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A STATISTICAL MODEL FOR HORIZONTAL MASS FLUX OF ERODIBLE SOIL *

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ABSTRACT

It is shown that the mass flux of erodible soil transported horizontally by a statistically distributed wind flow has a statistical distribution. Explicit expression for the probability density function, p.d.f., of the flux is derived for the case in which the wind speed has a Weibull distribution. The statistical distribution for a mass flux characterized by a generalized Bagnold formula is found to be Weibull for the case of zero threshold speed.

Analytic and numerical values for the average horizontal mass flux of soil are obtained for various values of wind parameters, by evaluating the first moment of the flux density function.

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I. INTRODUCTION

Following Bagnold's [1] and Chepil's [2] classic works on mass transport rate of wind-blown soil, a number of authors have obtained various formulae relating the horizontal mass flux of erodible soil to the observed and threshold wind speeds (for a recent review see Greeley and Iversen [3]). A generalized Bagnold formula which incorporates most of these expressions as special cases may be written as

$$q = q_0 (u - u_T)^n u^m$$

in which u and u_T are the observed and threshold wind speeds, n, m and q_0 are constants whose values depend on soil and flow properties.

Field observations and wind data reveal that wind erosion is commonly caused by a highly variable wind flow. One method of characterizing the variability of the wind is to assume a statistical distribution for the wind speed. In particular, Johnson [4] has shown that the two-parameter Weibull distribution fits wind data reasonably well.

Given a mass flux formula and a probability density function for the wind speed, one can compute the average loss of soil over a certain period. Such computation has recently been carried out by Skidmore [5] and Gillette [6]. Skidmore [5] modified Chepil's formula for the rate of erosion of damp material and obtained analytic expression for the wind erosion climatic factor for the case in which the wind speed has a Rayleigh distribution. Gillette [6] used in his analysis a special case of the generalized Bagnold formula given above ($n = 2$, $m = 1$) and the Rayleigh distribution function for the wind speed.

The first objective of this paper is to show that the horizontal mass flux of erodible soil transported by a statistically distributed wind has a statistical distribution and to obtain explicit expression for the flux distribution for the case in which the wind speed has a Weibull distribution. The analysis is carried out in Sec.II. The second objective is to derive from the general expression of the flux probability density function analytic and numerical values for the average soil loss for the case in which the horizontal flux is governed by the generalized Bagnold formula given above. The evaluation is carried out in Sec.III. The results obtained in this section generalize those found by Gillette [6]. Sec.IV contains some concluding remarks.

II. PROBABILITY DISTRIBUTION FOR THE FLUX

Consider a highly variable wind flow over a flat surface of erodible material. Assume that the wind speed has a Weibull distribution whose probability density function is given by

$$f(u) = \frac{k}{c}\left(\frac{u}{c}\right)^{k-1} \exp\left\{-\left(\frac{u}{c}\right)^{k}\right\} , \tag{1}$$

where k and c are the shape and scale parameters.

Let $q(u)$ be the horizontal mass flux of soil transported per unit time per unit area as a result of the wind flow. Let u_T be the threshold wind speed for wind erosion. In general

$$q(u) = K(u)\, I_{(u_T,\infty)}(u) , \tag{2}$$

where $I_{(u_T,\infty)}(u)$ is the indicator function of (u_T,∞) and $K(u)$ is a strictly increasing function of u for $u > u_T$.

For any $q > 0, \exists$ a unique $u = u(q)$ such that $K(u(q)) = q$. Thus, in view of (1), we can easily deduce that the flux $q(u)$ has the commulative distribution function

$$G(q) = \begin{cases} 0 & \text{for } q < 0 \\ \Delta & \text{for } q = 0 \\ 1 - \exp\left\{-\left(\frac{u(q)}{c}\right)^{k}\right\} & \text{for } q > 0 \end{cases} , \tag{3}$$

where

$$\Delta = 1 - \exp\left[-\left(\frac{u_T}{c}\right)^{k}\right] . \tag{4}$$

The probability density function of the flux, $g(q) = G'(q)$ is then given by

$$g(q) = \Delta\, \delta_0(q) + \frac{k}{c}\left[\frac{u(q)}{c}\right]^{k-1} u'(q)\, \exp\left[-\left(\frac{u(q)}{c}\right)^{k}\right] \tag{5}$$

in which δ_0 is the Dirac delta function.

