"Energy Dependence of the Neutron Multiplicity $P_v$ in Fast Neutron Induced Fission of $^{235,238}U$ and $^{239}Pu$"

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Introduction

For many technical purposes it suffices to know only the first moment, $<v>$, ("nubar"), of the neutron multiplicity probability, $P_v$, that $v$ prompt neutrons are emitted in a fission.

Certain applications however do require knowledge of the higher moments of the $P_v$ distribution. A case in point and the immediate reason for the present work are several methods for the analysis of nuclear material by what amounts to an autocorrelation performed on the pulse train from a detector exposed to the unknown sample. Then it can be shown that the second factorial moment $<v(v-1)> = \sum v(v-1)P_v$ is proportional to the fission rate in the sample, and that the third factorial moment $<v(v-1)(v-2)> = \sum v(v-1)(v-2)P_v$ can be of use in disentangling spontaneous fission (which can be related directly to the amount of material present), from induced fission, which is only partly related with the amount of material, since it is also a function of geometry, density, and other artifacts.

Calculating the effects of induced fission in the sample, i.e., fission caused by either spontaneous fission generated neutrons or those produced by $(\alpha,n)$ processes, or, in the case of certain nuclear material assay systems, by neutrons from an external source bombarding the sample, clearly requires knowledge of these moments for neutron energies ranging from the thermal to the fast (fission) neutron energy region.¹

¹This work was performed under the auspices of the U.S. Department of Energy, Contract No. DE-AC02-76CH00016.
While it has been possible to glean from the published literature much information (though never enough!) for nuclides undergoing spontaneous fission induced by thermal or tens of keV neutrons, we are unaware of any published \( P_v \) data for fission induced by neutrons in the few MeV range which are crucial to calculations of fast induced fission correlation.

We became aware though of extensive unpublished work by Frehaut and collaborators in which neutron multiplicities were derived for the fast neutron induced fission of \(^{235}\text{U}\), \(^{238}\text{U}\), and \(^{239}\text{Pu}\), three most interesting nuclides from the nuclear material assay standpoint. The reason this \( P_v \) data, despite its uniqueness and importance, was unpublished is that it is obviously flawed, due mainly it is presumed, to poor counting statistics and the way in which counting statistical uncertainties propagate in the transformations connecting experimentally observed neutron multiplicities with the \( P_v \) distributions (see below). Certain of the \( P_v \) at some energies are negative, physically and mathematically impossible for a probability, and, considered as a function of energy, the \( P_v \) often exhibit fluctuations obviously related to poor counting statistics rather than to any real physical process.

**Scope and Methodology**

Our task then has been to salvage from this data kindly furnished by Frehaut (and to our knowledge the only such existing), the multiplicity \( P_v \) as a function of incident neutron energy \( E_n \). The overall procedure may be described as data smoothing guided by general physical and mathematical principles. These will be described immediately below and justified with particular cases cited later on.

Though the data furnished covers up to \( E_n \sim 25 \text{ MeV} \), results will be quoted only over the range 0-10 MeV, as in the region 10-15 MeV and certainly for the
higher energies the data become prohibitively unreliable even though smoothing processes are employed. Another problem is that the furnished data only begin at $E_n = 1.36$ MeV. Fortunately comparatively accurate thermal neutron induced fission cross sections do exist for ($^{235}$U + n) and ($^{239}$Pu + n); more on this point later.

Any of the $P_v$ as a function of $E_n$ has the appearance of a bell-shaped curve or a portion of one. For the smaller $v$ values only the decreasing tail of the bell appears, starting at $E_n = 0$. For intermediate values of $v$ the whole bell shaped curve appears except the ascending part cut off at $E_n = 0$; the function rises to a maximum and then declines, presumably approaching zero for high enough $E_n$. The $P_v$, being probabilities, must sum to unity at any energy. Therefore, the only way for $\langle v \rangle$ to increase monotonically, as it is well known to do from experimental results, is for there to be an increase in the $P_v$ for higher values of $v$ with a corresponding decrease in the relative importance of those $P_v$ for smaller values of $v$. There is no evidence from the data of Frehaut et al. for any more complicated behaviour of any of the $P_v$ as a function of $E_n$ than this.

