CONF-860766--2

DE87 004578

September, 1986 EFI 86-56

SUPERMANIFOLDS AND SUPER RIEMANN SURFACES*

Jeffrey M. Rabin

Enrico Fermi Institute and Department of Mathematics
The University of Chicago
Chicago, IL 60637

ABSTRACT

The theory of super Riemann surfaces is rigorously developed using Rogers' theory of supermanifolds. The global structures of super Teichmüller space and super moduli space are determined. The super modular group is shown to be precisely the ordinary modular group. Super moduli space is shown to be the gauge-fixing slice for the fermionic string path integral.

*Based on lectures given at the NATO Advanced Research Workshop on Super Field Theories, Vancouver 1986.
Research supported by the NSF (PHY 83-01221) and DOE (DE-AC02-82-ER-40073).

CONF-860766--2

SUPERMANIFOLDS AND SUPER RIEMANN SURFACES*

Jeffrey M. Rabin

Enrico Fermi Institute and Department of Mathematics
The University of Chicago
Chicago, IL 60637

1. Introduction

The theory of Riemann surfaces [1-3] has recently become an important mathematical tool for string theorists. The world sheet of a bosonic string is in fact a Riemann surface, and the g-loop contribution to an amplitude in string theory can be expressed as an integral over the moduli space of Riemann surfaces of genus g [4-6]. Such a representation of the amplitudes allows one to use powerful techniques from algebraic geometry to study their analytic properties, investigate their finiteness, and even to compute them in terms of theta functions [7-9].

Since it is the superstring rather than the bosonic string which is physically relevant and possibly finite, there is great interest in generalizing the algebraic geometry to include the fermionic coordinates of superspace [10,11]. Several authors have described a notion of "super Riemann surface" which is appropriate for this purpose [12-15]. However, the intuitive concept of a supermanifold in the physics literature is not a sufficient foundation for a theory of super Riemann surfaces because it describes only the local geometry of superspace and not its global topology. The theory of super Riemann surfaces, like that of Riemann surfaces, makes use of several topological constructions including universal covering spaces, quotient spaces, and homotopy groups. Therefore one requires a theory of supermanifolds which is sufficiently rigorous to give meaning to these constructions and to ensure that they have their usual mathematical properties. In this paper I will develop the theory of super Riemann surfaces using as foundation the theory of supermanifolds due to A. Rogers [16]. Other supermanifold theories exist [17], but Rogers' is both the most general and the closest to the physicist's intuitive view of superspace as a manifold with some anticommuting coordinates. It also has the great advantage that a supermanifold is in fact an ordinary manifold, so that the

*Based on lectures given at the NATO Advanced Research Workshop on Super Field Theories, Vancouver 1986.
Research supported by the NSF (PHY 83-01221) and DOE (DE-AC02-82-ER-40073).

topological constructions have their standard meanings. Because of its extreme generality, Rogers' theory includes many topologically exotic supermanifolds which are not physically useful. This can be viewed as a disadvantage of the theory which could be avoided by using a less general one, or as an advantage in that it makes explicit the properties which are actually assumed in physical applications.

In Section 2 I review the connection between bosonic strings and Riemann surfaces, specifically the identification of the gauge-fixing slice for the functional integral with the moduli space of Riemann surfaces. This motivates the definition of a super Riemann surface, which must be such that the super moduli space coincides with the gauge-fixing slice for the fermionic string path integral. Section 3 develops Rogers' theory of supermanifolds, which allows one to discuss topological properties of super Riemann surfaces rigorously. It also clarifies some obscure points in the physics literature, such as whether the commuting coordinates of superspace are ordinary numbers or have nilpotent parts (the latter is true), and explains why the anticommuting dimensions must be topologically trivial in physical applications. Section 4 proves the basic properties of super Riemann surfaces, beginning with the relation between a super Riemann surface and an ordinary Riemann surface with spin structure, and continuing through the determination of the dimension and global structure of super moduli space. As a concrete example of this abstract discussion, the case of super tori (genus 1) is worked out explicitly. Section 5 is a brief survey of results which can be obtained with more advanced techniques, and open problems. Most of the work described here was done with Louis Crane [18,19].

2. Strings and Riemann Surfaces

Polyakov's closed bosonic string theory [20] is a theory of maps $X^\mu(\sigma^a)$ from the two-dimensional world sheet of the string with metric h^{ab} into Euclidean spacetime R^{26}, with action

$$S = \int d^2\sigma \sqrt{h}\; h^{ab}\, \partial_a X^\mu\, \partial_b X_\mu. \tag{2.1}$$

Quantization involves functional integration over the fields h^{ab} and X^μ. At g loop order we integrate over world sheets of genus g. The integral over X^μ is Gaussian, so only the integration over metrics is nontrivial. However, the action (2.1) has a large gauge symmetry: it is invariant under reparametrizations or diffeomorphisms of the world sheet as well as conformal rescalings of the metric $h^{ab} \to \Omega^2 h^{ab}$. To prevent overcounting of equivalent field configurations, we seek a gauge-fixing slice in the space of metrics which contains exactly one representative of each gauge equivalence class of metrics, and integrate over this slice. The slice will be a realization of the space of metrics modulo diffeomorphisms and conformal rescalings.

