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WHAT IS NEW IN THE STUDY OF DIFFERENTIAL EQUATIONS BY GROUP THEORETICAL METHODS ? *

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ABSTRACT

Several recent developments have made the application of group theory to the solving of differential equations more powerful than it used to be. The ones discussed here are : 1. The advent of symbol manipulating computer languages that greatly simplify the construction of the symmetry group of an equation 2. Methods of finding all subgroups of a given Lie symmetry group 3. The theory of infinite dimensional Lie algebras 4. The combination of group theory and singularity analysis.

1. INTRODUCTION

The idea of applying Lie group theory to solve differential equations is as old as Lie group theory itself [1]. An extensive literature exists on this subject, [2,...,5] but until quite recently, group theory has in this respect been underused. The reason may be several misconceptions in the minds of potential users of group theory, such as : "It is as difficult to find the symmetry group of an equation as it is to solve it" ; "Group theory only provides randomly occuring particular solutions" ; "Group theory is only useful for linear equations". We hope to show that such objections should be put to rest.

The symmetry group of a system of differential equations is, roughly speaking, a group of transformations of the independent and dependent variables, leaving the set of all solutions invariant. Once the symmetry group of a system of equations is known, it can be used to generate new solutions from old ones (often physically interesting ones from trivial ones).

It can be used to classify solutions into conjugacy classes and to classify and simplify differential equations. An important application is symmetry reduction : the reduction of an ordinary differential equation (ODE) to a lower order one, the reduction of a partial differential equation (PDE) to one with fewer independent variables.

This presentation is restricted to Lie point symmetries (for generalized symmetries see e.g. P. Olver's talk in the same Proceedings) and concentrates on applications to nonlinear partial differential equations. We hope to bring out the new features mentioned in the abstract [6-16].

2. CONSTRUCTION OF THE SYMMETRY GROUP

The algorithm for finding the group of Lie point transformations of a system of differential equations goes back to S. Lie and is formulated in an accessible form e.g. by P. Olver [2]. Computer programs implementing the algorithm are available (e.g.) in REDUCE [6] and MACSYMA [7]). The algorithm (and programs) apply to the system of equations

$$\Delta^{i}(x, u^{(k)}) = 0, \qquad i = 1, \dots, m$$
$$x = (x_1, \dots, x_n), \qquad u = (u_1, \dots, u_m), \tag{1}$$

where x are the independent variables, u the dependent ones and $u^{(k)}$ denotes the set of all derivatives of u upto order k.

The local Lie group of point transformations is given by the formulas

$$\tilde{x} = \Lambda_g(x, u), \qquad \tilde{u} = \Omega_g(x, u), \tag{2}$$

where the functions Λ_g and Ω_g are such that $\tilde{u}(\tilde{x})$ satisfies (1), whenever $u(x)$ does. Generalized symmetries (contact transformations, Lie Bäcklund transformations [4]) would allow the functions Λ_g and Ω_g to depend on the derivatives of $u(x)$ as well.

In order to find Λ_g and Ω_g (g denotes the group element) we use a Lie algebraic approach. An element of the Lie algebra corresponding to the Lie group given by (2) will have the form

$$V = \sum_{i=1}^{n} \eta_i (x, u) \frac{\partial}{\partial x_i} + \sum_{\ell=1}^{m} \phi_\ell (x, u) \frac{\partial}{\partial u_\ell} , \quad (3)$$

where the coefficients $\eta_i (x, u)$ and $\phi_\ell (x, u)$ are determined from the condition

$$pr^{(k)} V [\Delta^i (x, u^{(k)})] = 0, \quad \text{whenever} \quad \Delta^\ell (x, u^{(k)}) = 0, \ \ell = 1,...,m. \quad (4)$$

In (4) $pr^{(k)} V$ is the k-th prolongation of the vector field V (k is the order of the equation), i.e. an operator of the form

$$pr^{(k)} V = V + \sum_{\ell=1}^{m} \sum_{J} \psi_\ell^J (x, u^{(k)}) \frac{\partial}{\partial u_\ell^J} . \quad (5)$$

The label J runs over all values such that (5) includes differentiation with respect to all derivatives of u_ℓ upto order k. The coefficients ψ_ℓ^J are not free : they are completely determined by the functions η_i and ϕ_ℓ in (3) (for recursive formulas determining higher order prolongations in terms of lower ones, see e.g. [2,6,7]. The computer implemented algorithm consists of the following steps :

1. Construct $pr^{(k)}V$ (the result depends only on the number of independent and dependent variables and on the order of the equation)

2. Apply $pr^{(k)}V$ to the considered system (1) as in (4). In (4) x_i, u_ℓ and u_ℓ^J are viewed as independent variables, the unknowns are η_i and ϕ_ℓ.