In what follows we shall assume that the horizontal flux $q(u)$ is governed by the generalized Bagnold formula

$$q = q_0 (u - u_T)^n\, u^m . \tag{6}$$

If $u_T = 0$, Eq.(6) gives

$$u = \left(\frac{q}{q_0}\right)^{\frac{1}{n+m}} ,$$

and the substitution of this into (5) leads to

$$g(q) = \frac{k'}{c'}\left(\frac{q}{c'}\right)^{k'-1} \exp\left\{-\left(\frac{q}{c'}\right)^{k'}\right\} , \tag{7}$$

where

$$k' = \frac{k}{n+m} \quad \text{and} \quad c' = q_0\, c^{n+m} . \tag{8}$$

Thus for zero threshold wind speed the probability density function of the flux is Weibull with parameters k' and c' .

III. AVERAGE HORIZONTAL FLUX

The r^{th} moment of the probability density function of the flux, $g(q)$, is given by

$$\langle q^r \rangle = \int_0^\infty q^r\, g(q)\, dq . \tag{9}$$

For $u_T = 0$, the substitution of (7) into (9) gives

$$\langle q^r \rangle_{u_T=0} = c'^r\, \Gamma(1 + \frac{r}{k'}) = q_0^r\, c^{(n+m)r}\, \Gamma\left(1 + \frac{r(n+m)}{k}\right) . \tag{10}$$

The average horizontal flux of soil particles is obtained by calculating the first moment of the probability density function, $g(q)$. Substituting (5) into (9) and setting $r = 1$, we get

$$\langle q \rangle = \frac{k}{c}\int_0^\infty \left(\frac{u}{c}\right)^{k-1} q\, e^{-\left(\frac{u}{c}\right)^k} u'(q)\, dq . \tag{11}$$

Upon making the change of integration variable from q to u, where u and q are related by Eq.(6), Eq.(11) becomes

$$\langle q \rangle = q_0 \left(\frac{k}{c}\right) \int_{u_T}^\infty \left(\frac{u}{c}\right)^{k-1} (u - u_T)^n\, u^m\, e^{-\left(\frac{u}{c}\right)^k} du . \tag{12}$$

Furthermore, defining $x = u/\bar{u}$, where

$$\bar{u} = \int_0^\infty u \; f(u) \; du = c \; \Gamma(1 + \frac{1}{k}) \tag{13}$$

is the average wind speed, and introducing the new integration variable x in (12), we get

$$E = \left\langle \frac{q}{q_0} \right\rangle = k \left(\frac{\bar{u}}{c}\right)^k \bar{u}^{n+m} \int_0^\infty (x-R)^n \; x^{m+k-1} \; e^{-\left(\frac{\bar{u}}{c}\right)^k x^k} dx \; , \tag{14}$$

where

$$R = u_T/\bar{u} \quad .$$

Exact analytic values for the integral in (14) can be obtained for the following special cases:

i) <u>$k = 1$</u>

In this case (14) reduces to

$$E = \left\langle \frac{q}{q_0} \right\rangle = \bar{u}^{n+m} \; R^{\frac{n+m}{2}} \; \Gamma(m+1) \; e^{-\frac{R}{2}} \; W_{\frac{n-m}{2}, \; -\frac{(n+m+1)}{2}}(R) \tag{15}$$

where W is the Whittacker function.

For $n = 1$, $m = 2$ (15) has the value

$$E = \bar{u}^3 \; e^{-R} \; (R^2 + 4R + 6) \quad . \tag{16}$$

ii) <u>$k = 2, \; n = 1, \; m = 2$</u>

In this case (14) takes the form

$$E = \frac{\pi}{2} \; \bar{u}^3 \int_R^\infty (x-R) \; x^3 \; e^{-\frac{\pi}{4} x^2} \; dx \quad . \tag{17}$$

Introducing the new variable $y = \frac{\pi x^2}{4}$, (17) transforms into

$$E = \left(\frac{2}{\sqrt{\pi}} \bar{u}\right)^3 \int_{\lambda^2}^\infty (y^{1/2} - \lambda) \; y \; e^{-y} \; dy, \quad \lambda^2 = \frac{\pi}{4} R^2 \quad . \tag{18}$$

The integral in Eq.(18) can be evaluated in closed form [7] to give

$$E = \left(\frac{2}{\sqrt{\pi}} \; \bar{u}\right)^3 \frac{1}{2} \left\{ \frac{3\sqrt{\pi}}{2} (1 - \Phi(\lambda)) + \lambda e^{-\lambda^2} \right\} . \tag{19}$$

For other values of k, n and m, Eq.(14) has been evaluated numerically and the results are summarized in Figs.1(a)-1(e) and 2(a)-2(g).