This slice is in fact the moduli space of Riemann surfaces of genus g. A Riemann surface is a two-dimensional surface with a given complex structure: a covering by local coordinate charts so that each point in a given chart is assigned a complex coordinate z and the transition functions relating the coordinates in overlapping charts are analytic. Two Riemann surfaces are considered equivalent if they are related by an analytic diffeomorphism. It turns out that not all complex structures on a given surface are equivalent, and in fact the set of all complex structures on a surface of genus g is itself a complex manifold (except at some singular points) of complex dimension 0 for $g=0$, 1 for $g=1$, and $3g-3$ for $g>1$. This manifold is called the moduli space for genus g. It is a topologically complicated space, and it is useful to define a related but topologically trivial space called Teichmüller space. The fundamental group or first homotopy group $\pi_1(M)$ of a surface M is generated by $2g$ noncontractible closed curves, but a specific set of $2g$ generators can be chosen in many different ways. A Riemann surface together with a choice of generators for its fundamental group is called a marked Riemann surface and defines a point in Teichmüller space. Each Riemann surface is represented by a single point of moduli space but by a discrete infinity of points in Teichmüller space. Moduli space can be viewed as a quotient of Teichmüller space by a discrete group which identifies all points representing the same Riemann surface. This group is the modular, or mapping class, group for genus g.

What is the connection between gauge equivalence classes of metrics and Riemann surfaces? A metric on an orientable surface determines a Riemann surface structure in the following way. Any metric in two dimensions is locally conformally flat. This means that the surface can be covered by charts such that the metric in each chart is conformal to the flat metric, $h^{ab} = \Omega^2 \delta^{ab}$. When two such charts overlap, their coordinates are conformally related. If σ^a are coordinates in such a chart, define $z = \sigma^1 + i\sigma^2$. Since a conformal transformation is an analytic function of z, this set of charts defines a Riemann surface structure. It can be shown that two metrics determine the same Riemann surface iff they are related by diffeomorphisms and conformal rescalings, and conversely that any Riemann surface is defined by some metric [3]. Therefore the gauge-fixing slice is precisely the moduli space. Questions about the behavior of the string integrand on the gauge-fixing slice can be translated into questions about the geometry of moduli space, or into questions about the action of the modular group on Teichmüller space.

The fermionic string can be formulated as a superfield $X^\mu(\sigma^a,\theta^\alpha)$ coupled to two-dimensional supergravity [21]. The action is

$$S = \int d^2\sigma\, d^2\theta\, sdet\, E\ \nabla_\alpha X^\mu\, \nabla^\alpha X_\mu, \tag{2.2}$$

where E_M^A is the super vierbein and ∇_α the corresponding covariant derivative. Again, quantization requires functional integration over X^μ and E_M^A, and we seek a gauge-fixing surface for the super reparametrization and superconformal symmetries of the action. It is known that any supergravity geometry in two dimensions is locally superconformal to flat superspace [21], meaning that coordinates can be found in which the super vierbein takes the form,

$$E_M^a = \Omega e_M^a, \tag{2.3a}$$

$$E_M^\alpha = \sqrt{\Omega}\, e_M^\alpha - i e_M^a \gamma_a^{\alpha\beta} D_\beta \sqrt{\Omega}. \tag{2.3b}$$

Here $\Omega(\sigma,\theta)$ is the conformal factor and e_M^A and D_A are the flat super vierbein and covariant derivative. These local coordinates define complex coordinates by $z = \sigma^1 + i\sigma^2$, $\theta = \theta^1 + i\theta^2$. In this way we obtain a complex supermanifold of dimension (1,1) whose transition functions are superconformal maps. Such a supermanifold is a super Riemann surface (SRS). The moduli space of super Riemann surfaces should be the gauge-fixing surface for the functional integral, because supergravity geometries related by reparametrizations and superconformal transformations define the same super Riemann surface. This is the motivation for the rigorous theory of SRS's to be developed in Section 4.

3. Supermanifolds

A supermanifold should be a space having both even and odd coordinates. I will discuss the case of immediate interest in which there is one even complex coordinate z and one odd complex coordinate θ, but generalization is easy.