Therefore a basic premise of the fitting procedure was that each $P_v$ for any $v$ was considered to be a smooth function of neutron energy describable by a low order least squares fit polynomial in $E_n$. In fact, in the region 0-10 MeV it will be seen below there was no need for polynomials higher than the fourth order. In some cases these polynomial fits are good representations of the data for energies beyond 10 MeV, though in other cases the fit rapidly becomes unusable after 10 MeV.

In the fitting procedure the attempt was made, however, to use data beyond $E_n = 10$ MeV to help establish the trend in the data up to that point.
Similarly, thermal neutron data for $^{235}\text{U}$ and $^{239}\text{Pu}$ was used to anchor the start of the various $P_v$ curves for these nuclides, since the Frehaut data starts only at $E_n = 1.36$ MeV. Not surprisingly, there is no thermal neutron data for $(^{238}\text{U} + n)$ as the cross sections are too small below the effective threshold for induced fission. Here instead the fit obtained to the data between $E_n = 1.36$ MeV and $E_n = 10$ MeV was extrapolated down to $E_n = 0$ to yield an estimate for the multiplicity distribution between $E_n = 0$ and $1.36$ MeV for the system $(^{238}\text{U} + n)$.

After least squares fitting the $P_v$ as a polynomial in $E_n$, the $P_v$ set at a given energy will depart slightly from the normalization condition $\sum P_v = 1$, and so they were renormalized.

The average value for $v$, $<v> = \sum vP_v$, has been determined as a function of $E_n$ by independent experiment to a greater accuracy than can be determined from the $P_v$. These experiments basically determined a gross count rate $G$ in terms of a detector efficiency $\varepsilon$, and a source strength $q$:

$$G = \varepsilon<v>q. \quad (1)$$

The renormalized $P_v$ at any given energy predict a $<v> = \sum vP_v$ close to but not precisely equal to the best available values for $<v>$ determined as a function of $E_n$ from Eq. (1). The differences were reconciled by considering them formally as though they had arisen due to error or uncertainties in the efficiency of a (hypothetical) detector in an experiment determining the $P_v$.

In such an experiment, the observed neutron multiplicity, $Q_n$, that $n$ neutrons are observed from a given fission event is related to the $P_v$ by

$$Q_n = \sum P_v(n)\varepsilon^n(1-\varepsilon)^{v-n}, \quad (2)$$
where $\varepsilon$ is the detector efficiency. The $P_v$ are obtained from this expression by inverting the relation:

$$P_v = \sum Q_n(h)\varepsilon^{-n}(\varepsilon-1)^{n-v}. \quad (3)$$

As applied in this instance the normalized $P_v$ derived from the least squares fitting the Frehaut et al. data were used to obtain a hypothetical set $Q_n$ based on an assumed value for $\varepsilon$. The normalized set $P_v$ defines a value $<v> = \sum vP_v$. A set $P_v'$ such that it would yield a value $<v>' = \sum vP_v'$ considered more correct can be though of as being related to a detector efficiency $\varepsilon'$ such that $<v>' \varepsilon' = <v> \varepsilon$, since experimentally $<v>$ and $\varepsilon$ are inversely related. Then the hypothetical $Q_n$ can be used to obtain the set $P_v'$ corresponding to $\varepsilon'$ from

$$P_v' = \sum Q_n(h)(\varepsilon')^{-n}(\varepsilon'-1)^{n-v}. \quad (4)$$

Note that the values $Q_n$ and $\varepsilon$ may be considered dummy variables involved in a transformation from a set $P_v$ to a set $P_v'$. This transformation (Eqs. 2,3,4) preserves normalization, and, it can be shown, the values of such quantities (independent of $\varepsilon$) such as $<v(v-1)>/\langle v \rangle^2$, $<v(v-1)(v-2)>/<v>^3$, etc.