3. Obtain a system of unconstrained equations by making full use of equation (1) and its differential consequences. Having m equations, we eliminate m components $v_1, ..., v_m$ of the vector $u^{(k)}$ from the system (1), to obtain $v_i = S_i(x, w)$ (where $u^{(k)} = (v, w)$. The derivatives of v_i are obtained from the differential consequences of (1) (for details and a discussion of the restrictions, see Ref. 7).

4. Eliminate all v_i and their derivatives from the system (4).

5. Obtain a system of "determining equations" by setting equal to zero the coefficients of all functiononally independent expressions in the remaining derivatives u_ℓ^J $(J \neq 0)$.

6. Solve the determining equations for $\eta_i (x, u)$ and $\phi_\ell (x, u)$.

The programs [6,7] essentially realize steps 1-5 and to varying degrees step 6. Steps 1-5 of the algorithm involve cumbersome manipulations even for relatively simple equations. Having them done by a computer is a great saving. The user must only specify the equations, the dependent and independent variables, and the variables v_i to be eliminated in step 4.

3. SYMMETRY GROUP OF THE DAVEY-STEWARTSON EQUATIONS

3.1 The Equations

We shall now turn to the study of a specific system of equations, namely the Davey-Stewartson equations (DSE's) [17]

$$\begin{aligned} i \psi_t + \psi_{xx} + \varepsilon_1 \psi_{yy} &= \varepsilon_2 |\psi|^2 \psi + \psi w , \\ w_{xx} + \delta_1 w_{yy} &= \delta_2 (|\psi|^2)_{yy}. \end{aligned} \quad (6)$$

In (6) we have $\psi(x, y, t) = u (x, y, t) + i v (x, y, t)$ and u, v and w are real functions of x, y and t ; $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$ and δ_1, δ_2 are constants (the subscripts of ψ and w denote partial derivatives). Important features of the DSE's are :

a. They are physically interesting (they describe the propagation of two-dimensional waves in water of finite depth and also the interaction of electromagnetic waves with a plasma). For one-dimensional propagation we have $\psi = \psi (x, t)$, $w = 0$ and the DSE's reduce to the nonlinear Schrödinger equation.

b. They are integrable by inverse scattering techniques [18] and have been shown to allow soliton and multisoliton solutions [19].

c. Their group of Lie point symmetries is infinite-dimensional. It is this last point that we shall elaborate on here. The presentation will be somewhat sketchy. For all details see Ref. 13.

3.2 The Symmetry Algebra

Applying the algorithm of Section 2 (and the program of Ref. 7) we find that the general element (3) of the symmetry algebra for the DSE's has the form [13)]

$$V = X(f) + Y(g) + Z(h) + W(\ell), \qquad (7a)$$

$$X(f) = f(t)\,\partial_t + \frac{1}{2} f'(t)\,(x\,\partial_x + y\,\partial_y - u\,\partial_u - v\,\partial_v - 2\,w\,\partial_w) - \frac{1}{8}(x^2 + \epsilon_1 y^2)\,[\,f''(t)\,(v\,\partial_u - u\,\partial_v) + f'''(t)\,\partial_w\,]$$

$$Y(g) = g(t)\,\partial_x - \frac{1}{2}\,x\,[\,g'(t)\,(v\,\partial_u - u\,\partial_v) + g''(t)\,\partial_w\,] \qquad (7b)$$

$$Z(h) = h(t)\,\partial_y - \frac{1}{2}\,\epsilon_1\,y\,[\,h'(t)\,(v\,\partial_u - u\,\partial_v) + h''(t)\,\partial_w\,]$$

$$W(\ell) = \ell(t)\,(v\,\partial_u - u\,\partial_v) + \ell'(t)\,\partial_w$$

Here g, h and ℓ are arbitrary functions of time t (in $C^\infty(U)$, $U \subset R$) ; the primes denote time derivatives. For $\delta_1 = -\epsilon_1$ the function f(t) is also arbitrary ; for $\delta_1 \neq -\epsilon_1$ we have $f(t) = (a + b\,t)^2$, where a and b are constants. We recall that $\psi = u + iv$, $u, v \in R$.

