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ABSTRACT

An equation is derived for the calculation of the radioactivity of a source from the results of coincidence counting, taking into account the dead-time losses and accidental coincidences. The derivation is an extension of the method of J. Bryant [*Int. J. Appl. Radiat. Isot., 14:143, 1963*]. The improvement on Bryant's formula has been verified by experiment.

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COINCIDENCE METHODS; DEAD TIME; MATHEMATICAL MODELS; PARTICLE DECAY; PROBABILITY; RADIATION DETECTION; RADIATION DETECTORS; RADIOACTIVITY; SENSITIVITY.

EDITORIAL NOTE

From 27 April 1987, the Australian Atomic Energy Commission (AAEC) is replaced by Australian Nuclear Science and Technology Organisation (ANSTO). Serial numbers for reports with an issue date after April 1987 have the prefix ANSTO with no change of the symbol (E, M, S or C) or numbering sequence.

CONTENTS

1. INTRODUCTION

Some of the particles and photons emitted by atoms in a radioactive source are absorbed within the source. Others are not observed by the detectors. The count-rate is thus always less than the disintegration rate. However, coincidence counting which is independent of detection efficiency can be used to obtain the disintegration rate for those nuclides which, for example, emit a β particle and a γ photon. The equipment includes a coincidence counter which records the fact that a disintegration has been observed by both the β and the γ detectors. Let N be the disintegration rate in becquerel, N_β the β count-rate, $N\gamma$ the γ count-rate and N_c the coincidence count-rate. If corrections, such as those due to dead-time and resolving time, can be neglected, then

$$N = \frac{N_\beta N_\gamma}{N_c} \quad .$$

The derivation of this equation is given in a report by the US National Council on Radiation Protection and Measurements [NCRP 1985:76].

For every disintegration observed, a dead-time is imposed on the particular detector; this time is greater than the time taken by the detector to process an observed event. Events occurring during the dead-time are not recorded. When a disintegration is detected by both detectors, the coincidence mixer does not receive the β pulse and the γ pulse simultaneously. A coincidence is recorded if the interval between the two pulses is less than the resolving time set in the coincidence mixer. However, if a β pulse from one disintegration and a γ pulse from another disintegration arrive at the mixer within the resolving time, an accidental coincidence is recorded.

Because of dead-time losses and accidental coincidences, the equation given above is only approximately true. A more elaborate equation, such as that of Bryant [1963], is therefore required. Count-rate-dependent corrections are discussed in the NCRP report [NCRP 1985:85].

A coincidence counting correction formula is derived for the case of equal, non-extending dead-times in the two detector channels. The derivation is similar to that of Bryant's equation. The nuclear disintegrations considered produce two coincident radiations which are observed by detectors A and B. The efficiency of detector A with respect to one of these radiations is denoted by E_A, and the efficiency of B with respect to the other is denoted by E_B. E_A and E_B are the probabilities of detection of a disintegration by A and B, respectively, when these are live. For simplicity, it is assumed that when a disintegration is detected by both detectors to give a true coincidence, they respond simultaneously, *i.e.* the dead-times are exactly coincident.

2. STATES OF THE DETECTORS

At any arbitrary instant, the detectors will be in one of the following states. In state 1, both detectors are live, as at the instants I_1, I'_1, I''_1, and I'''_1 in **figure 1**. In state 2, A is live and B is dead, as at instant I_2, A having been live in the interval between I'_1, and I_2. In state 3, A is dead and B live, as at instant I_3, B having been live in the interval between I''_1 and I_3. In state 4, both detectors are dead, as at instant I_4, as a result of a genuine coincidence at I'''_1. The detectors have the same dead-time, denoted by T.