This procedure then produces a normalized set of $P_v'$ as a function of $E_n$ which yields the proper value of $<v>' = \sum vP_v'$ at any given $E_n$ and can be considered to be related to the same set of observables $Q_n$ as the original set $P_v$ that was smoothed and renormalized.

The $P_v'$ are expected to be close to the least squares fits to the $P_v$, and indeed do. In principle a new least squares fit could be made to the $P_v'$, the results renormalized, and again reconciled to the best values available for $<v>$. This process is rapidly convergent, so much so that the result-
ant changes in $P_v$ would be a marginal improvement considering the precision of the original data.

**Data Smoothing (Polynomial Fitting) Procedure**

The statistical uncertainties in the $P_v$ are naturally greater the fewer the observations of a given multiplicity are. The larger $P_v$ therefore have better statistical precision. Since all observed $Q_n$ with $n < v$ can be considered as stemming from a multiplicity $v$ with probability $P_v$ because the efficiency $\epsilon$ is less than 1 (see Eqs. 2 and 3), the error propagation from the observed $Q_n$ to the derived $P_v$ is not simple. Nevertheless there is a qualitative correspondence between the magnitude of the $P_v$ and the "smoothness" of the data points. Thus $P_3$, $P_4$, $P_5$, which are comparatively large in the region 0-10 MeV were relatively easy to fit while $P_0$, $P_7$, and $P_8$ were the most difficult. $P_0$ is small to begin with ($\approx 0.01$) at thermal energy and decreases rapidly to zero at roughly 10 MeV. $P_7$ and $P_8$ start out even smaller at thermal energies; while they increase rapidly, another difficulty that manifests itself is an increase in the general variability of the data at higher energies which is not explicable on the basis of the number of fissions observed at a given energy being inferior for high energies ($\approx 10$ MeV) compared to low energies ($\approx 2$ MeV). In fact, the higher energy data has significantly more fissions analyzed, for example, for $(^{235}\text{U} + n)$ there were 4,532 fissions analyzed for $E_n = 1.87$ MeV, but 11,374 fissions analyzed for $E_n = 9.74$, and these are typical of their respective neighboring values. (Incidentally, a typical modern $P_v$ experiment for spontaneous fissions or thermal neutron energies would involve $10^5$ or $10^6$ fissions. This points out the basic problem of the Frehaut data, lack of sufficient statistics.)
It is not practical to present every case treated, but the problems encountered and solutions adopted in reducing the data to useable form can best be illustrated with actual examples, using \((^{239}\text{Pu} + n)\) for \(v = 0\) to 8 and \((^{235}\text{U} + n)\) for \(v = 7,8\) as fairly representative.

These cases are illustrated in Fig. 1-10. The general format of these is to show the raw data (i.e. thermal plus Frehaut data points) above and the data plus the fitted polynomial in powers of \(E_n\) below.

The polynomial used was the lowest order which would give a good fit. Although the (Hewlett-Packard 9845) software for polynomial fitting routine also produced mathematical goodness of fit parameters, the criteria that proved practical in deciding the goodness of fit were those based in the general knowledge of how the \(P_v\) functions of energy behaved, and that the fitted curve should intercept the \(P_v\) axis close to the thermal value, since these are assumed to be one of two orders of magnitude more accurate than the typical data for \(E_n>0\).