It is easy to verify that the operators (7) form a closed set under commutation (e.g. we have $[\,X(f_1), X(f_2)\,] = X(f_1 f'_2 - f'_1 f_2)$). They form an *infinite dimensional Lie algebra since they* depend on arbitrary functions, rather than on a finite number of constants. The Lie algebra L of (7) allows a Levi decomposition

$$L \simeq S \ltimes N, \qquad (8)$$

where $S = \{X(f)\}$ is simple and $N = \{Y(g), Z(h), W(\ell)\}$ is the radical of N (which happens to be nilpotent).

Restricting f, ..., ℓ in (7) to be first order polynomials, we obtain an 8 dimensional subalgebra of L. This subalgebra contains all the physically obvious symmetries of the DSE's, namely translations, Galilei transformations, dilations and certain changes of the phase of ψ.

3.3 The Loop Structure of the Symmetry Algebra

The DSE's have been shown to be integrable [19)] in the case when $\delta_1 = -\epsilon_1$ and hence f(t) is arbitrary. The Lie algebra L in this case contains a specific loop structure, shared by all other known integrable nonlinear PDE's in 2 + 1 dimensions [11-13,20)]. To see this we expand f(t), ..., ℓ(t) into formal Laurent series and obtain a basis

$$\begin{aligned}
X(t^n) &= t^n\,\partial_t + \frac{1}{2}\,n\,t^{n-1}\,\Delta - \frac{1}{4}\,n\,(n-1)\,t^{n-2}\,A_1 - \frac{1}{4}\,n\,(n-1)(n-2)\,t^{n-3}\,W_1 \\
Y(t^n) &= t^n X - \frac{1}{2}\,n\,t^{n-1}\,A_2 - \frac{1}{2}\,n\,(n-1)\,t^{n-2}\,W_2 \\
Z(t^n) &= t^n Y - \frac{1}{2}\,\epsilon_1\,n\,t^{n-1}\,A_3 - \frac{1}{2}\,\epsilon_1\,n\,(n-1)\,t^{n-2}\,W_3 \\
W(t^n) &= t^n\,A_4 + n\,t^{n-1}\,W_4 \qquad (n \in Z)
\end{aligned} \qquad (9)$$

where we have put

$$\begin{aligned}
\Delta &= x\,\partial_x + y\,\partial_y - u\,\partial_u - v\,\partial_v - 2\,w\,\partial_w, \quad X = \partial_x,\ Y = \partial_y \\
A_1 &= \frac{1}{2}(x^2 + \epsilon_1 y^2)\,(v\,\partial_u - u\,\partial_v), \quad A_2 = x\,(v\,\partial_u - u\,\partial_v) \\
A_3 &= y\,(v\,\partial_u - u\,\partial_v), \quad A_4 = v\,\partial_u - u\,\partial_v \\
W_1 &= \frac{1}{2}(x^2 + \epsilon_1 y^2)\,\partial_w,\ W_2 = x\,\partial_w,\ W_3 = y\,\partial_w,\ W_4 = \partial_w\,.
\end{aligned} \qquad (10)$$

The operators (10) form the basis of an 11 dimensional solvable Lie algebra with an 8 dimensional abelian ideal $\{A_i, W_i, i = 1, \dots 4\}$. This algebra can be imbedded [13] into $s\ell(7, C)$ and we can identify the Lie algebra (9) as an infinite dimensional subalgebra of the affine loop algebra [10] :

$$\widetilde{s\ell}(7, C) \sim \{R(t, t^{-1}) \otimes s\ell(7, C)\} \oplus R(t, t^{-1}) \frac{d}{dt}, \qquad (11)$$

where $R(t, t^{-1})$ is the algebra of real Laurent polynomials in t.

3.4 The Symmetry Group

Formulas (7) provide us with the symmetry algebra of the DSE's ; integrating these vector fields, we obtain elements of the symmetry group. As an example, take $Y(g)$ of (7b). We must integrate the equations

$$\frac{d\tilde{x}}{d\lambda} = g(\tilde{t}), \quad \frac{d\tilde{y}}{d\lambda} = 0, \quad \frac{d\tilde{t}}{d\lambda} = 0,$$
$$\frac{d\tilde{u}}{d\lambda} = -\frac{1}{2}\tilde{x} g'(\tilde{t})\tilde{v}, \quad \frac{d\tilde{v}}{d\lambda} = \frac{1}{2}\tilde{x} g'(\tilde{t})\tilde{u}, \quad \frac{d\tilde{w}}{d\lambda} = -\frac{1}{2}\tilde{x} g''(\tilde{t}), \qquad (12)$$

