J. Kim, J.M. Wang and L.H. Yu for clarifying discussions on the coupled Vlasov-Maxwell equations.

REFERENCES

1. See e.g. J.D. Jackson, Classical Electrodynamics (Wiley, New York, 1975).
2. J.M.J. Madey, J. Appl. Phys. 42, 1906 (1971).
3. Much information on free electron lasers can be found in Physics of Quantum Electronics (Addison-Wesley, Reading, MA, 1978-1982) Vols. 5 and 7-9. Reviews oriented toward accelerator physicists have been given by C. Pellegrini, IEEE Trans. Nucl. Sci. NS-26, 3791 (1979), and P. Morton, Ibid. NS-28, 3125 (1981).
4. For background material on wigglers see Proc. Wiggler Workshop, SLAC, Eds. H. Winick and T. Knight, SSRP Report No. 77/05 (1977).
5. See also S. Krinsky, M.L. Perlman and R.E. Watson, Characteristics of synchrotron radiation and of its sources, in Handbook on Synchrotron Radiation, Vol. 1a, Ed. E.E. Koch (North-Holland,

Amsterdam, 1983); S. Krinsky, IEEE Trans. Nucl. Sci. NS-30, 3078 (1983).
6. H. Motz, J. Appl. Phys. 22, 527 (1951).
7. D.F. Alferov, Yu. Bashmakov and E.G. Bessonov, Eng. Transl. in Sov. Phys. Tech. Phys. 18, 1336 (1974).
8. B.M. Kincaid, J. Appl. Phys. 48, 2684 (1977).
9. Optical gain of infrared radiation from a carbon dioxide laser was observed by L.R. Elias, W.M. Fairbank, J.M.J. Madey, H.A. Schwettman and T.I. Smith, Phys. Rev. Lett. 36, 717 (1976).
10. Operation of a free electron laser oscillator was achieved by D.A.G. Deacon, L.R. Elias, J.M.J. Madey, G.J. Ramian, H.A. Schwettman and T.I. Smith, Phys. Rev. Lett. 38, 892 (1977).
11. N.M. Kroll and W.A. McMullin, Phys. Rev. A17, 300 (1978).
12. A. Gover and Z. Livni, Opt. Commun. 26, 375 (1978).
13. I.B. Bernstein and J.L. Hirshfeld, Phys. Rev. A20, 1661 (1979).
14. P. Sprangle, C.M. Tang and W.M. Manheimer, Phys. Rev. A21, 302 (1980).
15. C.C. Shih and A. Yariv, IEEE J. Quant. Electron. QE-17, 1387 (1981).
16. G. Dattoli, A. Marino, A. Renieri and F. Romanelli, IEEE J. Quant. Electron. QE-17, 1371 (1981).
17. R. Bonifacio, C. Pellegrini and L.M. Narducci, Opt. Commun. 50, 373 (1984).
18. K.J. Kim, in Proc. 7th Int. Free Electron Laser Conf., Granlibakken, CA (1985).
19. J.M. Wang and L.H. Yu, Ibid.
20. F.A. Hopf, P. Meystere, M.O. Scully and W.H. Louisell, Opt. Commun. 18, 413 (1976).
21. See e.g. the contributed papers to the Castelgandolfo 1984 FEL Conference, Nucl. Inst. Meth. A237 (1984).
22. M. Billardon, P. Elleaume, J.M. Ortega, C. Bazin, M. Bergher, M. Velghe, Y. Petroff, D.A.G. Deacon, K.E. Robinson and J.M.J. Madey, Phys. Rev. Lett. 51, 18, 1652 (1983).
23. R. Barbini et al., Proc. 1982 Bendor Free Electron Laser Conf., J. Physique, Colloque Cl, Suppl. to Vol. 44, No. 2 (1983).
24. J.M.J. Madey, in Free Electron Generation of Extreme Ultraviolet Coherent Radiation, Eds. J.M.J. Madey and C. Pellegrini, AIP Conf. Proc. 118, 12 (1984).
25. J. Bisognano, S. Chattopadhyay, M. Cornacchia, A. Garren, A. Jackson, K. Halbach, K.J. Kim, H. Lancaster, J. Peterson, M.S. Zisman, C. Pellegrini and G. Vignola, Particle Accelerators, in press.
26. See e.g. J.B. Murphy and C. Pellegrini in Ref. 21.
27. T.J. Orzechowski, B. Anderson, W.M. Fawley, D. Prosnitz, E.T. Scharlemann, S. Yarema, D. Hopkins, A.C. Paul, A.M. Sessler and J. Wurtele, Phys. Rev. Lett. 54, 889 (1985).
28. W.B. Colson, in Physics of Quantum Electronics, Vol. 5 (Addison-Wesley, Reading, MA, 1978).
29. W.B. Colson and S.K. Ride, in Physics of Quantum Electronics, Vol. 7 (Addison-Wesley, Reading, MA, 1980).
30. P. Luchini and S. Solimeno, Nucl. Inst. Meth. A237, 84 (1984).
31. J.M.J. Madey, Nuovo Cimento B50, 64 (1979).