The curve fitting routine did not allow for weighting the data points according to statistical uncertainty, but a simple subterfuge, entering a data pair more than once, produced the same effect. The only points weighted this way were the thermal values. It was found that giving the thermal value a weight of 5 or 10 tended to improve the agreement of the fitted curve at \(E_n = 0\), while not noticeably affecting the goodness of fit among the Frehaut data points which start at \(E_n = 1.36\) MeV. In a few cases, choice of the order of the polynomial between otherwise equivalent fits was decided on the basis of which gave best agreement with the thermal value. However, in all cases where there was a thermal value available, there was no difficulty in fitting a low order polynomial (fourth or less) to the thermal value and the Frehaut data, a point which will be commented on later.
Though the resulting fits are meant to be considered representations of the respective $P_\nu$ only in the region 0 to 10 MeV, data points for $E_n$ up to about 14 MeV were used in order to be able to make more certain the course of the function in the neighborhood of 10 MeV. In some of the Figs. (1-10), the fitted polynomial is shown extending beyond 10 MeV. In some cases, the particular $P_\nu$ continues to be represented by the fitted curve beyond 10 MeV, but in other cases the formal analytical extension of the polynomial is clearly non-physical, as the curve would depart from zero when it should remain there, or reverse direction when it should continue monotonically. This point will also be discussed later.

In a relatively few situations, data points were discarded when they seemed to lie off the tentatively fitted curves by what seemed to be significantly more than the typical variability in the data for that particular $P_\nu$ and seemed to interfere with the goodness of fit to the rest of the data.

Some particular remarks: The curve for $P_0$ (Fig. 1) illustrates the inadequacy of a finite order polynomial representation for more than a limited range, as the data indicates $P_0$ is statistically zero from about $E_n = 10$ MeV on. Similarly for $P_1$ (Fig. 2) where the best fit to the data over the whole region tends to change direction at $E_n = 13$ MeV, while (considering the data up to $E_n = 10$ MeV) it should asymptotically approach zero.

Fig. 5 ($P_w$) illustrates that a higher order least squares fitted polynomial, while a better fit by the least squares criterion, is not necessarily better physically, so that the third order polynomial was chosen as the better representation.

$P_8$ (Fig. 9) had two problems, the extreme scatter of the data points (upper part of the figure) though a trend is clearly visible, and the absence
of a thermal value to serve as an "anchor point." The effect of scatter was
greatly reduced by averaging groups of points with respect to ordinate and
abscissa, and fitting the curve to these points as plotted in the lower part of
the figure.

The fitted curve for $P_8$ predicts a small nonzero value for $P_0$ at $E_n = 0$
which is not inconsistent with the zero value reported considering the assigned
standard deviations for that data.

Fig. 10 illustrates $P_7$ and $P_8$ for $(^{235}U + n)$. A nonzero thermal value is
reported for $P_7$, but not for $P_8$, for this system. As can be seen, the Frehaut
data for $(^{235}U + n)$ is consistent with $P_8 = 0$ even beyond $E_n = 10$ MeV. How-
ever in agreement with the non-zero thermal value for $P_7$, the Frehaut data does
show a small non-zero component for $P_7$ from $E_n = 1.36$ to about 6 MeV, fol-
lowed by what seems to be an abrupt rise starting about $E_n = 8.5$ MeV, with
however much scatter to the data. It was decided that the abrupt rise is an
artifact due to chance occurrence of three consecutive low values in the region
$7 - 8$ MeV. Therefore a curve was fit which agrees with the thermal values
and extrapolates to an average of the Frehaut data in the region 10 to 11 MeV.

"Normalizing" the Smoothed Data

Here normalized will be used to mean that $\sum P_v = 1$ and $\sum vP_v = <v>$,
where $<v>$ is a prescribed value.

After smoothing each $P_v$ treated as a function of $E_n$ by least squares
fitting a polynomial to the data, the $P_v$ at a given energy will no longer
sum to unity although they are close, about a percent or so off.

The renormalized data are then subjected to the transformation according
to Eqs. (2), (3), and (4), subject to the condition that they produce the value
of $<v>$ appropriate to that energy.
The values of $\langle \nu \rangle$ vs. $E_n$ used in principle could have been those evaluated from all the literature. In the present situation it was thought better to ensure consistency and minimize uncertainties due to the as yet untried technique for evaluating the data, by using $\langle \nu \rangle$ for the three nuclides as determined by the same group that furnished the $P_\nu$ data.