subject to the initial conditions $\tilde{x}(\lambda = 0) = x$, etc. The result is

$$\tilde{x} = x + \lambda g(t) \qquad \tilde{y} = y \qquad \tilde{t} = t$$
$$\tilde{\psi}(\tilde{x}, \tilde{y}, \tilde{t}) = \psi(\tilde{x} - \lambda g(\tilde{t}), \tilde{y}\, \tilde{t}) \exp \frac{1}{2} i \lambda g'(\tilde{t}) [\tilde{x} - \frac{1}{2}\lambda g(\tilde{t})]$$
$$\tilde{w}(\tilde{x}, \tilde{y}, \tilde{t}) = w(\tilde{x} - \lambda g(\tilde{t}), \tilde{y}\, \tilde{t}) - \frac{1}{2}[\tilde{x} - \frac{1}{2}\lambda g(\tilde{t})] \lambda g''(\tilde{t}) \qquad (13)$$

We see that $Y(g)$ generates a transformation to a frame moving in the x direction with arbitrary velocity $\lambda g(t)$ (a Galilei transformation is obtained for $g(t) = t$).

3.5 Classification of Low-Dimensional Subalgebras

In order to perform symmetry reduction for the DSE's we need to find all subgroups of the symmetry group that have orbits of dimension 1 or 2 in the three-dimensional space of independent variables. Methods for classifying subalgebras of Lie algebras have been elaborated for finite-dimensional Lie algebras [8,9]. They are applicable to infinite dimensional Lie algebras as well [11,13]. Here we restrict ourselves to one-dimensional algebras and present only the results. A general element of the DS algebra (7) can, by a transformation of the DS group, be transformed into one of the following Lie algebras

$$L_{1,1} = \{X(1)\} \quad \text{if} \quad f(t) \neq 0$$
$$L^a_{1,2} = \{Y(1) + a Z(1)\} \quad \text{if} \quad f(t) = 0,\ h(t) = a g(t) \neq 0$$
$$L^h_{1,3} = \{Y(1) + Z(h(t))\} \quad \text{if} \quad f(t) = 0,\ h(t) \neq a g(t) \neq 0,\ h'(t) \neq 0 \qquad (14)$$
$$L_{1,4} = \{Z(1)\} \quad \text{if} \quad f(t) = g(t) = 0,\ h(t) \neq 0$$
$$L_{1,5} = \{W(t)\} \quad \text{if} \quad f(t) = g(t) = h(t) = 0,\ \ell'(t) \neq 0$$
$$L_{1,6} = \{W(1)\} \quad \text{if} \quad f(t) = g(t) = h(t) = 0 \quad \ell(t) = \ell_0 = \text{const.}$$

3.6 Symmetry Reduction

In order to reduce the DSE's to a two-dimensional system, we apply the method of symmetry reduction. In the present case this amounts to finding the invariants of the action of a one-dimensional subgroup of the symmetry group, when acting on the space of independent and dependent variables. In Lie algebraic terms, we do the following. Take a specific vector field V of the type (7) and request that it annihilate an auxiliary function F :

$$V F(x, y, t, u, v, w) = 0. \qquad (15)$$

This implies that F is a function of 5 variables $I_1, \dots, I_5$, the invariants of the subgroup e^V. We choose these invariants in such a manner that two of them, say $I_1 \equiv \xi\ (x, y, t)$, $I_2 \equiv \eta\,(x, y, t)$ depend on the independent variables only. The other three invariants $I_3 \equiv Q$, $I_4 \equiv \phi$, $I_5 \equiv \phi^*$ depend on all variables and we use then to express the dependent variables as

$$\begin{aligned} \Psi\,(x, y, t) &= \alpha\,(x, y, t)\,\phi\,(\xi, \eta) + \beta\,(x, y, t) \\ w\,(x, y, t) &= \alpha\,(x, y, t)\,Q\,(\xi, \eta) + \delta\,(x, y, t) \end{aligned} \tag{16}$$

In (16) the functions α, β, γ, δ, ξ and η are explicitly known. The complex and real functions $\phi\,(\xi, \eta)$ and $Q\,(\xi, \eta)$, respectively, are subject to a two-dimensional system of equations, obtained by substituting (16) into the DSE's (6).

