REFERENCE

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

IC/86/142

INTERNATIONAL
ATOMIC ENERGY
AGENCY

UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION

ON THE PROLONGATION STRUCTURE AND BACKLUND TRANSFORMATION FOR NEW NON-LINEAR KLEIN-GORDON EQUATIONS

A. Roy Chowdhury

and

Jayashree Mukherjee

IC/86/142

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON THE PROLONGATION STRUCTURE AND BACKLUND TRANSFORMATION FOR NEW NON-LINEAR KLEIN-GORDON EQUATIONS *

A. Roy Chowdhury **
International Centre for Theoretical Physics, Trieste, Italy

and

Jayashree Mukherjee
High Energy Physics Division, Department of Physics,
Jadavpur University, Calcutta - 700 032, India.

ABSTRACT

We have considered the complete integrability of two nonlinear equations which are some kind of extensions of usual Sine-Gordon and Sinh-Gordon equations. The first one is of non-autonomous version of Sinh-Gordon system and the second is closely related to the usual Sine-Gordon theory. The first problem indicates how (x,t) dependent non-linear equations can be treated in the prolongation theory and how a Backlund map can be constructed. The second one is a variation of the usual Sine-Gordon equation and suggests that there may be other equations (similar to Sine-Gordon) which are completely integrable. In both the cases we have been able to construct the Lax pair. We then construct an auto-Backlund map by following the idea of Konno and Wadati, for the generation of multisolution states.

MIRAMARE - TRIESTE
July 1986

* To be submitted for publication.

** Permanent address: High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta 700 032, India.

INTRODUCTION

In recent years there have been various attempts to extend the class of nonlinear equations that are completely integrable [1]. Already there are some indications that some of these equations may possess non-autonomous extensions with the property of integrability retained [2]. Two famous equations for which such extensions are known are the KdV and Nonlinear Schrödinger Equation. Here we propose a non-autonomous generalization of Sinh-Gordon equation, which may be thought of as simulating the effect of non-uniformity, inhomogeneity or dissipation upon the initial integrable system. The other equation we discuss is really a variant of the Sine-Gordon system. Our present study serves two main purposes. Firstly it indicates how a non-autonomous version of nonlinear Klein-Gordon system can be constructed and secondly what are the other kinds of nonlinearity that can be accomodated in a nonlinear Klein-Gordon system without destroying the integrability.

FORMULATION

(a) We start with the discussion of the non-autonomous Sinh-Gordon equation. The equation we propose to study is:

$$u_{xt} = b(t)\sinh u - g(t)\,\partial_x (x u_x) \tag{1}$$

In order to study the integrability of this equation, we try to find the Lax pair associated with Eq. (1) through the technique of prolongation structure [3]. To proceed with the calculation we firstly define some set of independent variables. Let us set:

$$u_x = p \tag{2}$$

Then Eqs.(1) and (2) are equivalent to the following sets of differential forms under proper sectioning. These two forms are written as:

$$\begin{aligned}\Delta_1 &= du\wedge dt - p\,dx\wedge dt\\ \Delta_2 &= -dp\wedge dx - b(t)\sinh u\,dx\wedge dt\\ &\quad - g(t)\,p\,dx\wedge dt + g(t)\,x\,dp\wedge dt\end{aligned} \tag{3}$$

It is not difficult to observe that these set of forms are closed under exterior differentiation, that is:

$$d\Delta_i = \sum \sigma^i_j \, \Delta_j \tag{4}$$

where σ^i_j are some functions of x and t. The Whalquist-Estabrook prescription [3] is to search for differential one forms ω_k defined through

$$\omega_k = dy_k + F_k dx + G_k dt \tag{5}$$

where y_k's are the prolongation variables. In Eq. (5) it is implied that F_k and G_k depend on $(u, p, x, t,$ and $y_k)$, in such a manner so that the following equality holds:

$$d\omega_k = \sum f^k_i \Delta_i + \sum_j \left(a^j_k dx + b^j_k dt\right)\wedge\omega_j \tag{6}$$

which essentially means that ω_k also belongs to the closed ideal generated by Δ_i and ω_k. Writing out Eq. (6) in full we get,

$$\begin{aligned} &F_u = 0 \\ &G_u + g(t)\,x F_p = 0 \\ &G_x - F_t + pG_u - b(t)\sinh u\, F_p + g(t)\,pF_p = -[F, G] \end{aligned} \tag{7}$$