If z and θ are to be coordinates rather than just symbols, they must be able to assume values in some number system so that different values of the coordinates can label different points. We take as this number system a Grassmann algebra B_L with L generators $\nu_1, \nu_2, \cdots, \nu_L$ obeying $\nu_i \nu_j = -\nu_j \nu_i$. The most general even and odd elements of this algebra have the form,

$$z = z_0 + z_{ij}\nu_i\nu_j + \cdots,$$

$$\theta = \theta_i \nu_i + \theta_{ijk}\nu_i\nu_j\nu_k + \cdots, \tag{3.1}$$

where the coefficients $z_{ij\cdots l}$ and $\theta_{ij\cdots l}$ are ordinary complex numbers, and all subscripts are in increasing order, $i<j<\cdots<l$. The coefficient z_0 is called the body of z, and the soul of z is $z - z_0$ [22]. θ has no body and is pure soul. The fact that there are 2^L independent complex coefficients in the expansions (3.1) suggests the following idea. Take an ordinary complex manifold M of dimension 2^L. In each chart it has 2^L complex coordinates, which we call the body coordinate z_0 and the soul coordinates $\theta_i, z_{ij}, \cdots$. Then the series (3.1) define one even and one odd complex Grassmann coordinate in each chart, and different values of these coordinates do indeed label different points of M. Obviously this can be done on any complex manifold M. But we will now demand that the transition functions of M have special analytic properties when expressed in terms of the Grassmann coordinates. Not every M will admit coordinates with these properties; the ones that do will be called complex supermanifolds.

The transition functions relating coordinates in overlapping charts have the form $\tilde{z} = \tilde{z}(z,\theta)$, $\tilde{\theta} = \tilde{\theta}(z,\theta)$. We require these functions to have the properties normally assumed of superfields in physics. They must take the form,

$$\tilde{z} = f(z) + \theta\zeta(z), \quad \tilde{\theta} = \psi(z) + \theta g(z), \tag{3.2}$$

where the B_L-valued component functions f, ζ, ψ, g are required to have Taylor expansions in powers of the soul of z, for example,

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \cdots, \tag{3.3}$$

with $f(z_0)$ analytic. Such functions will be called superanalytic. Notice that the series (3.3) terminates because $z - z_0$ is nilpotent, so convergence is not a problem. If we were discussing real rather than complex supermanifolds, $f(z_0)$ would be required to be smooth rather than analytic, and the transition functions would then be called supersmooth or G^∞.

Eq. (3.3) clears up a confusing point in the literature. The even coordinates of superspace are always treated as ordinary numbers despite the fact that they clearly cannot be: z cannot be an ordinary number both before and after a supersymmetry transformation $\tilde{z} = z + \theta\eta$, $\tilde{\theta} = \theta + \eta$. The present formalism resolves this problem: only the body of z is an ordinary number, not z itself. But then why can superfields be manipulated as if their even arguments were ordinary numbers? The answer is that according to (3.3) a superanalytic function is completely determined when its components are known for soulless values of z. Functions of z are in 1-1 correspondence with functions of z_0 and this correspondence is preserved by algebraic manipulations. In other words, a Grassmann-valued function of z_0 can always be analytically continued to a function of z with the same algebraic properties.

One expects on physical grounds that an ordinary manifold M_0 can be obtained from a supermanifold M by "throwing away" all the soul coordinates. If M is a supergravity superspace, M_0 should be physical spacetime; if M is the super world sheet of a fermionic string, M_0 should be the physical world sheet. We will see that M_0, called the body of M, does not generally exist unless the topology of M is restricted.

In each chart of M, consider the surfaces of constant z_0. When charts overlap we have

$$\begin{aligned}\tilde{z} &= f(z) + \theta\zeta(z) \\ &= f(z_0) + (z - z_0)f'(z_0) + \cdots + \theta\zeta(z).\end{aligned} \tag{3.4}$$

Since every term on the right is pure soul except the first, $\tilde{z}_0$ depends only on z_0. This means that a surface of constant z_0 is also a surface of constant $\tilde{z}_0$, so these surfaces fit together smoothly to give the leaves of a foliation, called the soul foliation. Every point of M lies on exactly one constant body surface. A topological space M_0 is obtained by identifying all points of M which lie on the same surface. Unfortunately M_0 need not be a smooth manifold or even Hausdorff.

As a simple example, consider the Euclidean plane with coordinates denoted x_0 and θ_1. This is a real, rather than complex, supermanifold over the trivial Grassmann algebra B_1, with real Grassmann coordinates $x = x_0$ and $\theta = \theta_1 v_1$. The leaves of the soul foliation are the lines of constant x_0 and the body can be thought of as the x_0 axis. The quotient space of the plane by the group of integer translations along the coordinate axes is a torus, which is also a supermanifold. The leaves are now circles going around the torus in the θ_1 direction, and the body is a circle in the x_0 direction. The quotient of the plane by the group of integer translations along two axes having irrational slope, however, is a torus whose soul foliation consists of spirals each of which is dense in the entire torus! In this case M_0 is a non-Hausdorff topological space, and no body manifold exists. Intuition suggests that only supermanifolds with bodies can be physically relevant.