32. N.M. Kroll, in Physics of Quantum Electronics, Vol. 8 (Addison-Wesley, Reading, MA, 1982).
33. S. Krinsky, J.M. Wang and P. Luchini, J. Appl. Phys. 53, 5453 (1982).
34. See e.g. S. Krinsky, Phys. Rev. A31, 1267 (1985).
35. L.D. Landau and E.M. Lifshitz, The Classical Theory of Fields (Addison-Wesley, Reading, MA, 1962).
36. This transformation is used by A. Bambini, A. Renieri and S. Stenholm, Phys. Rev. A19, 2013 (1979). However, their work is in a moving reference system, not the laboratory system.
37. The neglected terms have been shown to be significant for sufficiently large values of the Pierce parameter ρ by I. Gjaja and A. Bhattacharjee, Optics Commun. 58, 201 (1986).
38. See e.g. E. Jerby and A. Gover, IEEE J. Quant. Electron. QE-21, 1041 (1985).
39. L.H. Yu and S. Krinsky, in Proc. AIP Conf. on Short Wavelength Coherent Radiation, Monterey, CA, 1986; K.J. Kim, LBL preprint (1986); S.Krinsky and L.H. Yu, BNL preprint (1986).
40. G.T. Moore, in Proc. Int. Workshop on Coherent and Collective Properties of Electrons and Electron Radiation, Como, Italy, 1984, Nucl. Inst. Meth. A239, 19 (1985).
41. E.T. Scharlemann, A.M. Sessler and J.S. Wurtele, Phys. Rev. Lett. 54, 1925 (1985).
42. See e.g. R. Bonifacio, F. Casagrande and L. DeSalvo, Phys. Rev. A33, 2836 (1986).

APPENDIX A

RELATION BETWEEN DISTRIBUTION AND COLLECTIVE VARIABLES

Let $z_j(t)$ be the position and $\eta_j(t) = [\gamma_j(t) - \gamma_o]/\gamma_o$ be the energy deviation of the j^{th} electron at time t, where j = 1, ..., N labels the different electrons. Consider the distribution F defined by

$$F(\eta,z,t) = \frac{1}{N}\sum_{j=1}^{N} \delta\left(\eta - \eta_j(t)\right)\,\delta\left(z - z_j(t)\right) . \qquad \text{(A1)}$$

The collective variables introduced in Section 4 are

$$x_k = \frac{1}{N}\sum_{j=1}^{N} \theta_{kj}\, e^{-i\psi_{kj}^{(0)}}$$

and

$$\rho y_k = \frac{1}{N}\sum_{j=1}^{N} \eta_j\, e^{-i\psi_{kj}^{(0)}} .$$

The phase ψ_{kj} was defined by

$$\psi_{kj} = (k + k_w)z_j(t) - kct .$$

We now write the time variation of the position $z_j(t)$ as

$$z_j(t) = z_j^{(0)} + \beta_o ct + \zeta_j(t) , \quad \text{(A2)}$$

where $z_j^{(0)}$ is the initial position at $t = 0$ of the j^{th} electron, β_o is the longitudinal velocity in the absence of the radiation field [see Eq. (6)], and $\zeta_j(t)$ describes the bunching due to the radiation. The phase θ_{kj} is given by $\theta_{kj} = (k + k_w)\zeta_j(t)$.

