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BOUND STATES IN QUANTUM FIELD THEORY  
AND COHERENT STATES: A FRESH LOOK

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BOUND STATES IN QUANTUM FIELD THEORY  
AND COHERENT STATES: A FRESH LOOK \*

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ABSTRACT

We consider here bound state equations in quantum field theory where the state explicitly includes radiation quanta as constituents with the number of such quanta not fixed. The fully interacting system is dealt with through equal time commutators/anticommutators of field operators. The multiparticle channel for the radiation field is approximated through coherent state representations.

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Usually one considers bound states in field theory with a fixed number of particles. However, in field theory particle number need not be conserved, and in particular the bound state as an eigenstate of the Hamiltonian need not be an eigenstate of the number operators. In the present analysis, we shall recognise this fact for the definition of the bound state, and develop a nonperturbative framework accordingly.

To motivate, let us first consider a toy model in quantum mechanics. Let the Hamiltonian be

$$H = \epsilon c^\dagger c + \omega a^\dagger a + g c^\dagger c (a^\dagger + a). \quad (1)$$

In the above,  $c$  stands for "matter" annihilation operator,  $a$  for the "radiation" annihilation operator, and we take the usual quantum conditions  $[c, c^\dagger]_+ = [a, a^\dagger] = 1$ . Thus matter has been taken as fermionic and radiation as bosonic. The state  $|\text{vac}\rangle$  is defined through  $a |\text{vac}\rangle = c |\text{vac}\rangle = 0$ . Let us now make the substitution  $a = a' - (g/\omega) c^\dagger c$ . We then have  $[a', a'^\dagger] = 1$ . However,  $a'$  does not commute with  $c$  or  $c^\dagger$ . Equation (1) then simplifies to

$$H = \epsilon c^\dagger c + \omega a'^\dagger a' - (g^2/\omega)(c^\dagger c)(c^\dagger c). \quad (2)$$

Let us next consider a state  $|B_n\rangle = f(a^\dagger) c^{f_n} |\text{vac}\rangle$  with  $n (=0,1)$  fermions and arbitrary number of radiation quanta such that  $a' |B_n\rangle = 0$ . With  $a = \delta/\delta a^\dagger$  this leads to the differential equation

$$(\delta/\delta a^\dagger + (g/\omega)n) f(a^\dagger) = 0, \quad (3)$$

such that, with  $A$  as a normalisation constant,

$$|B_n\rangle = A \exp(-n(g/\omega)a^\dagger) (c^\dagger)^n |\text{vac}\rangle \quad (4)$$

is an eigenstate of  $H$  with eigenvalue  $n\epsilon - n^2(g^2/\omega)$ . We may also construct other eigenstates of  $H$  as

$$|m, n\rangle = (a^\dagger + ng/\omega)^m |B_n\rangle, \quad (5)$$

where,  $H|m,n\rangle = (m\omega + n\epsilon - n^2 g^2/\omega) |m,n\rangle$ . We took matter as fermionic so that energy is bounded below. Here  $|B_1\rangle$  is a coherent state<sup>1</sup> with  $a|B_1\rangle = -(g/\omega) |B_1\rangle$ . Also there are an infinity of radiation quanta with the probability for  $k$  quanta given as

$$p_k = (g/\omega)^{2k}/k! \exp(-g^2/\omega^2). \quad (6)$$

We may here note that for adequately large  $g^2/\omega$ , there is a phase transition and the single fermion state  $|B_1\rangle$  with its radiation cloud constitutes the physical vacuum.

The above reveals the relevance of multiradiation quanta, simulating field theory, which we now proceed to consider. For illustrating the dynamics, we take a nonrelativistic Hamiltonian density at  $t=0$  given as, with  $\varphi(\vec{x}) = (2\omega_{\vec{x}})^{-1/2} (a(\vec{x}) + a(\vec{x})^\dagger)$ ,

$$\begin{aligned} H(\vec{x}) = & c_1(\vec{x})^\dagger \epsilon_{1\vec{x}} c_1(\vec{x}) + c_2(\vec{x})^\dagger \epsilon_{2\vec{x}} c_2(\vec{x}) + a(\vec{x})^\dagger \omega_{\vec{x}} a(\vec{x}) \\ & + (\epsilon_1 c_1(\vec{x})^\dagger c_1(\vec{x}) + \epsilon_2 c_2(\vec{x})^\dagger c_2(\vec{x})) \varphi(\vec{x}). \end{aligned} \quad (7)$$