IV. CONCLUDING REMARKS

We have obtained in Sec.II an explicit expression for the probability density function describing the random variation of the horizontal mass flux of erodible soil transported by a variable wind speed satisfying a two-parameters Weibull distribution.

The probability density function is used to obtain the average soil loss, E, for the case in which the horizontal flux is proportional to $(u - u_T)^n u^m$. The values obtained show that for a fixed Weibull parameter k and a fixed threshold speed u_T, the average soil transported increases as m increases and n decreases (for values of n and m satisfying $n + m = 3$). On the other hand, for fixed values of n and m the average soil transported decreases as the Weibull parameter k increases for any value of the threshold speed u_T. The case $k = 2$, $n = 1$, $m = 2$ (Fig.(1)) agrees with the results of Gillette [6].

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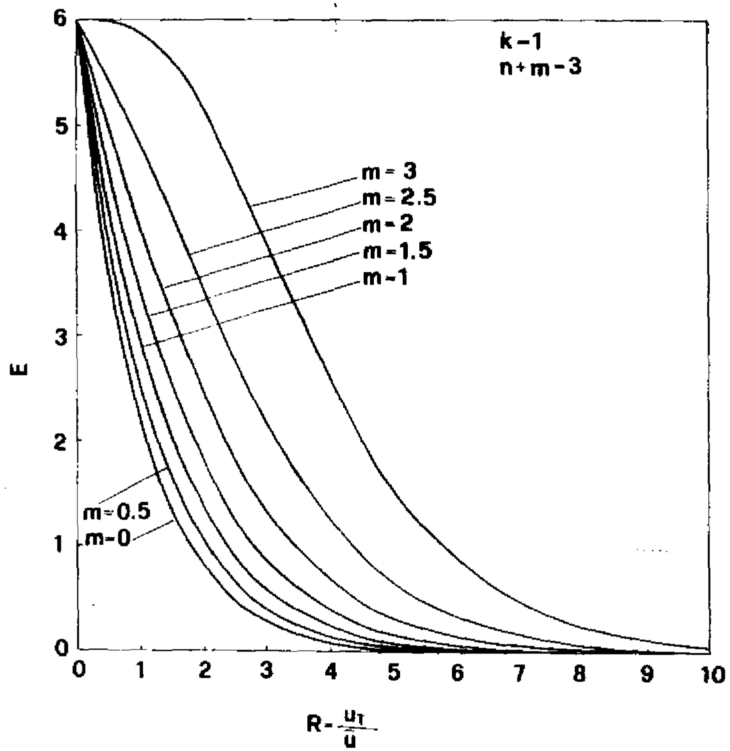
FIGURE CAPTIONS

Figs.1(a)-1(e): Average horizontal flux E plotted against k = threshold speed/average wind speed, for values of Weibull parameter k ranging from 1 to 3.

Figs.2(a)-2(g): Average horizontal flux E plotted against k = threshold speed/average wind speed, for various values of n and m.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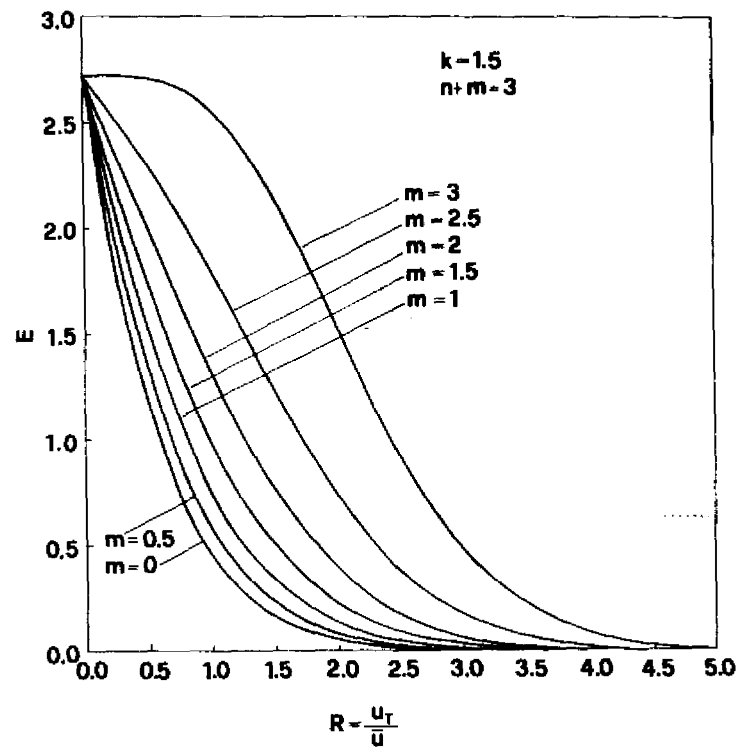
n+m=3
m=3
m=2.5
m=2
m=1.5
m=1
m=0.5
m=0
E
R = uT/ū