The time evolution of ψ_{kj} is

$$\begin{aligned} \psi_{kj} &= (k + k_w)\left[z_j^{(0)} + \beta_o ct + \zeta_j(t)\right] - kct \\ &= (k + k_w)z^{(0)} + \omega_w(1 - \frac{k}{k_o})t + (k + k_w)\zeta_j(t) \\ &= \psi_{kj}^{(0)} + \dot{\psi}_{ko}t + \theta_{kj}(t) . \end{aligned} \quad \text{(A3)}$$

Defining the spatial distribution

$$f(z,t) = \int d\eta F(\eta,z,t) = \frac{1}{N}\sum_{j=1}^{N} \delta\left[z - z_j(t)\right] , \quad \text{(A4)}$$

and using the Fourier expansion for the delta-function, one obtains

$$f(z,t) = \int\frac{dk}{2\pi}\, e^{i(k+k_w)(z-\beta_o ct)}\, \frac{1}{N}\sum_{j=1}^{N} e^{-i(k+k_w)z_j^{(0)}}\, e^{-i\theta_{kj}(t)} . \quad \text{(A5)}$$

In the linear regime, $\exp(-i\theta_{kj}) \simeq 1 - i\theta_{kj}$, hence

$$f(z,t) \simeq \int\frac{dk}{2\pi}\, e^{i(k+k_w)(z-\beta_o ct)}\left(f_k^{(0)} - ix_k\right) , \quad \text{(A6)}$$

where x_k is the collective variable of Eq. (81) and

$$f_k^{(0)} = \frac{1}{N}\sum_{j=1}^{N} e^{-i(k+k_w)z_j^{(0)}} \quad \text{(A7)}$$

is the Fourier transform of the initial spatial distribution of the electrons. Equation (A6) shows that the collective variable x_k is simply related to the Fourier transform of the zeroth moment, $\int d\eta F(\eta,z,t)$, of the distribution F.

Now consider the first moment,

$$\int d\eta\, \eta\, F(\eta,z,t) = \frac{1}{N}\sum_{j=1}^{N} \eta_j\, \delta\left[z - z_j(t)\right] . \quad \text{(A8)}$$

Again employing the Fourier expansion for the delta-function, one finds

$$\int d\eta\ \eta\ F(\eta,z,t) = \int \frac{dk}{2\pi}\, e^{i(k+k_w)(z-\beta_o ct)} \frac{1}{N} \sum_{j=1}^{N} \eta_j e^{-i\psi_{kj}^{(0)}} e^{-i\theta_{kj}} \ . \quad \text{(A9)}$$

Keeping only the first-order term,

$$\int d\eta\ \eta F(\eta,z,t) \simeq \int \frac{dk}{2\pi}\, e^{i(k+k_w)(z-\beta_o ct)} \rho y_k \ , \quad \text{(A10)}$$

showing that ρy_k is simply the Fourier transform of the first moment of the distribution F.

An excellant approach to the theory of free electron lasers is the use of the coupled Vlasov-Maxwell equations.[11,20] The Vlasov equation for the distribution F is

$$\frac{\partial F}{\partial t} + \dot{z}\,\frac{\partial F}{\partial z} + \dot{\eta}\,\frac{\partial F}{\partial \eta} = 0 \ . \quad \text{(A11)}$$