Physics imposes further restrictions on the topology of the leaves of the soul foliation. These restrictions follow from the theorem that any G^∞ function on a supermanifold must be constant along any compact leaf of the soul foliation [18]. In string theory this would imply that the map $X^\mu(\sigma,\theta)$ must send each compact leaf to a point. But then there is no embedding of the super world sheet in any flat superspace, so the super world sheet cannot represent a string moving in spacetime if it contains compact leaves. To prove the theorem, let F be a real G^∞ function on M. Expand F in terms of the basis of B_L as $F = F_0 + F_i v_i + \cdots$, and consider any of the coefficient functions $F_{ij \ldots l}$ on the compact leaf. If it is nonconstant then it necessarily achieves a maximum on the leaf. However, one can see from Eqs. (3.2) and (3.3) that a G^∞ function is always a polynomial function of the soul coordinates. In fact it is linear in each soul coordinate separately: because each soul coordinate appears together with at least one v_i which squares to zero, no higher power of a soul coordinate can appear. Elementary calculus (the second derivative test) shows that such a function cannot have maxima, but only saddle points. Therefore each $F_{ij \ldots l}$ is constant on the leaf. Evidently the supermanifolds of interest in physics should contain no compact leaves and should have bodies. Because there is no complete classification of supermanifolds, it is not known whether these requirements eliminate all possibilities for nontrivial topology in the soul directions, but examples of additional possibilities have not been found.

The simplest way to guarantee that a supermanifold M will have a body and be free of compact leaves is to demand that it have the DeWitt topology [22]. This means that each coordinate chart must be the Cartesian product of an open set in the z_0 plane with the entire complex planes of the soul coordinates. The leaves of a DeWitt supermanifold are complex vector spaces C^k, $k = 2^L - 1$, with trivial topology, and a body always exists. A set of charts for the body is obtained by projecting the charts of M onto the z_0 plane, and the transition functions for the body are the bodies $\tilde{z}_0 = f_0(z_0)$ of the transition functions of M, Eq. (3.2). M is then a fiber bundle over M_0 with fiber C^k. All supermanifolds used in physics are implicitly assumed to have the DeWitt topology. Historically this was due to the absence of any definition permitting any other topology, but as we have seen there are good physical reasons for the choice.

Finally, there are some technical points connected with the size of the Grassmann algebra B_L. Nothing in the mathematical theory depends on the value of L, which should be thought of as a large but finite integer. However, for physical applications it is necessary to take the $L \to \infty$ limit. For example, Green's functions containing more than L fermionic fields vanish identically if L is finite, because the product of more than L odd elements in B_L is zero. The limit is well understood and is fully discussed in [23].

Another technical problem which arises for finite L is an ambiguity in the definition of the derivative ∂_θ. It should be defined so that $\partial_\theta[f(z) + \theta\zeta(z)] = \zeta(z)$, but ζ is ambiguous because adding a multiple of $v_1 v_2 \cdots v_L$ to ζ does not change $\theta\zeta$. The ambiguity can be removed by insisting that the components $f(z_0), \zeta(z_0)$ of all G^∞ functions take values in the subalgebra B_{L-1} obtained by deleting the generator v_L. This method of correcting the problem is necessary if ∂_θ is to obey the Leibniz rule [23].

We now have a theory of supermanifolds which is fully rigorous and gives us control over their topological properties, yet which reduces in practical calculations to the physicist's standard calculus of superfields. We can now develop the theory of SRS's on a firm mathematical basis.

4. From Super Riemann Surfaces to Super Moduli Space

Let M be a complex supermanifold of dimension (1,1). In each chart we have the flat covariant derivative

$$D = D_\theta = \partial_\theta + \theta\partial_z, \quad D^2 = \partial_z, \tag{4.1}$$

as well as the flat frame field

$$e^z = dz + \theta d\theta, \quad e^\theta = d\theta. \tag{4.2}$$

(My convention for superforms is that $d\theta$ commutes with itself but anticommutes with dz and with θ.) The transition functions have the superanalytic form,

$$\tilde{z} = f(z) + \theta\zeta(z), \quad \tilde{\theta} = \psi(z) + \theta g(z). \tag{4.3}$$

Following Eq. (2.3a) we define a superconformal map as a superanalytic transformation (4.3) under which e^z is multiplied by a function Ω. A simple calculation shows that (4.3) is superconformal iff

$$\zeta = g\psi, \quad g^2 = f' + \psi\psi', \tag{4.4}$$

and one finds that $\Omega = (D\tilde{\theta})^2$. We say that M is a super Riemann surface if its transition functions are superconformal,

$$\begin{aligned} \tilde{z} &= f + \theta\psi\sqrt{f'}, \\ \tilde{\theta} &= \psi + \theta\sqrt{f' + \psi\psi'}. \end{aligned} \tag{4.5}$$

Unless otherwise stated, all SRS's are assumed to have the DeWitt topology.