The following notations may be noted. In the above we have expressed the Hamiltonian density in terms of the fermion and boson creation and annihilation operators with e.g. the obvious algebra

$$\begin{aligned} [c_1(\vec{x}), c_1(\vec{y})^\dagger]_+ &= [c_2(\vec{x}), c_2(\vec{y})^\dagger]_+ = [a(\vec{x}), a(\vec{y})^\dagger] \\ &= \delta(\vec{x}-\vec{y}) \end{aligned} \quad (8)$$

Further,  $\epsilon_{1\vec{x}}$ ,  $\epsilon_{2\vec{x}}$  and  $\omega_{\vec{x}}$  are differentiation operators corresponding to the respective free field Hamiltonians, and are defined through the Fourier transform space. The model corresponds to two fermions interacting with a scalar meson. The spins are suppressed, but the kinetic relativistic corrections may be present through  $\epsilon_{r\vec{x}} = (-\vec{\nabla}_{\vec{x}}^2 + m_r^2)^{1/2}$  and  $\omega_{\vec{x}} = (-\vec{\nabla}_{\vec{x}}^2 + \mu^2)^{1/2}$ . We shall also have in addition counter terms corresponding to the self-energy of the fermions.

As earlier, we define  $|\text{vac}\rangle$  through  $c_r(x) |\text{vac}\rangle = 0 = a(x) |\text{vac}\rangle$ .

The bound state of two fermions and an arbitrary number of radiation quanta will be considered. This will be an eigenstate of the Hamiltonian  $H = \int d\vec{x} H(\vec{x})$  where the eigenstate  $|B\rangle$  will have the form  $|B_0\rangle + |B_1\rangle + |B_2\rangle + \dots$  with  $|B_n\rangle$  describing a state with  $n$  radiation quanta. The problem will thus involve the coupling of infinitely many channels which is not possible to solve. We shall hence use here the approximation of the multiparticle states being the coherent states, and shall proceed to construct such states.

We start with the fiducial state  $|S_0(\vec{x}, \vec{y})\rangle = c_1(\vec{x})^\dagger c_2(\vec{y})^\dagger |\text{vac}\rangle$ . We next consider the operator  $G^\dagger = \int c_r(\vec{z})^\dagger A_r(\vec{z})^\dagger c_r(\vec{z}) d\vec{z}$  with summation over repeated index  $r$ , and with  $A_r(\vec{z})^\dagger = f_r(\vec{z}-\vec{j}) a(\vec{j})^\dagger d\vec{j}$ .  $f_r(\vec{z})$  ( $r=1,2$ ) are two functions of space coordinates  $\vec{z}$ . The justification for this choice of operators will occur later. Using equations (8) we now see that

$$\begin{aligned} |S(\vec{x}, \vec{y})\rangle &\equiv N \exp(G^\dagger) |S_0(\vec{x}, \vec{y})\rangle \\ &= N c_1(\vec{x})^\dagger \exp(A_1(\vec{x})^\dagger) \exp(A_2(\vec{y})^\dagger) c_2(\vec{y})^\dagger |\text{vac}\rangle \\ &\equiv c_1(x)^\dagger c_2(y)^\dagger |R(\vec{x}, \vec{y})\rangle. \end{aligned} \quad (9)$$

In the above,  $N$  is a normalisation constant which we shall determine. We may interpret  $\exp(A_1(\vec{x})^\dagger)$  <sup>that</sup> creates the radiation quanta attached to the fermion at  $\vec{x}$ , and similarly for  $\exp(A_2(\vec{y})^\dagger)$ . Such an identification will help us to recognise the self-energy contributions of the fermions. We note that  $[a(\vec{z}), A_r(\vec{x})^\dagger] = f_r(\vec{x}-\vec{z})$ , which yields that

$$a(\vec{z}) |S(x, y)\rangle = (f_1(\vec{x}-\vec{z}) + f_2(\vec{y}-\vec{z})) |S(x, y)\rangle. \quad (10)$$

It was our purpose to construct such an eigenstate.

We shall define a bound state of zero momentum in the form

$$|B(\vec{0})\rangle = (2\pi)^{-3/2} \int u(\vec{x}-\vec{y}) |S(\vec{x}, \vec{y})\rangle d\vec{x} d\vec{y}. \quad (11)$$

This has three arbitrary functions;  $u(\vec{x}-\vec{y})$  which will correspond to the old Schrodinger wave function, and,  $f_r(\vec{x})$  ( $r=1,2$ ) which will decide the nature of the radiation quanta in the bound state.