The rate of change of the energy is given by

$$\dot{\eta} = \frac{e}{\gamma_o mc^2}\, \vec{v}_\perp \cdot \vec{E} \ , \quad \text{(A12)}$$

and $\dot{z} \simeq \beta_o c$, where the longitudinal and transverse velocities were given earlier in Eqs. (6 and 7). The electric field $\vec{E}$ is determined from the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \vec{E} = \frac{4\pi}{c^2}\frac{\partial \vec{j}}{\partial t} \ , \quad \text{(A13)}$$

and the loop is closed by expressing the current $\vec{j}$ in terms of the distribution according to

$$\vec{j} = e\int \vec{v}_\perp \cdot F\, d\eta \ . \quad \text{(A14)}$$

We shall not consider the analysis of these equations, but simply refer the reader to the literature. The use of the Vlasov equation has received much attention, and as we shall show, the results obtained in the linear regime of exponental growth are equivalent to those of Section 4. Having two formulations of the problem may provide some additional insight in the study of the saturation of the instability due to nonlinearities.

Since the collective variables x_k and y_k have been shown to be simply Fourier components of the zeroth and first moments of the distribution, the equations satisfied by these variables can in fact be derived by taking moments of the Vlasov equation. Although we shall not explicitly do this here, we shall show that the partial differential equation for the slowly varying amplitude of the radiated electric field derived by Wang and Yu[19] from the Vlasov equation can be derived from the equations of Section 4. Recall the linearized equations:

$$x_k' = (1 - 2\rho\delta_k)y_k\ , \qquad y_k' = -\sum_p A_p D_{k-p}$$

$$A_k' = i\delta_k A_k - ix_k - \rho y_k + f_{k+k_w}\ .$$

The prime denotes $\partial/\partial\tau = (2\rho\omega_w)^{-1}\,\partial/\partial t$. Differentiating Eq. (85) twice and employing Eqs. (83) and (90), we obtain

$$A_k''' = i\delta_k A_k'' + \rho\sum_p A_p' D_{k-p} + i(1 - 2\rho\delta_k)\sum_p A_p D_{k-p}\ . \qquad \text{(A15)}$$

Remember that D_k is the Fourier transform of the initial bunch distribution, ignoring the high frequency shot noise [see Eq. (89)], and $\delta_k = (k_o - k)/2k_o\rho$.

The radiated electric field is determined by A_k via

$$\vec{E} = \sqrt{2\pi\gamma_o mc^2 n_o\rho}\ e^{ik_o(z-ct)}\sum_k \left(\frac{\hat{x} + i\hat{y}}{\sqrt{2}}\, A_k\, e^{i(k-k_o)(z-v_ot)} + \text{c.c}\right), \text{(A16)}$$

where $v_o = \beta_o c = c\left[1 - (1 + K^2)/2\gamma_o^2\right]$ and $k_o = k_w/(1 - \beta_o)$. Equation (A16) follows directly from Eqs. (59), (68), and (78). To make contact with Wang and Yu,[19] let us introduce the slowly varying envelope function

$$\tilde{E}(\xi,t) = \sum_k A_k(t)e^{i(k-k_o)\xi} = \sum_k A_k(t)e^{-2i\rho k_o\delta_k\xi}\ , \qquad \text{(A17)}$$

where

$$\xi = z - v_o t\ . \qquad \text{(A18)}$$

We also define

$$\tilde{D}(\xi) = \sum_k D_k e^{ik\xi}\ . \qquad \text{(A19)}$$

Then, noting that

$$\tilde{D}(\xi)\tilde{E}(\xi,t) = \sum_k \left(\sum_p A_p(t)D_{k-p}\right) e^{i(k-k_o)\xi}\ , \qquad \text{(A20)}$$

it is straightforward to derive the following partial differential equation for $\tilde{E}(\xi,t)$ from Eq. (A15):

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial t} + (c - v_o)\frac{\partial}{\partial\xi}\right)\tilde{E} = \omega_w^2(2\rho)^3\left(\frac{1}{2}\frac{\partial}{\partial t} + (c - v_o)\frac{\partial}{\partial\xi} + i\omega_w\right)(\tilde{D}\tilde{E}). \qquad \text{(A21)}$$

If one replaces the factor of 1/2 on the right-hand side of (A21) by unity, one obtains the envelope equation derived by Wang and Yu[19] from the Vlasov-Maxwell equations. The term with the slightly differing coefficient is in fact a negligible term for our considerations, so the results of the two approaches are equivalent.