These equations of F, Ge and some other simple considerations immediately lead to

$$\begin{aligned} F &= p\,X_1(y_i) + \sigma(t)\,X_2(y_i) \\ G &= -g(t)\,x\,p\,X_1(y_i) - g(t)\,\sigma(t)\,x\,X_2(y_i) \\ &\quad + \cosh u\; X_3(y_i) - \sinh u\; X_4(y_i) \end{aligned} \tag{8}$$

where $X_i(y_i)$ denotes the dependence of F, G on the prolongation variables y_k which up till now are unknown. Substituting these forms in the last equation of (7) we get:

$$\begin{aligned} [X_1, X_3] &= -X_4 \\ [X_1, X_4] &= -X_3 \\ [X_2, X_3] &= 0 \end{aligned} \tag{9}$$

along with the condition

$$\frac{d\sigma}{dt} = -g(t)\,\sigma(t) \tag{10}$$

and also

$$\sigma(t)\,[X_2, X_4] = b(t)\,X_1 \tag{11}$$

The result shown in Eq. (9) is usually referred to as the incomplete Lie algebra. To attain a closure of this algebra we use the various Jacobi-identities to get:

$$X_2 = X_3$$

$$[X_1, X_2] = -X_4 \;; \quad [X_2, X_4] = kX_1$$

$$[X_4, X_1] = X_2$$

with

$$k = \frac{b(t)}{\sigma(t)} \tag{12}$$

It is usually the dimension of the matrix representation of this Lie algebra which fixes up the number of the prolongation variables y_k.

Using the simplest 2 x 2 matrix representation [4] we observe that the x-part of the Lax equation is given as:

$$y_x = \left\{ p\begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} + \sigma(t)\frac{\sqrt{k}}{2}\, i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} y \tag{13}$$

The inverse problem posed by Eq. (13) can be solved following the usual procedure of AKNS except that some modifications will be needed for the explicit dependence via $\sigma(t)$ and $g(t)$. If we now set $\frac{y_1}{y_2} = \phi$ then from (13) it follows that

$$\phi_x = -\frac{p}{2}\left(1 - \phi^2\right) - \frac{i}{2}\,\sigma(t)\sqrt{k}\left(1 + \phi^2\right) \tag{14}$$

It is now a routine exercise to deduce the infinite number of conservation laws from (14).

On the other hand we observe, following Konno and Wadati [5], that this Riccati equation remains form invariant if and only if we effect the following change of variables:

$$\phi' = \frac{1}{\phi}$$

$$p' = -p - 4\frac{\phi_x}{1-\phi^2} \tag{15}$$

The second part of (15) is equivalent to:

$$u_x' = -u_x - 4\frac{\partial}{\partial x}\tanh^{-1}\phi \tag{16}$$

Solving (16) for ϕ we get

$$\phi = -\tanh\left(\frac{u+u'}{4}\right) \tag{17}$$

Now eliminating ϕ between (17) and (14) we get the Backlund transformation:

$$u_x' - u_x = 2i\,\sigma(t)\sqrt{K}\cosh\left(\frac{u+u'}{2}\right) \tag{18}$$

It is interesting to observe that this form of Backlund transformation is very much similar to those of the ordinary Sine or Sinh-Gordon equation the only difference being the explicit time dependent factor on the right-hand side of (18). So the composition law for two consecutive BT can also be formulated along the same lines.