As an example, let V in (15) be the general vector field (7) with $f \neq 0$ (in the considered region of t) i.e. an element of the most general subalgebra conjugate to $L_{1,1}$ of (14). In order to determine the invariants I_μ we must solve the characteristic system of ODE's, namely in this case,

$$\frac{dt}{f} = \frac{dx}{\frac{1}{2}\,x\,f' + g} = \frac{dy}{\frac{1}{2}\,y\,f' + h} = \frac{dw}{-f'\,w - \frac{1}{8}\,(x^2 + \epsilon_1 y^2) f''' - \frac{1}{2}\,x g'' - \frac{1}{2}\,\epsilon_1\,y\,h'' + \ell'}$$

$$= \frac{du}{-\frac{1}{2}\,f'\,u - [\,\frac{1}{8}\,(x^2 + \epsilon_1\,y^2)\,f'' + \frac{1}{2}\,x\,g' + \frac{1}{2}\,\epsilon_1\,y\,h' - \ell\,]\,v}$$

$$= \frac{dv}{-\frac{1}{2}\,f'\,v + [\,\frac{1}{8}\,(x^2 + \epsilon_1\,y^2)\,f'' + \frac{1}{2}\,x\,g' + \frac{1}{2}\,\epsilon_1\,y\,h' - \ell\,]\,u}\ .$$

Integrating the first two equations we find

$$x\,f^{1/2} - \int_0^t g(s)\,[\,f(s)\,]^{-3/2}\,ds, \quad \eta = y f^{-1/2} - \int_0^t h(s)[\,f(s)\,]^{-3/2}\,ds\ . \tag{17a}$$

The remaining ones yield

$$\Psi\,(x,y,t) = \phi\,(\xi, \eta) f^{-1/2} \exp i\,[\,\frac{1}{8}\,(x^2 + \epsilon_1 y^2)\,\frac{f'}{f} + \frac{1}{2f}\,(xg + \epsilon_1\,y\,h) - \frac{1}{2}\int \frac{\epsilon_1\,h^2 + g^2 + 2\,\ell\,f}{f^2}\,ds\,] \tag{17b}$$

$$w\,(x, y, t) = Q\,(\xi, \eta)\,f^{-1} - \frac{1}{8}\,f^{-2}\,(ff'' - \frac{1}{2}\,f'^2)\,(x^2 + \epsilon_1\,y^2) - \frac{1}{4}\,x\,f^{-2}\,(2\,g'f - gf') - \frac{1}{4}\,\epsilon_1 y f^{-2}\,(2h'f - hf') + \frac{1}{4}\,f^{-2}\,(g^2 + \epsilon_1\,h^2 + 4\,\ell\,f).$$

For $\delta_1 = -\epsilon_1$ we put (17) into the DSE's (6) and obtain the reduced two-dimensional system

$$\begin{aligned} \phi_{\xi\xi} + \epsilon_1\,\phi_{\eta\eta} &= \epsilon_2\,|\,\phi\,|^2\,\phi + \phi\,Q \\ Q_{\xi\xi} - \epsilon_1\,Q_{\eta\eta} &= \delta_2\,(\,|\,\phi\,|^2)_{\eta\eta} \end{aligned} \tag{18}$$

Thus, any solution (ϕ, Q) of (18), when substituted into (17) will yield a class of solutions of the DSE's, depending on 4 arbitrary functions of t. Notice that a "trivial" constant solution of (18)

$$(\phi_0, Q_0 = -\epsilon_2\,|\,\phi_0\,|^2) \tag{19}$$

provides a quite nontrivial class of solutions of the DSE's.

Reductions by other subalgebras in the list (14) lead to different two-dimensional equations than (18) and to different types of solutions of the DSE's [13]. Reductions by two-dimensional subalgebras [13] would lead to systems of ODE's, but we shall not go into this here. The system (18) merits an independent study.

4. SOLUTIONS OF THE CLASSICAL ϕ^6 FIELD EQUATIONS

4.1 Introductory Comments

In this Section we shall use an example to show how the combination of group theory with singularity analysis can be used to obtain exact analytic solutions of nonlinear PDE's that do not necessarily belong to the class of "integrable" equations. The example is itself of considerable physical interest, namely the nonlinear Klein-Gordon equation (NLKGE) :

$$\Box_\varepsilon \phi = -2(a_2 \phi + 2 a_4 \phi^3 + 3 a_6 \phi^5), \quad a_6 \neq 0 \tag{20}$$

where

$$\Box_\varepsilon = \frac{\partial^2}{\partial x_0^2} + \varepsilon \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad \varepsilon = \pm 1 \tag{21}$$

and a_2, a_4, a_6 are constants.