APPENDIX B

RECOVERY OF LOW-GAIN RESULTS FROM GENERAL EXPRESSION

In Section 3, stimulated emission was considered in the low-gain regime. The small signal gain G was calculated in Eq. (40), based on the approximation that the increment of the radiation field during a single pass can be neglected in determining the electron's energy loss. Later, in Section 4, a more general expression for the change in the electromagnetic field was derived [Eq. (96)], which takes into account the variation of the electromagnetic field within a single pass, which is necessary in the high-gain regime. Here, we show how to recover[17] the low-gain result from the general expression of Eq. (96), which we write in the form

$$A_k(\tau) = A_k(0) \left(\frac{\mu_1^2 e^{i\mu_1\tau}}{(\mu_1 - \mu_2)(\mu_1 - \mu_3)} + \frac{\mu_2^2 e^{i\mu_2\tau}}{(\mu_2 - \mu_1)(\mu_2 - \mu_3)} + \frac{\mu_3^2 e^{i\mu_3\tau}}{(\mu_3 - \mu_1)(\mu_3 - \mu_2)} \right) . \quad \text{(B1)}$$

We consider the Pierce parameter to be small, $\rho \ll 1$, and take $\delta_k \gg 1$, but $\rho\delta_k \ll 1$. [Recall the definition of δ_k in Eq. (87)]. The eigenfrequencies μ_1, μ_2, μ_3, can be determined from

$$\mu^3 - \delta_k \mu^2 + 1 = 0 , \quad \text{(B2)}$$

whose solutions are approximately given, for $\delta_k \gg 1$, by

$$\mu_1 \simeq \delta_k \left(1 - \frac{1}{\delta_k^3}\right) , \qquad \mu_2 \simeq \frac{1}{\sqrt{\delta_k}} \left(1 + \frac{1}{2\delta_k^{3/2}}\right) ,$$

$$\mu_3 \simeq \frac{1}{\sqrt{\delta_k}} \left(-1 + \frac{1}{2\delta_k^{3/2}}\right) . \quad \text{(B3)}$$

Using these results in Eq. (B1) , one finds

$$A_k(\tau) \simeq A_k(0) \left[\left(1 + \frac{2}{\delta_k^3}\right) e^{i\delta_k\tau} - \frac{1}{2\delta_k^{3/2}} \left(1 + \frac{2}{\delta_k^{3/2}}\right) e^{i\tau/\sqrt{\delta_k}} + \frac{1}{2\delta_k^{3/2}} \left(1 - \frac{2}{\delta_k^{3/2}}\right) e^{-i\tau/\sqrt{\delta_k}} \right] . \quad \text{(B4)}$$

The small signal gain is defined by

$$G = \frac{|A_k(\tau)|^2 - |A_k(0)|^2}{|A_k(0)|^2} , \tag{B5}$$

hence

$$G \lesssim \frac{4}{\delta_k{}^3} + \frac{\sin^2 (\tau/\sqrt{\delta_k})}{\delta_k{}^3} - \frac{2}{\delta_k{}^{3/2}} \sin \frac{\tau}{\sqrt{\delta_k}} \sin\delta_k\tau$$

$$- \frac{4}{\delta_k{}^3} \cos \frac{\tau}{\sqrt{\delta_k}} \cos\delta_k\tau \quad . \tag{B6}$$

The result of Eq. (40) is now obtained in the limit $\tau << \sqrt{\delta_k}$,

$$G \lesssim \frac{4}{\delta_k{}^3} \quad (1 - \frac{\delta_k\tau}{2} \sin \delta_k\tau - \cos \delta_k\tau) \quad . \tag{B7}$$

*Research carried out under the auspices of the U.S.Department of Energy under Contract No. DE-AC02-76CH00016.