We now determine the normalisation constant N in equation (9) from the condition

$$\langle R(\vec{x}, \vec{y}) | R(\vec{x}, \vec{y}) \rangle = 1. \quad (12)$$

We first note that by (8)

$$\begin{aligned} [A_R(\vec{x}), A_S(\vec{y})^\dagger] &= \int f_R(\vec{x}-\vec{z})^* f_S(\vec{y}-\vec{z}) d\vec{z} \\ &= h_{RS}(\vec{x}-\vec{y}), \end{aligned} \quad (13)$$

such that (12) yields

$$N_{xy}^{-2} = \exp(f(\vec{x}-\vec{y})) \quad (14)$$

with

$$f(\vec{x}-\vec{y}) = h_{11}(\vec{0}) + h_{22}(\vec{0}) + h_{12}(\vec{x}-\vec{y}) + h_{21}(\vec{y}-\vec{x}). \quad (15)$$

We next take the formal normalisation for zero momentum states as  $\langle B(\vec{0}) | B(\vec{0}) \rangle = \delta(\vec{0})$ , which yields the conventional normalisation

$$\int |u(\vec{x})|^2 d\vec{x} = 1. \quad (16)$$

We shall now consider the expectation value of the Hamiltonian density of equation (7) for the state as in equation (11), and then minimise this to obtain mass. We substitute

$$h[u, f_1, f_2] = t + h_M + h_i \quad (17)$$

where we have taken the expectation value  $(2\pi)^3 \langle B(\vec{0}) | \mathcal{H}(\vec{0}) | B(\vec{0}) \rangle$ .

The individual terms  $t$ ,  $h_M$  and  $h_i$  are respectively the expectation values of the fermionic kinetic part, the "meson" or radiation field part, and, the interaction part of equation (7).

These functionals are next to be evaluated.

For the evaluation of  $t$ , we first note that

$$\begin{aligned} &\langle R(\vec{x}', \vec{y}') | R(\vec{x}, \vec{y}) \rangle \\ &= N_{x'y'} N_{xy} \exp[h_{11}(\vec{x}'-\vec{x}) + h_{22}(\vec{y}'-\vec{y}) + h_{12}(\vec{x}'-\vec{y}) + h_{21}(\vec{y}'-\vec{x})]. \end{aligned} \quad (18)$$

Using this, and with some algebra, we get that

$$\begin{aligned} t &= \int u(-\vec{y})^* \epsilon_{1\vec{x}} [\exp(h_{11}(-\vec{x}) - h_{11}(\vec{0}) + (1/2)(h_{12}(\vec{y}) - h_{12}(\vec{y}-\vec{x})) \\ &\quad \cdot u(\vec{x}-\vec{y}))]_{\vec{x}=0} d\vec{y} \\ &+ \int u(\vec{x})^* \epsilon_{2\vec{y}} [\exp(h_{22}(\vec{y}) - h_{22}(\vec{0}) + (1/2)(h_{21}(-\vec{x}) - h_{21}(\vec{y}-\vec{x})) \\ &\quad \cdot u(\vec{x}-\vec{y}))]_{\vec{y}=0} d\vec{x}. \end{aligned} \quad (19)$$

In case we can approximate the above limits inside the differentiation, we obtain that

$$t = \int u(\vec{x})^* (\epsilon_{1\vec{x}} + \epsilon_{2\vec{x}}) u(\vec{x}) d\vec{x}. \quad (20)$$

Using equations (10) and (12), evaluation of  $h_M$  gives

$$h_M = \int |u(\vec{x}-\vec{y})|^2 d\vec{x} d\vec{y} (f_1(\vec{x})^* + f_2(\vec{y})^*) (\omega_{\vec{x}} f_1(\vec{x}) + \omega_{\vec{y}} f_2(\vec{y})). \quad (21)$$

Next, evaluation of  $h_i$  gives

$$\begin{aligned} h_i &= \int |u(-\vec{y})|^2 d\vec{y} (e_1 ((2\omega_{\vec{x}})^{-(1/2)} f_1(\vec{x})^* + (2\omega_{\vec{y}})^{-(1/2)} f_2(\vec{y})^*))_{\vec{x}=0} \\ &+ \int u(\vec{x})^* d\vec{x} (e_2 ((2\omega_{\vec{x}})^{-(1/2)} f_1(\vec{x})^* + (2\omega_{\vec{y}})^{-(1/2)} f_2(\vec{y})^*))_{\vec{y}=0} \\ &+ \text{h.c.} \end{aligned} \quad (22)$$

Here we have used that  $\varphi(\vec{x}) = (2\omega_{\vec{x}})^{-(1/2)} (a(\vec{x}) + a(\vec{x})^*)$  and equations (10) and (12).