This equation occurs in condensed matter theory in the study of critical phenomena (e.g. in Landau-Ginzburg theory [21]) and also in relativistic field theory. A case of special interest occurs when $a_2 = a_4 = 0$, $-6 a_6 \equiv a \neq 0$, the tricritical point, where the equation reduces to

$$\Box_\varepsilon \phi = a \phi^5. \tag{22}$$

In order to solve (20) (or (22)) we use the following general method :

1. Find the symmetry group.
2. Find all subgroups with generic orbits of codimension 1 in the underlying Euclidean ($\varepsilon = 1$) or Minkowski ($\varepsilon = -1$) space.
3. Use the subgroups to reduce the equation to an ODE.
4. Solve the ODE using Painlevé analysis.

We mainly wish to illustrate the last point in which algebraic computing can again play an important role.

For all details see Ref 16.

4.2 <u>The Symmetry Group, its Subgroups and Symmetry Reduction</u>

Applying the algorithm of Section 2 we can easily find the symmetry group of the NLKGE for any fixed n. It turns out that for $(a_2, a_4) \neq (0, 0)$ the symmetry group is only the obvious one, namely the Euclidean group $E(n+1)$ for $\varepsilon = 1$ and the Poincaré group $P(n, 1)$ for $\varepsilon = -1$. For $a_2 = a_4 = 0$, i.e. equation (22), the group is larger. For $n \neq 2$ the group is Sim (n + 1), or Sim (n,1), i.e. the similitude group of the corresponding Euclidean or Minkowski space, respectively. For $n = 2$ the group is even larger : the corresponding conformal group O (4, 1) or O (3,2), respectively. We shall in this lecture restrict ourselves to $n = 3$ and $\varepsilon = -1$.

The Lie algebra sim (3,1) is generated by translations P_μ, rotations M_{ab}, Lorentz boosts M_{0a}, and dilations D. The corresponding vector fields V are :

$$P_\mu = \frac{\partial}{\partial x_\mu}, \quad M_{ab} = x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a}, \quad M_{0a} = -x_0 \frac{\partial}{\partial x_a} - x_a \frac{\partial}{\partial x_0}$$

$$D = \sum_\mu x_\mu \frac{\partial}{\partial x_\mu} - \frac{1}{2}\phi \frac{\partial}{\partial \phi} \qquad \mu = 0, 1, 2, 3, \quad a, b = 1, 2, 3 \tag{23}$$

The Poincaré algebra p (3,1) is obtained by omitting the dilation D. All subalgebras of p (3,1) and sim (3,1) are known [8] ; the subalgebras of p (n,1) corresponding to subgroups of P (n,1) with orbits of codimension 1 in M(n,1) are known [22] for all n.

Reductions by subgroups of P (3,1) lead to the ODE [16,22]

$$\ddot F + \frac{k}{\xi}\dot F = -2\lambda(a_2 F + 2 a_4 F^3 + 3 a_6 F^5) \tag{24}$$

where

$$\phi(x) = \rho(x) F(\xi) \qquad x = (x_0, x_1, x_2, x_3) \tag{25}$$

$$\xi = \xi_k = (x_0^2 - x_1^2 - \dots - x_k^2)^{1/2}, \quad \xi = r_k = (x_1^2 + \dots + x_{k+1}^2)^{1/2},$$

$$\xi = \delta_0 = x_1 \cos\theta + x_2 \sin\theta + f$$

or $\xi = \delta_1 = [\,(x_1 + f_1)^2 + (x_2 + f_2)^2\,]^{1/2}$.

The dots in (24) denote derivatives with respect to the argument ; θ, f, f_1, and f_2 are arbitrary functions of $x_0 + x_3$ (for a discussion of "degenerate variables" such as δ_0 and δ_1, see Ref. [16,23]). For subgroups of $P(3,1)$ we have $\rho(x) = 1$ in (25). The value of k in (24) is given by the subscript for ξ_k $(0 \le k \le 3)$, r_k $(0 \le k \le 2)$ and δ_a $(a = 0,1)$; $\lambda = 1$ for ξ_k, $\lambda = -1$ for r_k and δ_a.