Fig.1a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n+m=3
m=3
m=2.5
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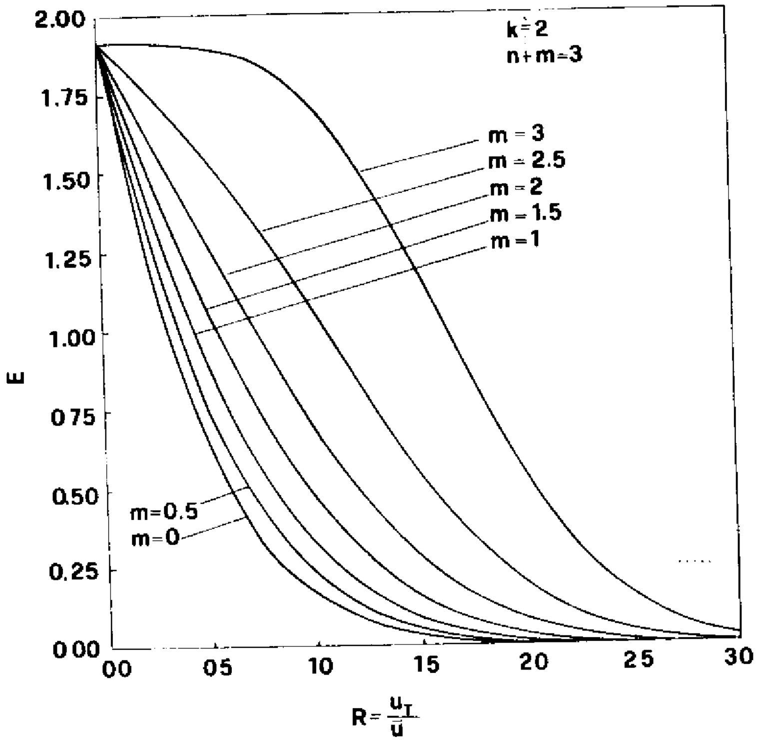
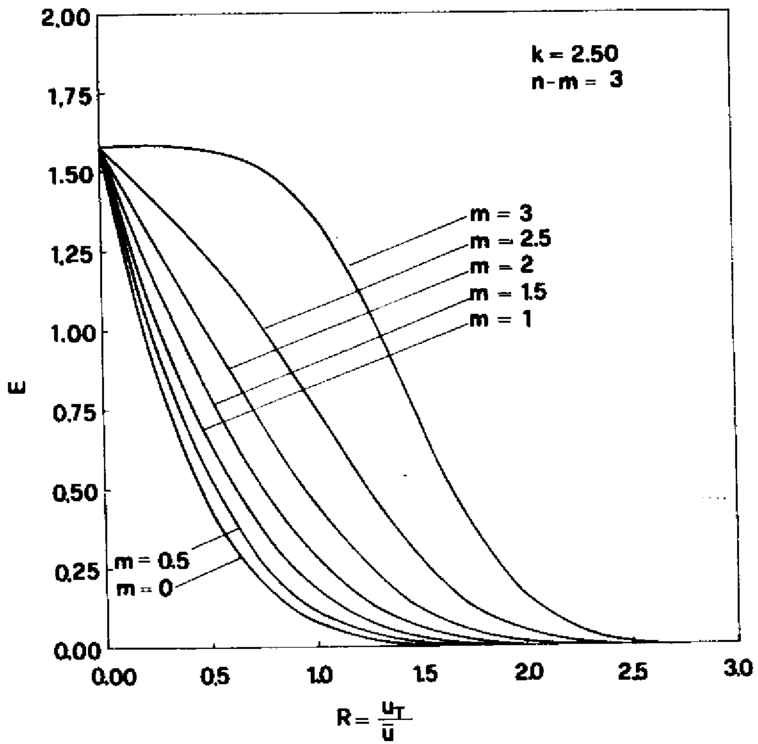
m=1.5
m=1
m=0.5
m=0
E
R = uT/ū