The body M_0 of a SRS M is a Riemann surface whose transition functions are $\tilde{z}_0 = f_0(z_0)$. Then $f'_0(z_0)$ are the transition functions for the tangent bundle of M_0 (holomorphic tangent vectors transform by this factor under a change of coordinates), and a choice of signs for the square roots $\sqrt{f'_0(z_0)}$ defines a spin structure on M_0: a consistent transformation law for spinors under changes of coordinates. Recall that there can be many inequivalent spin structures on a nonsimply connected manifold [24,25]. Roughly speaking, spinors can be chosen to be either periodic or antiperiodic around each of the $2g$ noncontractible closed curves on a surface of genus g, so that there are 2^{2g} spin structures. Because the body of $\sqrt{f'(z)}$ is $\sqrt{f'_0(z_0)}$, such a choice of signs is implicit in Eqs. (4.5). Therefore a SRS determines a particular spin structure on its body. Conversely, given a Riemann surface M_0 with a particular spin structure, a SRS M can be constructed in a canonical way. The charts for M

are the Cartesian products of the charts of M_0 with the entire complex planes of the soul coordinates. If $f(z_0)$ are transition functions for M_0 then the transition functions of M have $\psi(z) = 0$ and $f(z)$ the Grassmann analytic continuation of $f(z_0)$, with the signs of the square roots in Eqs. (4.5) chosen according to the given spin structure. A SRS constructed in this way will be called canonical, and we will see that not all SRS's are of this type.

We now begin the task of classifying all SRS's, with the goal of obtaining a picture of the super moduli space which is the gauge-fixing surface for the fermionic string path integral. If M is a SRS, construct its universal covering space $\hat{M}$. This is a simply connected manifold which locally looks just like M. The standard theory of covering spaces [1] shows that $\hat{M}$ is also a SRS, and $M = \hat{M}/G$ where G is a group of superconformal transformations isomorphic to $\pi_1(M)$. Furthermore, since M is a bundle with topologically trivial fibers, any noncontractible loops in M are due to the topology of M_0, meaning that $\pi_1(M) = \pi_1(M_0)$. The classification problem now splits into two parts: classifying all simply connected SRS's $\hat{M}$, and determining the groups G which can act on them.

A simply connected SRS $\hat{M}$ has a simply connected Riemann surface $\hat{M}_0$ as its body. According to the classical uniformization theorem [1] there are only three simply connected Riemann surfaces: the complex plane C, the upper half plane U, and the Riemann sphere C^* (thought of as the plane plus the point at infinity). Each of these surfaces has a unique spin structure, so exactly one canonical SRS can be constructed over each of them, to be denoted SC, SU, and SC^*. I claim that these are the only simply connected SRS's. To verify that a given SRS $\hat{M}$ is one of these three, one would examine its transition functions. If all the functions $\psi(z)$ are zero and all the $f(z_0)$'s are soulless, the SRS is canonical and must be one of these three. If these conditions do not hold, the SRS may still be canonical, because the transition functions can always be changed by redefining the coordinates in the charts. The question becomes whether some superconformal coordinate redefinitions can set to zero all the ψ's and the souls of the f's. If so, the SRS was canonical but this fact was obscured by a poor original choice of coordinates. Such questions about coordinate redefinitions can be answered by the methods of sheaf cohomology [26]. In the present case they show that the desired coordinate redefinitions can always be found, which proves that SC, SU, and SC^* are indeed the only simply connected SRS's [19].

To complete the classification we must determine the groups G of superconformal automorphisms of the simply connected SRS's. Begin with SC^*. The body of a superconformal

automorphism of SC^* must be a conformal automorphism of C^*, namely a Möbius transformation

$$\tilde{z}_0 = \frac{a_0 z_0 + b_0}{c_0 z_0 + d_0}. \tag{4.6}$$

The soul of the superconformal automorphism is restricted only by the condition that it be well defined at $z_0 = -d_0/c_0$, which is mapped to the point at infinity by (4.6). The behavior of a map at infinity is studied by using the transition functions of SC^* to switch from $(\tilde{z},\tilde{\theta})$ to new coordinates $(-1/\tilde{z},\tilde{\theta}/\tilde{z})$ which must be finite. This is true only for maps of the form,

$$\tilde{z} = \frac{az+b}{cz+d} + \theta\frac{\gamma z+\delta}{(cz+d)^2},$$
$$\tilde{\theta} = \frac{\gamma z+\delta}{cz+d} + \frac{\theta}{cz+d}(1+\frac{1}{2}\delta\gamma). \tag{4.7}$$

The transformations (4.7) form the supergroup of superconformal automorphisms of SC^*. It has three independent even parameters (since a choice of normalization can impose the constraint $ad - bc = 1$) and two odd ones, and will be denoted $SPL(2,C)$. Its super Lie algebra is the subalgebra of the Neveu-Schwarz algebra generated by L_0, $L_{\pm 1}$, and $G_{\pm 1/2}$ [12,27].