For $a_2 = a_4 = 0$ the situation is much richer and subgroups of Sim (3,1) involving dilations lead to a whole series of further reductions. The reduction formula in all cases has the form (25) and leads to an ODE of the form

$$(\alpha_1 + \alpha_2\,\xi + \alpha_3\,\xi^2)\,F + (\beta_1 + \beta_2\,\xi)\,F + \gamma\,F + (\mu_1 + \mu_2\,\xi)\,F^5 = 0 \qquad (26)$$

The functions $\rho(x)$ and $\xi(x)$, as well as the constants α_i, β_i, γ, and μ_i depend on the choice of subgroup.

As an example (1 out of 22 types) consider the subgroup corresponding to the subalgebra $D + \frac{1}{2} M_{03}$, $M_{23} + M_{02} - P_0 - P_3$, P_1 ; in (25) and (26) we have

$$\rho(x) = \left(\frac{9}{4a}\right)^{1/4} [\,x_2 - \frac{1}{4}(x_0 + x_3)^2\,]^{-1/2},$$

$$\xi(x) = \frac{1}{8}\,[6\,(x_3 - x_0) + 6\,x_2\,(x_0 + x_3) - (x_0 + x_3)^3]\,[x_2 - \frac{1}{4}(x_0 + x_3)^2]^{-3/2} \qquad (27)$$

$$\alpha_2 = \beta_1 = \mu_2 = 0, \quad \alpha_1 = \alpha_3 = \mu_1 = 1, \qquad \beta_2 = \frac{7}{3}, \qquad \gamma = \frac{1}{3}\ .$$

The remaining question is how to solve nonlinear ODE's like (24) or (26).

4.3 Painlevé Analysis of the Obtained ODE's

An ODE is said to have the Painlevé property if none of its solutions have moving critical points. We recall that a critical point is a singularity (in the complex ξ plane), other than a pole (i.e. a branch point, an essential singularity,...). It is moving, if its position depends on the initial conditions (i.e. is not determined by the equation itself). Linear equations always have the Painlevé property, nonlinear ones only rarely. Painlevé and Gambier have investigated second order equations of the form

$$\ddot{F} = f(\dot{F}, F, \xi), \qquad (28)$$

where f is rational in $\dot{F}$ and F and analytical in ξ (see Ince [24] for the results). The main result is that if an equation of the type (28) has the Painlevé property, then there exists a transformation

$$w(\eta) = \frac{\alpha(\xi)\,H(\xi) + \beta(\xi)}{\gamma(\xi)\,H(\xi) + \delta(\xi)}\,, \quad \eta = \eta(\xi) \qquad (29)$$

such that $w(\eta)$ satisfies one of 50 standard equations. Of these, 6 have solutions in terms of the Painlevé transcendants [24], the others can be solved in terms of elliptic functions, or elementary ones.

A simple (and algorithmic) test exists [14] that determines whether a given ODE satisfies certain necessary conditions for having the Painlevé property. To perform the test, expand the solution $F(\xi)$ into a power series about a singular point ξ_0 = const. :

$$F(\xi) = \sum_{j=0}^{\infty} f_j\,\tau^{\,j+\alpha}, \quad \tau = \xi - \xi_0, \qquad f_0 \neq 0, \quad f_j = \text{const.} \qquad (30)$$

Substitute (30) into the equation (28) and collect powers of τ.

Necessary conditions for the Painlevé property are :

a. α is a negative integer

b. The coefficients f_j satisfy a recursion relation of the form

$$P(j)\, f_j = \Phi(f_0, f_1, \dots, f_{j-1}, \xi_0), \tag{31}$$

where $P(j)$ is a polynomial with $m-1$ positive integer roots (m is the order of the ODE). The values of j for which we have $P(j) = 0$ are called "resonances". If j is a resonance, then (31) does not determine f_j.

c. The "resonance condition" $\Phi(f_0, f_1, \dots, f_{j-1}, \xi_0) = 0$ must be satisfied identically at each resonance (for arbitrary values of ξ_0 and of the coefficients f_j at earlier resonances).

If conditions a), b) and c) are satisfied, then (30) contains m arbitrary constants (ξ_0 and the m-1 resonance values of f_j) and can be the general solution. If α is not a negative integer, but a negative rational number, $\alpha = -p/q$ with p and q mutually prime positive integers, then the equation for $H(\xi) = [F(\xi)]^q$ may have the Painlevé property.