Fig.1b



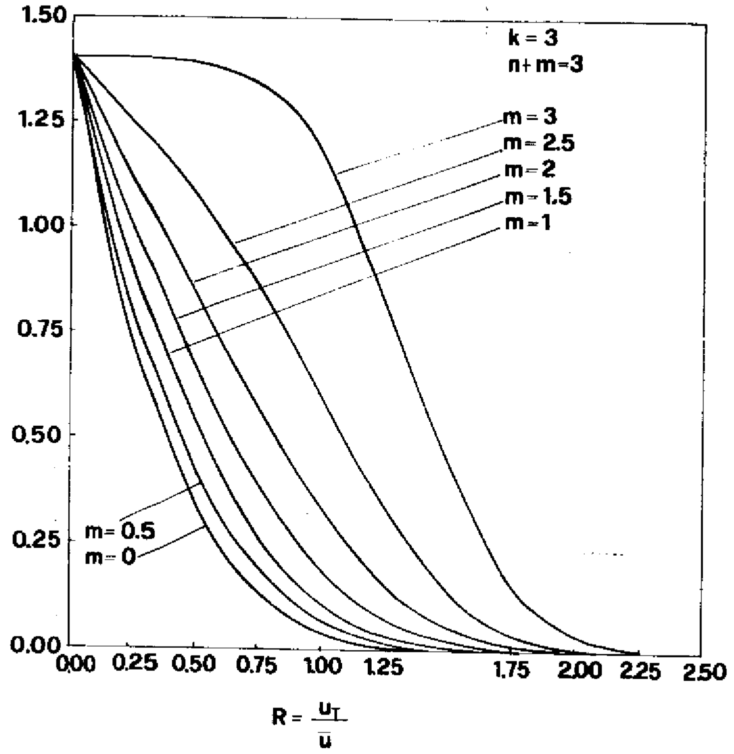
k=2
n+m=3
m=3
m=2.5
m=2
m=1.5
m=1
m=0.5
m=0
E
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$R=\frac{u_T}{\bar{u}}$

Fig.1c



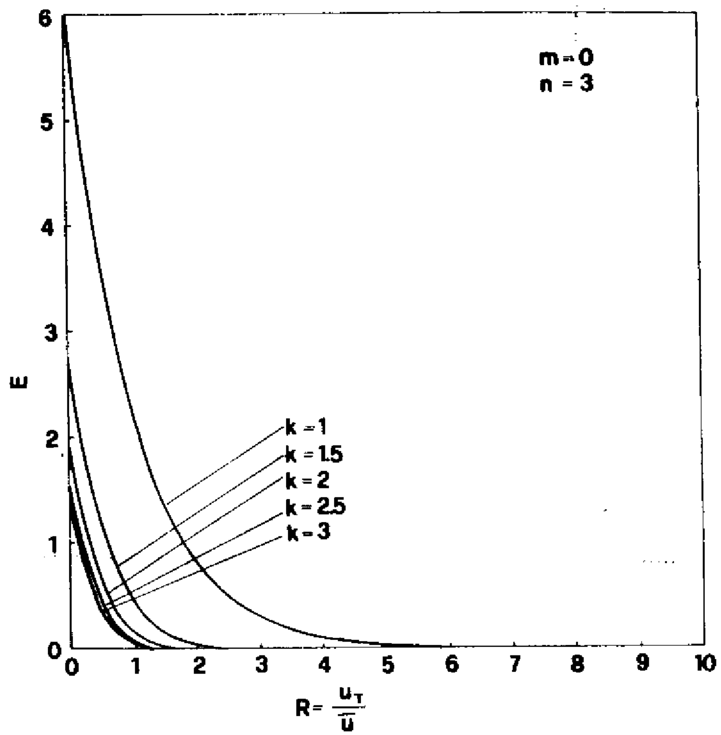
k = 2.50
n-m = 3
m = 3
m = 2.5
m = 2
m = 1.5
m = 1
m = 0.5
m = 0
E
2,00
1,75
1.50
1,25
1.00
0,75
0,50
0,25
0.00
0.00
0.5
1.0
1.5
2.0
2.5
3.0
$R=\frac{u_T}{\bar{u}}$