The groups of superconformal automorphisms of SC and SU are much larger than $SPL(2,C)$. These SRS's do not contain the point at infinity, so finiteness at this point is no longer required. Any superconformal map whose body is a Möbius transformation is an automorphism of these SRS's. For example,

$$\tilde{z} = z + 1 + \theta\eta z^n, \quad \tilde{\theta} = \theta + \eta z^n, \tag{4.8}$$

is an automorphism of SC which does not belong to $SPL(2,C)$ for $n>1$.

The classification theorem for SRS's states that any SRS is a quotient of SC, SU, or SC^* by a group G of superconformal automorphisms. Such a quotient space does not automatically have the DeWitt topology, however. If G_0 is the group of Möbius transformations which are the bodies (4.6) of elements of G, then G_0 must act properly discontinuously on the body of $\hat{M}$ to get the DeWitt topology. This means that each point of $\hat{M}_0$ must have a neighborhood which does not intersect any of its images by elements of G_0. (Roughly speaking, G_0 should not have fixed points or points which are "nearly fixed".) Supersymmetry provides an example of what goes wrong if this condition is not met. The supersymmetry transformation

$$\tilde{z} = z + \theta\delta, \quad \tilde{\theta} = \theta + \delta, \tag{4.9}$$

fixes the body of every point of SC, transforming only the soul. Taking the quotient of SC by this transformation will identify points with the same body but different souls. This curls up the soul fibers, violating the DeWitt topology.

To complete the argument that the moduli space of SRS's is the gauge-fixing slice for the fermionic string, it must be shown that every SRS admits a supergravity geometry. This is in fact only true if the group G is a subgroup of $SPL(2,C)$, so the extra automorphisms of SC and SU have no physical relevance. A supergravity geometry on M can be defined by a frame field on $\hat{M}$ which is invariant under G up to a phase, since $U(1)$ is the tangent space group of two-dimensional supergravity. On SC such a frame field is given by (4.2), which is invariant up to a phase under a subgroup of $SPL(2,C)$ to be determined below but not under any other superconformal transformations. On SU the frame field

$$E^z = (\mathrm{Im}\, z + \frac{1}{2}\theta\bar{\theta})^{-1}\,(dz + \theta d\theta),$$

$$E^\theta = (\mathrm{Im}\, z + \frac{1}{2}\theta\bar{\theta})^{-1/2}\, d\theta + \frac{1}{2}(i\theta - \bar{\theta})(\mathrm{Im}\, z + \frac{1}{2}\theta\bar{\theta})^{-3/2}\,(dz + \theta d\theta), \tag{4.10}$$

is invariant up to a phase only under a subgroup $SPL(2,R)$ obtained by restricting the even parameters a,b,c,d of $SPL(2,C)$ to be real and the odd ones to obey $\bar{\gamma} = i\gamma$, $\bar{\delta} = i\delta$. SRS's which admit these frame fields will be called metrizable, so that the gauge-fixing slice for the fermionic string is the moduli space of metrizable SRS's.

To illustrate these ideas I will now work out the classification of super tori, SRS's whose bodies have genus 1. A super torus is obtained as the quotient of SC by a subgroup G of $SPL(2,C)$ which is isomorphic to the fundamental group of a torus. Therefore G has two commuting generators of the form (4.7). The generators leave the flat super vierbein invariant up to a phase only if $c = 0$, $a^2 = 1$, and $\gamma = 0$; and if G_0 acts properly discontinuously as well then $b_0 \neq 0$. The generators then take the form,

$$\tilde{z} = z + ab + \theta\delta, \quad \tilde{\theta} = a(\theta + \delta), \tag{4.11}$$

where $a = \pm 1$. We represent the two generators by the ordered triples $A = (a,b,\delta)$ and $A' = (a',b',\delta')$. The choice of signs for a and a' determines one of the four spin structures on the torus. The commutator of the generators is

$$A'^{-1}A^{-1}A'A = [1,\, (aa' + a + a' - 1)\delta\delta',\, (1 - a')\delta - (1 - a)\delta']. \tag{4.12}$$

Note that this commutator is a pure soul transformation. This means that if we dropped the requirement that the generators commute, the quotient space would not have the DeWitt topology.