We shall now use the limit (20) for  $t$ , and equations (21) and (22), and extremise with respect to  $f_1$  and  $f_2$ . Thus, e.g.

$(\delta/\delta f_1(\vec{x})^*)(t + h_M + h_i) = 0$  yields that

$$\begin{aligned} &\int |u(\vec{x}-\vec{y})|^2 (\omega_{\vec{x}} f_1(\vec{x}) + \omega_{\vec{y}} f_2(\vec{y})) d\vec{y} \\ &+ e_2 (2\omega_{\vec{x}})^{-(1/2)} |u(\vec{x})|^2 + e_1 (2\omega_{\vec{x}})^{-(1/2)} \delta(\vec{x}) = 0. \end{aligned} \quad (23)$$

We can easily verify that the above as well as the other similar extremisation equations for  $f_1$  and  $f_2$  are satisfied when

$$f_1(\vec{x}) = -e_1 / (\sqrt{2} \omega_{\vec{x}}^{3/2}) \delta(\vec{x}). \quad (24)$$

As mentioned earlier, all these equations are defined through Fourier transforms and are to be regularised when necessary, and, for appropriate contributions, self-energy terms are to be subtracted.

We shall now use equation (24) and identify the self-energy contributions. Equation (21) has respectively the self-energy and the potential contributions given as

$$h_M^S = (1/2)(e_1^2 + e_2^2) (\omega_{\vec{x}}^{-3/2} \delta(\vec{x})) (\omega_{\vec{x}}^{-1/2} \delta(\vec{x})) d\vec{x}, \quad (25)$$

and

$$h_M^V = e_1 e_2 \int |u(\vec{x})|^2 (\omega_{\vec{x}}^{-2} \delta(\vec{x})) d\vec{x}. \quad (26)$$

Similarly, the self-energy and the potential contributions from equation (22) are respectively given as

$$h_i^S = -(e_1^2 + e_2^2) (\omega_{\vec{x}}^{-2} \delta(\vec{x}))_{\vec{x} \rightarrow 0}, \quad (27)$$

and

$$h_i^V = -2e_1 e_2 \int |u(x)|^2 (\omega_{\vec{x}}^{-2} \delta(\vec{x})) d\vec{x}. \quad (28)$$

We assume that the terms in (25) and (27) are cancelled by appropriate self-energy counter terms. This gives us the potential as

$$v = h_M^V + h_i^V \\ = -e_1 e_2 \int |u(\vec{x})|^2 (\omega_{\vec{x}}^{-2} \delta(\vec{x})) d\vec{x}. \quad (29)$$

Thus the conventional potential term is contributed both from the free radiation part as well as the interaction part in a simple manner. Now a variation with respect to  $u(x)^*$  yields the familiar eigenvalue equation

$$(\epsilon_{1\vec{x}} + \epsilon_{2\vec{x}}) u(\vec{x}) - \frac{e_1 e_2}{4\pi |\vec{x}|} u(\vec{x}) = E u(\vec{x}) \quad (30)$$

when we shall carry out the corresponding algebra.<sup>2</sup> This ends the "conventional" nature of the present theory, and we shall now proceed to show the extra physics content of the present approach.

Firstly we see that the probability for there being  $k$  radiation

quanta contained in the bound state parallel to (6) is given as<sup>3</sup>

$$P_k = \int |u(\vec{x})|^2 (f(\vec{x}))^k / k! \exp(-f(\vec{x})) d\vec{x}. \quad (31)$$

We may also obtain the average number of radiation quanta as

$$N_R = \sum k P_k = \int |u(\vec{x})|^2 f(\vec{x}) d\vec{x}. \quad (32)$$

We may also find the momentum squared carried by the radiation quanta. Taking the expectation value of  $a(\vec{x})^\dagger (-\nabla_{\vec{x}}^2 a(\vec{x}))_{\vec{x} \rightarrow 0}$  we thus formally obtain,

$$\langle P_R^2 \rangle = \int |u(\vec{x}-\vec{y})|^2 (\nabla_{\vec{x}}^2 f_1 + \nabla_{\vec{y}}^2 f_2)^2 d\vec{x} d\vec{y}. \quad (33)$$

The above expression also contains divergent expressions which are to be subtracted. After doing so, we obtain the corresponding contribution as

$$\langle P_R^2 \rangle = -e_1 e_2 \int |u(\vec{x})|^2 (\nabla_{\vec{x}}^2 \omega_{\vec{x}}^{-3} \delta(\vec{x})) d\vec{x} \\ = e_1 e_2 / (2\pi)^3 \int d\vec{k} d\vec{k}' \tilde{u}(\vec{k}')^* \tilde{u}(\vec{k}'-\vec{k}) \vec{k}^2 / (\omega(\vec{k}))^3. \quad (34)$$

We note that this is Yukawa coupling so that  $e_1$  and  $e_2$  have the same sign.