Fig.1d



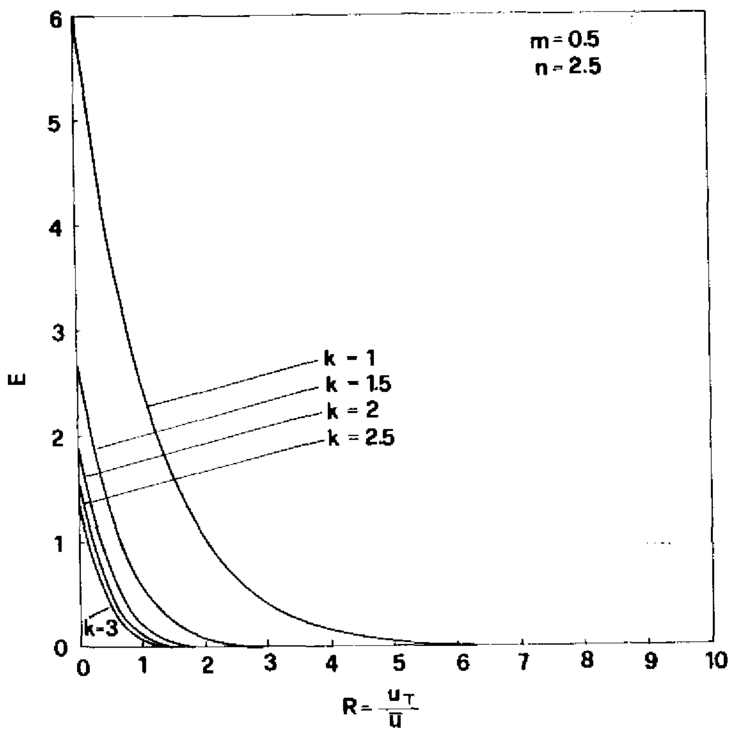
k = 3
n+m=3
m = 3
m= 2.5
m= 2
m= 1.5
m= 1
m= 0.5
m= 0
E
R = $\frac{u_T}{\bar{u}}$

Fig.1e



m = 0
n = 3
k = 1
k = 1.5
k = 2
k = 2.5
k = 3
E
R = $\frac{u_T}{\bar{u}}$

Fig.2a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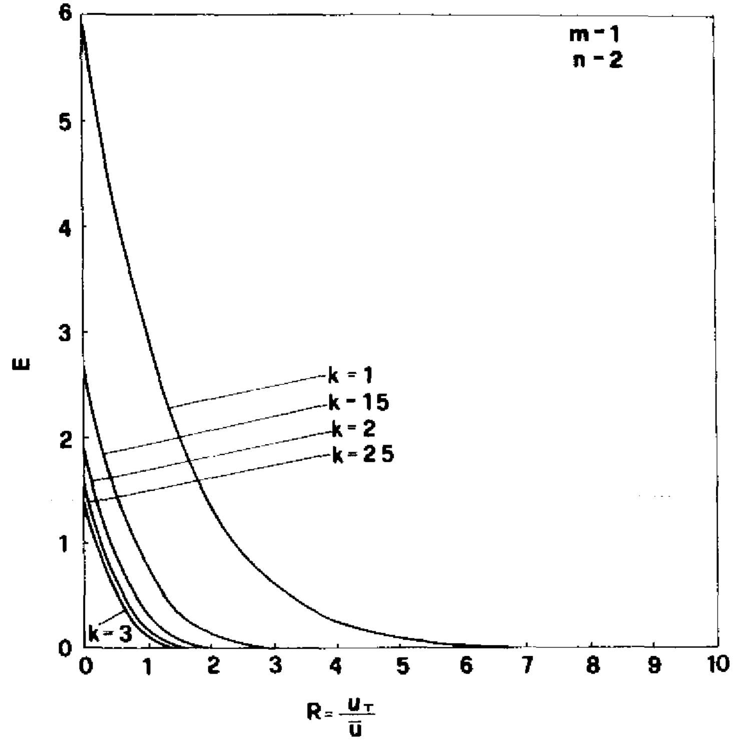
n = 2.5
k = 1
k = 1.5
k = 2
k = 2.5
k=3
E
R = $\frac{u_T}{\bar{u}}$
0
1
2
3
4
5
6
7
8
9
10