A second choice made was to smooth the $\langle \nu \rangle$ data rather than use the quoted experimental points. This is because as long as one stays away from an energy region roughly less than 1 MeV, there is no indication of any structure in $\langle \nu \rangle$ vs $E_n$; the relation seems quite linear, with the exception of what seems to be a definite jog between 4 and 7 MeV for ($^{235}U + n$) (Fig. 11). Thus, with the exceptions noted, deviations from linearity could be considered statistical (Figs. 12 and 13).

The process of normalizing in both senses the smoothed distributions is illustrated in Tables I and II, for ($^{235}U + n$) and ($^{239}Pu + n$) respectively, using the data at $E_n = 0$. Column (a) contains the smoothed data at $E_n = 0$ normalized only in the sense $\sum P_\nu = 1$. Column (b) contains the data normalized additionally so that $\sum \nu P_\nu = \langle \nu \rangle$, where the value of $\langle \nu \rangle$ was evaluated independently. Column (c) shows $P_\nu$ at $E_n = 0$ from our earlier evaluation of thermal $P_\nu$ data, together with the standard deviations assigned at that time.

Final Results and Evaluation

The final results of this process of data evaluation are presented in tabular form (Tables III, IV, V) and graphically (Fig. 14 and 15) to show the general truncated (on the abscissa scale) bell-shape. The $E_n = 0$ $P_\nu$ for ($^{238}U + n$) are obtained of course by extrapolation of the fitted functions. The ($^{238}U + n$) curves are qualitatively similar to those for ($^{239}Pu + n$) and so are not presented.
The important question at this point is how accurately this salvaged data represents reality. There is only $P_v$ data for $E_n = 0$ and then only for $(^{235}\text{U} + n)$ and $(^{239}\text{Pu} + n)$. As mentioned above, the smoothed data of Frehaut extrapolates very well to those $E_n = 0$ values in the two cases where such are available, even before weighting those points (although in recognition of their superior statistical precision and accuracy, they were accordingly weighted in the final fitting process). This argues that the lack of smoothness in the Frehaut data is statistical and that there may not be any serious systematic errors. Then a smoothing procedure which tends to level statistical fluctuations would have validity.

Another indication of validity in the procedure is that in the two cases (Tables I and II) where the thermal values obtained by the data processing procedure could be compared with $P_v$ values obtained independently, the two sets are well within the uncertainties assigned to the independently obtained $P_v$ values, even without weighting the $E_n = 0$ values.*

Finally, each $P_v$ set was derived independently on at least two occasions widely enough separated in time so that some of the subjective details of the process, such as which points to drop from the fitting procedure, how far beyond 10 MeV should points be included, what was the lowest order best fit, etc., were forgotten. Nevertheless, aside from minor differences due to our acquiring improved skills at fitting, the respective derived $P_v$ sets were all essentially equivalent well within the attributed uncertainties.

* The data in Tables I and II do have weighting for the thermal values, which of course increases the apparent agreement.
It would be very desirable to compare the Frehaut data as processed above with $P_v$ values derived from new experiments. However, it was the lack of such experiments together with the low probability in the present state of reactor physics or nuclear data research that such experiments would be performed that was a major reason for the present work.

All things considered, we do feel the present set of data will prove adequate for safeguards work, and the kind of methodology used may even indicate how to design future $P_v$ experiments so as to most economically and efficiently utilize experimental time and facilities.

The lack of a thermal $P_v$ set to "anchor" the fitted curves does however reduce the reliability of $(^{238}\text{U} + n)$ compared to the others.

Further Remarks

A reason for our interest in $P_v$ is the hope that some information regarding the systematics of neutron multiplicity in fission would result. This could be important both from the standpoint of improving the theory of the fission process and to compensate for the lack of experimental data in many instances.

The more we have delved into the details of $P_v$ the more it seems to be so that such systematics as are noticed (e.g. Terrell, et al.) are true only in a fairly approximate sense and are not necessarily accurate enough relations to be useful in many technical applications, such as neutron correlation counting.