We may note that in equation (15)  $f(\vec{x}-\vec{y})$  has a divergent contribution through  $h_{11}(0)$  and  $h_{22}(0)$ , and the "convergent" part  $f_c(\vec{x}-\vec{y})$  is given by

$$f_c(\vec{x}) = h_{12}(\vec{x}) + h_{21}(-\vec{x}) \\ = \frac{e_1 e_2}{(2\pi)^3} \int \frac{e^{i\vec{k}\cdot\vec{x}}}{\omega(\vec{k})^3} d\vec{k} \\ = \frac{e_1 e_2}{2\pi^2} K_0(\mu r). \quad (35)$$

In (35),  $\mu$  is the effective mass or infrared cutoff of the radiation quanta,  $r = |\vec{x}|$  and,  $K_0$  is the Bessel function with imaginary argument<sup>4</sup> such that, for small  $r$ ,  $K_0(\mu r) \simeq \ln(1/\mu^2 r^2)$ , and

$K_0 \rightarrow 0$  as  $r \rightarrow \infty$ . In order to get a feeling for the contribution, let us use an explicit approximation

$$f(x) = \frac{2\alpha}{\pi} \ln \left( \frac{1 + \mu^2 r^2}{\mu^2 r^2} \right), \quad (36)$$

where we have substituted  $e_1 e_2 = \alpha, 4\pi$  with quantum electrodynamics in mind. One then obtains

$$N_R = \frac{2\alpha}{\pi} \int |u(\vec{x})|^2 \ln \left( \frac{1 + \mu^2 r^2}{\mu^2 r^2} \right) d\vec{x}. \quad (37)$$

We note that when  $\mu \rightarrow 0$ , (37) includes the usual infrared divergence which is to be tackled separately.<sup>5</sup>

The present description of radiation quanta inside the bound state has another interesting consequence. Let us define, using equations (10) and (24),

$$\begin{aligned} \varphi_{\alpha}(\vec{z}) &\equiv \langle B(\vec{0}) | \varphi(\vec{z}) | B(\vec{0}) \rangle \\ &= \int |u(\vec{x}-\vec{y})|^2 d\vec{x} d\vec{y} [ -e_1 \omega_x^{-2} \delta(\vec{x}-\vec{z}) - e_2 \omega_y^{-2} \delta(\vec{y}-\vec{z}) ]. \quad (38) \end{aligned}$$

The above correctly describes the classical potential inside the bound system when the sources are at  $\vec{x}$  and  $\vec{y}$ . Thus, the radiation quanta inside the bound state may be recognised as the quantum description of the confined classical field. These radiation quanta, as well as the two fermions, are obviously off the mass shell. We may picture the radiation quanta to act like glue to keep the fermions together. The conclusions are nonperturbative in the sense that only equal time commutators or anticommutators are used.

In conclusion, we may note the following:

- (i) We have shown that an approximation through multi-radiation quanta present in the bound state generates the conventional Schrodinger equation for the energy eigenvalues.
- (ii) The momentum carried by the radiation quanta is calculable. Thus, in a hadron, all the momentum will not be carried by the

quarks or antiquarks, but, the gluons may also carry a substantial fraction of the momentum. This has obvious relevance for deep inelastic collisions.<sup>6</sup>

(iii) For hadronic spectroscopy, gluon number in hadrons will be nonzero, and, can be large. Since gluons carry spin, there will be substantial spin corrections, and the static SU(6) models will be particularly bad regarding polarisation effects.

(iv) The gluons present in the hadron will simulate the potential which may be calculable for the heavy quarkonium system.<sup>7</sup> For this purpose more complicated coherent states could be relevant.<sup>8</sup>

In a deeper sense, the present formulation of the bound state is "true" field theory since here we take a bound state with particle non-conservation, which is a basic feature of field theory. The fact that it has many correct hints regarding hadronic phenomenology was thus probably to be expected.

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