Fig.2b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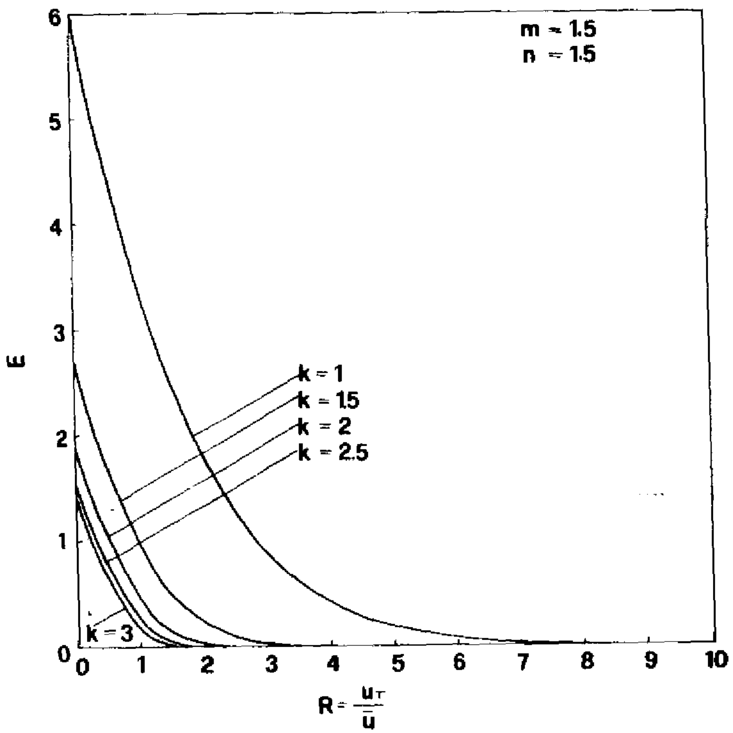
n = 2
k = 1
k = 1.5
k = 2
k = 2.5
k = 3
E
R = $\frac{u_T}{\bar{u}}$
0
1
2
3
4
5
6
7
8
9
10

Fig.2c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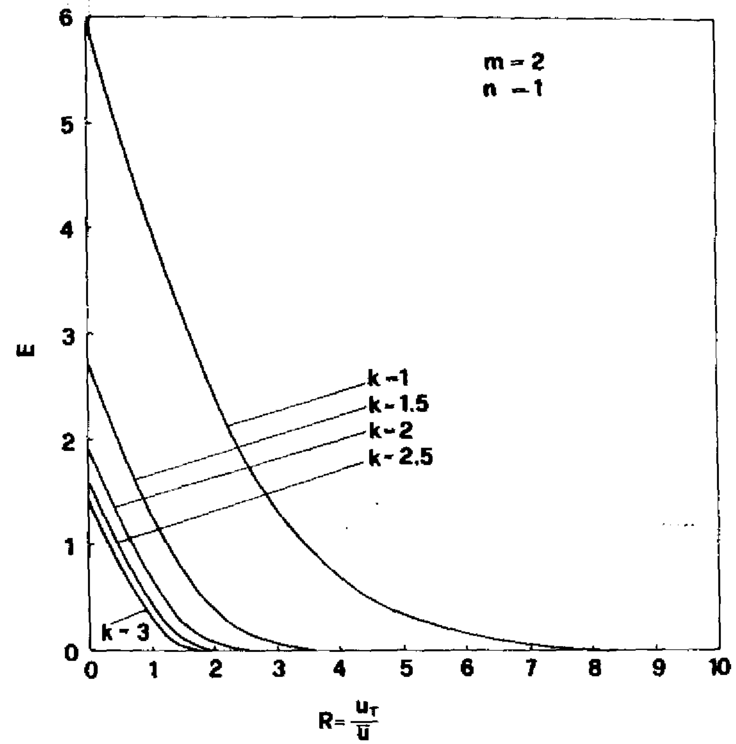
n = 1.5
k = 1
k = 1.5
k = 2
k = 2.5
k = 3
E
R = $\frac{u_T}{\bar{u}}$
0
1
2
3
4
5
6
7
8
9
10