We now consider the time part of the above BT. We now utilize the time part of the Lax pair and the variable ϕ to deduce:

$$\phi_t = A(1+\phi^2) + B(1-\phi^2) + C\phi \tag{19}$$

where

$$A = \frac{i\sqrt{K}}{2}g(t)\sigma(t)x - \frac{i\sqrt{K}}{2}\cosh u$$

$$B = \frac{g(t)x}{2}u_x$$

$$C = i\sqrt{K}\sinh u \tag{20}$$

Now eliminating ϕ between (19) and (18) we have

$$u_t + u_t' = -4A\cosh\frac{u+u'}{2} - 4B - 2C\sinh\frac{u+u'}{2} \tag{21}$$

So Eqs. (21) and (18) give the mapping from one solution to another of our Eq. (1).

(b) In our previous analysis we have shown how one can generalize the prolongation analysis to a space-time dependent case, and construct the space and time part of BT explicitly. In this section a new type of nonlinear pde is considered and it is also shown to be completely integrable and does possess a Backlund transformation. Actually the equation that we are going to consider is a Klein-Gordon type equation with a new kind of nonlinearity. The equation we consider is written as:

$$\alpha_{xt} + \frac{i\alpha_{xxt}}{\alpha_x} = -ie^{-i\alpha} \tag{22}$$

Eq. (22) is to be understood with its complex conjugate. As before we define new independent variables:

$$\alpha_x = p\,;\ \alpha_x^* = p^*\,;\ \alpha_{xt} = p_t = q\,;\ \alpha_{xt}^* = p_t^* = q^* \tag{23}$$

It is now not very difficult to write down the set of differential forms equivalent to (22) and its complex conjugate. These are:

$$\Delta_1 = d\alpha\wedge dt - p\,dx\wedge dt$$

$$\Delta_2 = -dp\wedge dx - q\,dx\wedge dt$$

$$\Delta_3 = d\alpha^*\wedge dt - p^*dx\wedge dt$$

$$\Delta_4 = -dp^*\wedge dx - q^*dx\wedge dt$$

$$\Delta_5 = -p\,dp\wedge dx + i\,dq\wedge dt + ie^{i\alpha}p\,dx\wedge dt$$

$$\Delta_6 = -p^*dp^*\wedge dx - i\,dq^*\wedge dt - ie^{i\alpha^*}p^*dx\wedge dt \tag{24}$$

One can now write down an equation equivalent to Eq. (6) and obtain as before the following conditions:

$$G_p = G_{p^*} = 0$$

$$F_q = F_{q^*} = F_\alpha = F_{\alpha^*} = 0$$

$$p G_\alpha + i p q G_q - q F_p + G_\alpha p^* - q^* F_{p^*} - i p^* q^* G_{q^*}$$
$$- e^{-i\alpha} p G_q - e^{i\alpha^*} p^* G_{q^*} = - [F, G] \qquad (25)$$

We now consider the following structures of F and G, compatible with (25)

$$F = p X_1(y_i) + p^* X_2(y_i) + X_3(y_i)$$

$$G = q X_4(y_i) + q^* X_5(y_i) + e^{-i\alpha} X_6(y_i)$$
$$+ e^{i\alpha^*} X_7(y_i) \qquad (26)$$

Substituting in the last equation of (25) we get:

$$[X_1, X_4] = -i X_4 \qquad [X_1, X_5] = 0$$

$$[X_1, X_6] = X_4 + i X_6 \qquad [X_1, X_7] = 0$$

$$[X_2, X_4] = 0 \qquad [X_2, X_5] = i X_5$$

$$[X_2, X_6] = 0 \qquad [X_3, X_4] = X_1$$

$$[X_3, X_5] = X_2 \qquad [X_3, X_6] = 0$$

$$[X_3, X_7] = 0 \qquad (27)$$

which is the resultant incomplete Lie algebra. Using the complete set of Jacobi identities we obtain the following multiplication table defining a closed Lie algebra:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	0	0	$x_4 + ix_6$	ix_4	0	$-ix_4 - ix_6$	0
x_2	0	0	$x_5 - ix_7$	0	$-ix_5$	0	$-x_5 + ix_7$
x_3	$-x_4 - ix_6$	$-x_5 + ix_7$	0	$-x_1$	$-x_2$	0	0
x_4	$-ix_4$	0	x_1	0	0	$-x_4$	0
x_5	0	ix_5	x_2	0	0	0	$-x_2$
x_6	$x_4 + ix_6$	0	0	x_4	0	0	0
x_7	0	$x_5 - ix_7$	0	0	x_2	0	0