As is true for Riemann surfaces, the groups G and $G^q = q^{-1}Gq$ represent equivalent SRS's for any $SPL(2,C)$ element q. This is shown by finding a superconformal diffeomorphism relating SC/G and SC/G^q. Starting from a point x in SC/G, choose any point y lying above it in SC, map y to $q^{-1}y$, and project down to SC/G^q. This map is readily seen to be superconformal, invertible, and independent of the choice of y. By an appropriate choice of q it is possible to conjugate the generators of G into a standard form with $b' = 1$ and $\delta' = 0$. If the spin structure is trivial, so that $a = a' = 1$, then the commutator (4.12) automatically vanishes, and the group is completely described by the two parameters b and δ. These parameters describe super tori with a distinguished choice of generators of $G = \pi_1(M)$, so they give global coordinates on super Teichmüller, rather than super moduli, space. It would appear that this space is a supermanifold of complex dimension (1,1), but this is not quite true. Conjugation by the $SPL(2,C)$ element q: $\tilde{z} = z$, $\tilde{\theta} = -\theta$ flips the sign of δ without changing b, showing that both signs of δ describe the same super torus. Except for the singular points with $\delta = 0$, the coordinates b and δ cover super Teichmüller space twice, showing it to be a super orbifold.

For the nontrivial spin structures, a and a' are not both 1; assume $a' = -1$. Then the commutator (4.12) vanishes only if $\delta = 0$, leaving just one free parameter b. The global structure of super Teichmüller space is therefore as follows. It has four disconnected pieces representing the four spin structures. One piece is a complex super orbifold of dimension (1,1), and the other three pieces are complex supermanifolds of dimension (1,0). Its body is four copies of the ordinary Teichmüller space of the torus, a one-dimensional complex manifold with coordinate b_0. Points of the body represent supertori of the canonical type, since their group parameters are soulless.

Having chosen the group G representing a super torus, we can change from A and A' to a new pair of generators. By conjugating these generators into the standard form, we can identify new values of b and δ which represent the same super torus. Such points of super Teichmüller space which represent the same SRS are related by an element of the super modular group, and the quotient of super Teichmüller space by this group is the super moduli space. For example, we can choose A'^{-1} and A as generators of G rather than A and A'.

For the trivial spin structure this leads to the transformation $b \to -1/b$, $\delta \to i\,\delta b^{-3/2}$ on super Teichmüller space. In this way the action of any super modular transformation on super Teichmüller space can be worked out. One finds in particular that the nontrivial spin structures mix under super modular transformations [25]. The structure of the super modular group is also easy to determine: because G is isomorphic to $\pi_1(M_0)$, changing the choice of generators for G is the same as changing the choice of generators for $\pi_1(M_0)$. But this is precisely what the ordinary modular group of a torus does. Therefore the super modular group is isomorphic to the ordinary modular group. The super moduli space will be a complex super orbifold whose body is a four-sheeted cover of ordinary moduli space.

The results for genus $g>1$ are similar (SC^* is the only metrizable SRS of genus 0). A metrizable SRS M of genus $g>1$ is the quotient of SU by a subgroup G of $SPL(2,R)$ having $2g$ generators. The generators are described by $6g$ even and $4g$ odd parameters. By a conjugation, two odd parameters can be set to zero and three even parameters can be fixed. The group relation

$$q_1 q_2 q_1^{-1} q_2^{-1} \cdots q_{2g-1} q_{2g} q_{2g-1}^{-1} q_{2g}^{-1} = 1 \tag{4.13}$$

obeyed by the generators q_i of the fundamental group of any surface then eliminates three more even and two more odd parameters. The only other restriction on the parameters comes from the requirement that G_0 act properly discontinuously on U, and this affects only the bodies of the parameters. Super Teichmüller space therefore has the DeWitt topology, since its soul coordinates are unrestricted. It has $6g-6$ real even and $4g-4$ real odd dimensions, and comes in 2^{2g} disconnected pieces. Each piece is a super orbifold, because flipping the signs of all the odd coordinates together describes the same SRS. Note that because the parameters of $SPL(2,R)$ are real, this construction only gives real coordinates on super Teichmüller space. The space itself is actually a complex super orbifold, but more sophisticated methods are needed to prove this. The super modular group is again isomorphic to the ordinary modular group and the body of super moduli space is a 2^{2g}-sheeted cover of ordinary moduli space.

5. Conclusions and Open Problems

The results of this paper provide a rigorous foundation for the mathematical theory of super Riemann surfaces and for their applications to superstrings. The connection between SRS's and ordinary Riemann surfaces with spin structure was explained, and the gauge-fixing slice for the fermionic string path integral was shown to be the super moduli space of metrizable DeWitt SRS's. The global structure of super Teichmüller space was determined and the super modular group was shown to be the ordinary modular group. The parameters of the group representing a SRS provide local coordinates on super moduli space. These results constitute the "elementary" theory of SRS's.