Fig.2d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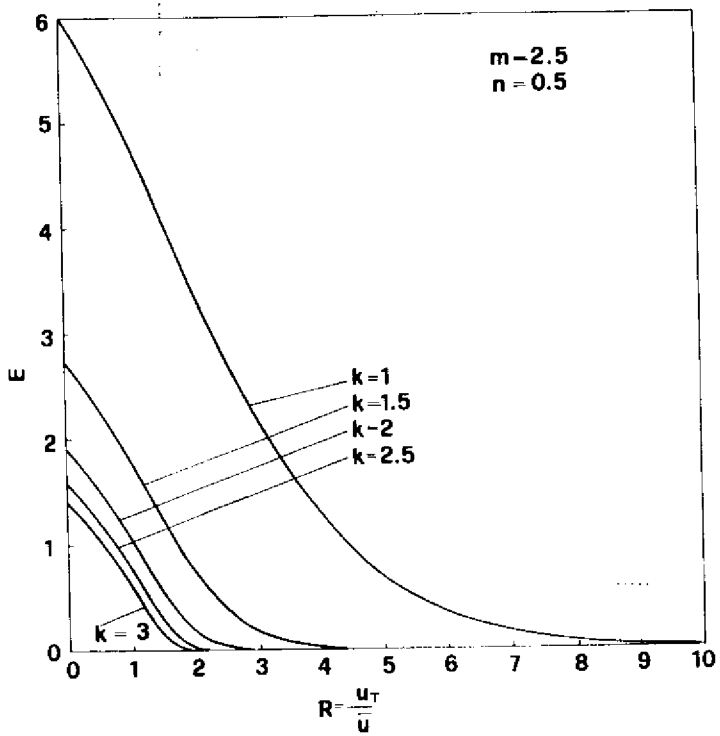
n = 1
k = 1
k = 1.5
k = 2
k = 2.5
k = 3
E
R = $\frac{u_T}{\bar{u}}$
0
1
2
3
4
5
6
7
8
9
10

Fig.2e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n = 0.5
k = 1
k = 1.5
k = 2
k = 2.5
k = 3
E
R = $\frac{u_T}{\bar{u}}$

Fig.2f

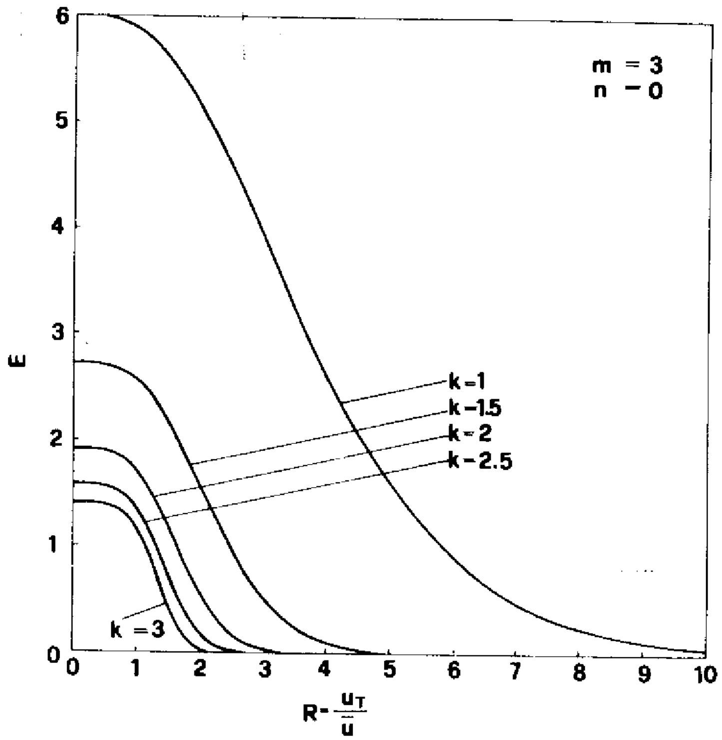
m = 3
n = 0
k = 1
k = 1.5
k = 2
k = 2.5
k = 3
E
R = $\frac{u_T}{\bar{u}}$

Fig.2g