Next we observe that a nonlinear realization of these generators can be written in the following form:

$$X_1 = i y_1 \frac{\partial}{\partial y_1} + i y_2 \frac{\partial}{\partial y_2}$$

$$X_2 = - i y_1 \frac{\partial}{\partial y_1}$$

$$X_3 = \left\{ -\frac{i}{2}\left(y_2 + \frac{1}{y_2}\right) y_1 + \frac{y_1^2}{2y_2} - \frac{y_2}{2} \right\} \frac{\partial}{\partial y_1} - \frac{i}{2}(y_2^2 + 1) \frac{\partial}{\partial y_2}$$

$$X_4 = \frac{y_1}{y_2} \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}$$

$$X_5 = i y_2 \frac{\partial}{\partial y_1}$$

$$X_6 = \frac{i y_1}{2}\left(y_2 + \frac{1}{y_2}\right) \frac{\partial}{\partial y_1} + \frac{i}{2}(y_2^2 + 1) \frac{\partial}{\partial y_2}$$

$$X_7 = \left(\frac{y_2}{2} - \frac{y_1 2}{2 y_2} \right) \frac{\partial}{\partial y_1} \qquad (28)$$

Now we can again define; $\phi = \frac{y_1}{y_2}$ and show that it satisfies:

$$\phi_x = - i \alpha_x^* \phi + \frac{1}{2} \phi^2 - \frac{1}{2} \qquad (29)$$

So that if we make a change of variable

$$\phi' = -\phi$$

$$\alpha_x' = -\alpha_x - 2i\frac{\partial}{\partial x}(\log \phi^*) \tag{30}$$

then Eq. (29) remains form invariant. Solving as before for ϕ we obtain

$$\phi = e^{-i\frac{\alpha' + \alpha^*}{2}} \tag{31}$$

whence eliminating ϕ between (30) and (31) we get,

$$\alpha_x' - \alpha_x = 2 \sin \frac{\alpha + \alpha'}{2}$$

which is nothing but the space part of the BT. For the construction of the time part we observe that ϕ also satisfies

$$\phi_t = D\phi^2 + \frac{1}{2} e^{i\alpha^*}$$

$$D = -\frac{1}{2} e^{-i\alpha} + i\alpha_{xt}$$

From which we can deduce

$$\alpha_t + \alpha_t' = 2\alpha_{xt} \cdot \exp\left[\frac{i}{2}(\alpha + \alpha')\right] - e^{-i\alpha} \sin\left(\frac{\alpha + \alpha'}{2}\right) \tag{32}$$

DISCUSSIONS

In our above analysis we have considered in detail the IST problems that can be associated with two new kinds of nonlinear Klein-Gordon equation. One of these equations indicates in which way the procedure of prolongation is to proceed if there be explicit space time dependence, and the other deals with a different kind of nonlinearity.

ACKNOWLEDGMENTS

One of the authors (A.R.C.) is grateful to Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. He also wishes to thank SAREC for supporting his visit.

REFERENCES

1) "Nonlinear Phenomena", Lecture Notes in Physics, Vol. 189 (Springer-Verlag, Berlin, New York) 21, 1416 (1980).

2) A. Roy Chowdhury and T. Roy, J. Math. Phys. 21, 1416 (1980);
A. Roy Chowdhury, K. Roy Chowdhury and G. Mahato, Lett. Math. Phys. 6, 423 (1982).

3) H.Q. Whalquist and F.B. Estabrook, J. Math. Phys. 16, 1 (1975).

4) N. Jacobson, Lie Algebras (Interscience Publishers).

5) K. Konno and M. Wadati, Prog. Theor. Phys. 53, 1652 (1975).