In the present paper it is seen that for two nuclides $<v>$ is quite linear over the range $E_n > 1$, say, for two of the nuclides, but must be represented by two lines joined by a smooth transition for the third nuclide. That same
nuclide has no discernable $P_8$ component in the studied region, whereas the other nuclides, only 3 and 4 nucleons away have small but definitely non-zero $P_8$ components.

On the other hand, a plot of the second factorial moment $\langle \nu(\nu-1) \rangle$ for the three nuclides reveals a curious consistency which we are at a loss to explain as being some artifact of the data smoothing process (Fig. 16). The three plots of $\langle \nu(\nu-1) \rangle$ each seem to be made up of two straight line segments with a smooth transition between. The extrapolations of these straight line portions seem to all intersect between 4 and 5 MeV.

These and other speculations can ultimately only be satisfied with more experimental work.
References

1(a) K. Böhnle, Nuclear Science and Engineering 90, 75 (1985).
(b) W. Hage and D. M. Cifarelli, Nuclear Science and Engineering 89, 159 (1985).

2. We have to date evaluated the $P_V$ data for thermal or spontaneous fission for the following nuclides based on what data could be gleaned from the literature or from private communications (all the evaluated data is available at least in the form of BNL reports): $^{257}$Fm, $^{252}$No, $^{242,244,246,248}$Cm, $^{246,250,252,254}$Cf, $^{236,238,239,240,241,242}$Pu, $^{233,235,238}$U.

3. We are very much indebted to J. Frehaut and coworkers at the Commissariat a l'Énergie Atomique, Centre d'Etudes de Bruyères le Chatel, for so kindly furnishing their data.

Figures and Tables

Fig. 1  $P_0$ vs. $E_n$ for $^{239}$Pu
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Fig. 15 $P_v$ vs. $E_n$ for $^{239}$Pu + $n$
Fig. 16 $\langle v(v-1) \rangle$ vs. $E_n$

Table I  Comparison of $P_v (E_n = 0)$ for $^{235}$U + $n$, derived in different ways.

Table II  Comparison of $P_v (E_n = 0)$ for $^{239}$Pu + $n$, derived in different ways.

Table III  $P_v$ vs. $E_n$ (MeV) for $^{235}$U.

Table IV  $P_v$ vs. $E_n$ (MeV) for $^{238}$U.

Table V  $P_v$ vs. $E_n$ (MeV) for $^{239}$Pu.
FIGURE 1

$P_0$ vs. $E_n$ for $^{239}$Pu

RAW DATA

FITTED CURVE (4th ORDER POLY.)

$E_n$ (MeV)

$P_0$

$0.012$

$0.010$

$0.008$

$0.006$

$0.004$

$0.002$

$0.000$

$2.0$ $4.0$ $6.0$ $8.0$ $10.0$ $12.0$ $14.0$
FIGURE 2

$P_1$ vs. $E_n$ (MeV) for $^{239}$Pu

RAW DATA

FITTED CURVE
(4th ORDER POLY.)

$P_1$

$E_n$ (MeV)
FIGURE 3

$P_2$ vs. $E_n$ for $^{239}\text{Pu}$

RAW DATA

FITTED CURVE
($4^{th}$ ORDER POLY)

$P_2$

$E_n$ (MeV)
FIGURE 4

$P_3$ vs. $E_n$ for $^{239}\text{Pu}$

RAW DATA

FITTED CURVE
($3^{rd}$ ORDER POLY.)

$E_n$ (MeV)
FIGURE 5

$P_4$ vs $E_n$ for $^{239}$Pu

RAW DATA

FITTED CURVE

(3rd ORDER POLY)

3rd ORDER

4th ORDER

$E_n$(MeV)
FIGURE 6

$P_5$ vs. $E_n$ for $^{239}$Pu

RAW DATA

FITTED CURVE (3$^{rd}$ ORDER POLY.)
FIGURE 7

$P_6$ vs. $E_n$ for $^{239}$Pu

**RAW DATA**

**FITTED CURVE**
(3\textsuperscript{rd} ORDER POLY.)
**FIGURE 8**

**$P_7$ vs. $E_n$ for $^{239}$Pu**

RAW DATA

Fitted Curve
(3$^{rd}$ ORDER POLY.)