A MACSYMA program exists that performs the Painlevé test for ODE's [15]. Most of the ODE's obtained in the symmetry reduction of the NLKGE do not pass the Painlevé test, however some interesting ones do. In all cases we find $\alpha = -\frac{1}{2}$ so we put

$$F(\xi) = [H(\xi)]^{1/2}. \tag{32}$$

Equation (24) is transformed into

$$\ddot{H} = \frac{1}{2H}\dot{H}^2 - \frac{k}{\xi}\dot{H} - 4\lambda(a_2 H + 2 a_4 H^2 + 3 a_6 H^3) \tag{33}$$

and (33) passes the Painlevé test in precisely the following cases :

$$\text{(i)}\,k = 0,\quad \text{(ii)}\,k = 2,\ a_2 = a_4 = 0,\quad \text{(iii)}\,k = 3,\ a_2 = a_0 = 0, \tag{34}$$

$$\text{(iv)}\quad a_4 = a_6 = 0.$$

Similarly, equation (26) implies $\alpha = -1/2$ and we make the transformation (32).

Whenever the Painlevé test is passed, we perform a further transformation of the form (29). It turns out that in the case under consideration it suffices to take $\beta = \gamma = 0$, $\delta = 1$ and to choose $\alpha(\xi)$ and $\eta(\xi)$ appropriately, in order to obtain the following equation for $W(\eta)$:

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + \frac{3}{2}W^3 + \frac{\mu}{2}W, \quad \mu = 0, \pm 1 \tag{35}$$

(for $\mu = 0$ this is the equation PXXIX, for $\mu \neq 0$, PXXX in Ince's notations [24]).

The first integral of (35) is

$$\dot{W}^2 = W^4 + C\,W + \mu W^2, \tag{36}$$

where C is an integration constant. We can rewrite (36) as

$$\frac{dW}{d\eta} = [\,W(W-W_1)(W-W_2)(W-W_3)\,]^{1/2}, \qquad W_i = \text{const.} \tag{37}$$

If all 4 roots of the polynomial under the square root are different, we have solutions in terms of Jacobi elliptic functions. In all other cases we obtain solutions in terms of elementary functions (kinks, solitary waves, various algebraic solutions, constant solutions for $W = W_i$ or $W = 0$).

A complete discussion and list of all solutions is given elsewhere [16b], here we shall just give some examples of solutions of equation (22) obtained by the methods described above.

a. A static, localized finite energy solution

$$\phi(x) = \left(\frac{3C}{a}\right)^{1/4}(x_1^2 + x_2^2 + x_3^2 + C)^{-1/2}\,; \tag{38}$$

C is an arbitrary constant. Further constants can be introduced by applying a general similitude transformation.

b. A solution in terms of "degenerate variables"

$$\phi(x) = \left\{-a\left[(x_1+f_1)^2+(x_2+f_2)^2\right]\right\}^{-1/4} \tanh \frac{1}{2}\left[\arctan \frac{x_2+f_2}{x_1+f_1} - \theta\right] \left\{1 - \tanh^2 \frac{1}{2}\left[\arctan \frac{x_2+f_2}{x_1+f_1} - \theta\right]\right\}^{-1/2} . \quad (39)$$

Here f_1, f_2 and θ are arbitrary functions of $x_0 + x_3$. As a function of the "angle" $\chi = \frac{1}{2}\left[\arctan \frac{x_2+f_2}{x_1+f_1} - \theta\right]$ this solution has the form of a kink, since it satisfies

$$\lim_{\chi \to \pm\infty} \phi = \pm \left\{-4\left[(x_1+f_1)^2+(x_2+f_2)^2\right]\right\}^{-1/4} . \quad (40)$$

Notice that (39) is multivalued since it is not a periodic function of χ.

5. CONCLUSIONS

The main conclusion is that group theory provides efficient, simple and algorithmic tools for obtaining solutions of differential equations. Its power is enhanced when combined with singularity analysis. A great number of the computations involved can be performed on a computer using a symbolic language.

While in general a system of differential equations may have only a trivial symmetry group, it turns out that most equations coming from physics are left invariant by reasonably large groups. In particular, all known "integrable" nonlinear PDE's in more than 1 + 1 dimensions (the Kadomtsev-Petviashvili equation, the modified Kadrmtrev-Petviashvili equation, the Davey-Stewartson equation, the three-wave equations) have infinite-dimensional symmetry groups with a very specific loop structure [11-13,20]. Many PDE's that do not belong to the integrable class also have infinite dimensional symmetry groups, that do not, however, have this loop structure (the von Karman equations are an example [25]).

It would seem that interesting new developments in the theory and

application of all types of symmetry groups of differential equations are to be expected in the near future.

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