More powerful results can be obtained by generalizing more advanced techniques from Riemann surface theory, specifically the theory of quasiconformal maps [2,3,19]. With these methods one can show that super Teichmüller space for genus $g>1$ actually has a complex structure. The complex structure comes from an embedding of super Teichmüller space in a certain space of superanalytic functions. A set of complex coordinates is provided by the parameters of a different kind of group representing the SRS, a Schottky group.

Many questions about SRS's remain open. As explicitly as possible, how does the super modular group act on the odd coordinates of super Teichmüller space, and on the spin structures? A proof of finiteness for superstring amplitudes via algebraic geometry probably depends on answering this question. First, finiteness is known to depend on cancellations between the contributions of different spin structures when these contributions are summed to give a modular invariant result. Second, finiteness probably depends on the existence of a compactification of super moduli space by adding points at infinity representing SRS's with certain singularities [9]. Such a compactification of ordinary moduli space is known to exist [28], but generalization of the proof depends on controlling the behavior of spin structures on singular Riemann surfaces. A possible approach to this problem is to study the boundary of the space of metrizable DeWitt SRS's within the larger space of all SRS's. Little is known about this larger space. A related problem is to describe SRS's in terms of period matrices. For Riemann surfaces the entries of these matrices are the integrals of analytic 1-forms around noncontractible loops, and they can serve as coordinates on moduli space. Modular transformations are conveniently described in terms of their effect on a period matrix, and the same should be true for super modular transformations. Finally, an abstract formulation of string theory in terms of the geometry of moduli space has been proposed [29]. The corresponding formulation for superstrings would be of interest.

References

[1] H.M. Farkas and I. Kra, *Riemann Surfaces*, Springer-Verlag, New York, 1980.

[2] L. Bers, Bull. London Math. Soc. **4** , 257 (1972).

[3] C.J. Earle, in *Discrete Groups and Automorphic Functions*, ed. W.J. Harvey, Academic Press, London, 1977.

[4] O. Alvarez, Nucl. Phys. **B216** , 125 (1983).

[5] J. Polchinski, Commun. Math. Phys. **104** , 37 (1986).

[6] E. D'Hoker and D.H. Phong, Nucl. Phys. **B269** , 205 (1986).

[7] G. Moore, "Modular forms and two-loop string physics", Harvard preprint HUTP-86/A038 (1986).

[8] A.A. Belavin and V.G. Knizhnik, Phys. Lett. **168B** , 201 (1986).

[9] R. Catenacci, M. Cornalba, M. Martellini, and C. Reina, Phys. Lett. **172B** , 328 (1986).

[10] E. D'Hoker and D.H. Phong, Nucl. Phys. **B278** , 225 (1986).

[11] G. Moore, P. Nelson, and J. Polchinski, Phys. Lett. **169B** , 47 (1986).

[12] D. Friedan, in the Proceedings of the Workshop on Unified String Theories, ed. D. Gross and M. Green, World Press, Singapore, 1986.

[13] E. Martinec, "Conformal field theory on a (super)Riemann surface", Princeton preprint (1986).

[14] M.A. Baranov and A.S. Shvarts, JETP Lett. **42** , 419 (1986).

[15] P. Nelson and G. Moore, Nucl. Phys. **B274** , 509 (1986).

[16] A. Rogers, J. Math. Phys. **21** , 1352 (1980).

[17] D. Leites, Russ. Math. Surv. **35** , 1 (1980).

[18] J.M. Rabin and L. Crane, Commun. Math. Phys. **100** , 141 (1985); **102** , 123 (1985).

[19] L. Crane and J.M. Rabin, "Super Riemann surfaces: uniformization and Teichmüller theory", University of Chicago preprint EFI 86-25, submitted to Commun. Math. Phys.

[20] A.M. Polyakov, Phys. Lett. **103B** , 207 (1981).

[21] P.S. Howe, J. Phys. A **12** , 393 (1979).

[22] B.S. DeWitt, *Supermanifolds*, Cambridge University Press, Cambridge, 1984.

[23] A. Rogers, Commun. Math. Phys. **105** , 375 (1986).

[24] R. Geroch, J. Math. Phys. **9** , 1739 (1968).

[25] L. Alvarez-Gaumé, G. Moore, and C. Vafa, "Theta functions, modular invariance, and strings", Harvard preprint HUTP-86/A017 (1986).

[26] K. Kodaira, *Complex Manifolds and Deformation of Complex Structures*, Springer-Verlag, New York, (1986).

[27] A. Neveu and J.H. Schwarz, Nucl. Phys. **B31** , 86 (1971).

[28] S. Wolpert, Ann. Math. **118** , 491 (1983).

[29] D. Friedan and S. Shenker, Phys. Lett. **175B** , 287 (1986).