$P_7$

$E_n$(MeV)
FIGURE 9

$P_B$ vs. $E_n$ for $^{239}$Pu

RAW DATA

CURVE FITTED TO "AVERAGED" DATA
(4th ORDER POLY.)
FIGURE 10

$P_8$ vs. $E_n$ for $^{235}\text{U}$

$P_7$ vs. $E_n$ for $^{235}\text{U}$

RAW DATA

FITTED CURVE
($4^{th}$ ORDER POLY.)
FIGURE 11

$\langle \nu \rangle$ FOR $^{235}\text{U} + n$

$E_n$ (MeV)

$\langle \nu \rangle$

$0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14$

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$
FIGURE 12

$\langle \nu \rangle$ FOR $^{238}\text{U} + n$

![Graph showing $\langle \nu \rangle$ for $^{238}\text{U} + n$](image-url)
\[ \langle \nu \rangle \] FOR $^{239}\text{Pu} + n$
Table I

Comparison of $P_v(E_n = 0)$ for ($^{235}$U + n), derived in different ways

a) curve fitted, $\langle v \rangle = 2.41052$.
b) curve fitted, normalized to $\langle v \rangle = 2.41400$.
c) consensus data, normalized to $\langle v \rangle = 2.41400$.

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$\langle v \rangle$ | 2.4105200 | 2.4140000 | 2.4140000 | .007 |
$\langle v(v-1) \rangle$ | 4.6231073 | 4.6364655 | 4.6382 | .0297 |
$\langle v(v-1)(v-2) \rangle$ | 6.7769876 | 6.8063813 | 6.8176 | .1683 |
$\langle v(v-1) \rangle/\langle v \rangle^2$ | .7956325 | .7956325 | .79593 | .00510 |
$\langle v \rangle^2 - \langle v^2 \rangle$ | 1.2230207 | 1.2230695 | 1.2248 | .0297 |
$\langle v^2 \rangle$ | 7.0336273 | 7.0504654 | 7.0522 | .0297 |
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\( \langle \nu(\nu-1)(\nu-2) \rangle \) 12.6350212 12.6169716 12.5447 0.0539
\( \langle \nu(\nu-1) \rangle / \langle \nu \rangle^2 \) 0.8161309 0.8161309 0.81528 0.00223
\( \langle \nu^2 \rangle - \langle \nu \rangle^2 \) 1.3550705 1.3551498 1.3481 0.0184
\( \langle \nu^2 \rangle \) 9.6343328 9.6265254 9.6195 0.0184
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**TABLE III.** $P_v$ vs. $E_n$ (MeV) for $^{235}U$
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<td>0.2840540</td>
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<td>0.3260192</td>
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<td>0.3368663</td>
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**TABLE IV.** $P_\nu$ vs. $E_p$ (MeV) for $^{239}$U
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<td>( v_{(v-1)}(v-2) )</td>
<td>( v_{(v-1)/(v-2)} )</td>
<td>( v_{(v-1)^2/v^2} )</td>
<td>( v_{(v-1)}(v-2) )</td>
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**TABLE V.** \( P_v \) vs. \( E_n \) (MeV) for \(^{239}\text{Pu}\)
$P_\nu$ vs. $E_n$ for $^{235}\text{U} + \text{n}$

$E_n$ (MeV)

$P_\nu$

$\nu = 0$

$\nu = 1$

$\nu = 2$

$\nu = 3$

$\nu = 4$

$\nu = 5$

$\nu = 6$

$\nu = 7$

FIGURE 14
\[ \langle \nu(\nu-1) \rangle \text{ vs. } E_n \]

- \( (239\text{Pu} + n) \)
- \( (235\text{U} + n) \)
- \( (238\text{U} + n) \)

\( E_n \) (MeV)