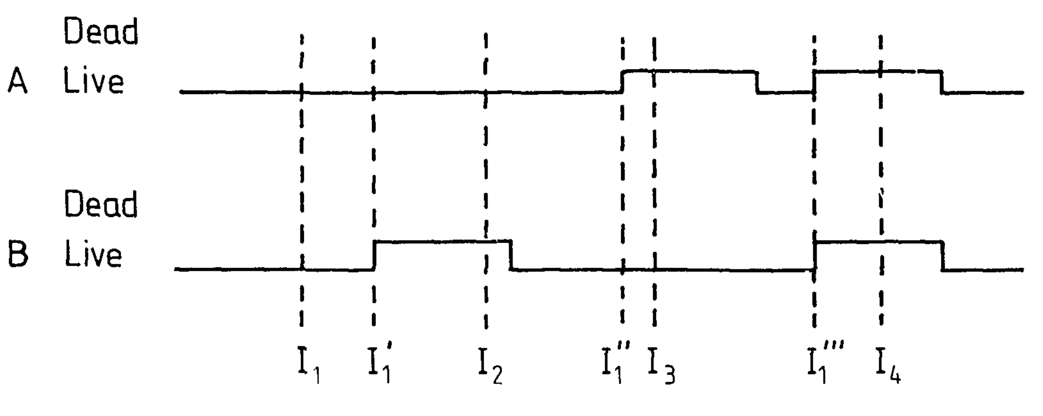

Figure 1 States 1, 2, 3 and 4 of the detectors

The detectors are in state 5A (**figure 2**) when A is dead and B has gone dead as a result of an earlier disintegration at a time between 0 and T before A went dead. State 5A is divided into (a) sub-state 5BA, in which both detectors are dead (instant I_{5BA}), B having gone dead before A, and (b) sub-state 3D (instant I_{3D}) in which B is live and A is dead as in state 3, but with B having been dead when A went dead.

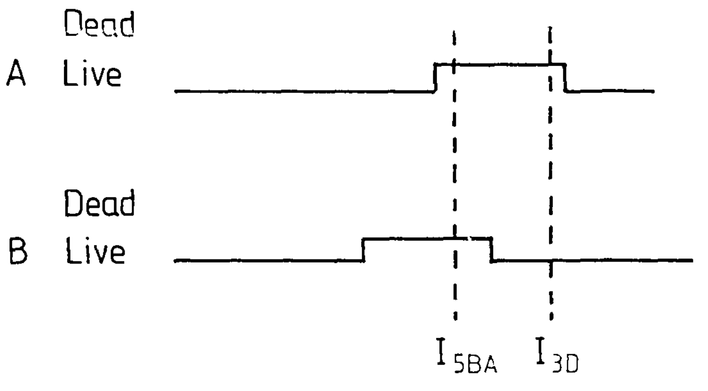

Figure 2 State 5A and sub-states 5BA and 3D

Similarly, the detectors are in state 5B (**figure 3**) when B is dead and A has gone dead between 0 and T before B went dead. In sub-state 5AB, both detectors are dead (instant I_{5AB}), and in sub-state 2D, detector A is live (instant I_{2D}).

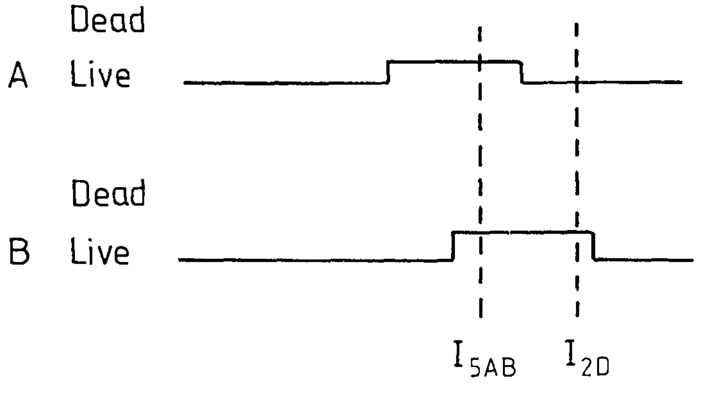

Figure 3 State 5B and sub-states 5AB and 2D

Let the probabilities of observing the states 1,2,3,4, 5A and 5B at any arbitrary instant be P_1, P_2, P_3, P_4, P_{5A} and P_{5B}, respectively. Let the probabilities of observing the sub-states 2D, 3D, 5AB and 5BA at any arbitrary instant be P_{2D}, P_{3D}, P_{5AB} and P_{5BA}, respectively.

3. PAIRS OF CONSECUTIVE DISINTEGRATIONS

Suppose that at the instant of the first of a pair of consecutive disintegrations the system is in a state or sub-state denoted by r, and at the instant of the second disintegration the system is in a state or sub-state denoted by s. The probability of this occurring is denoted by the symbol (r.s). It should be noted that between state or sub-state r of the first disintegration and state or sub-state s of the second disintegration, there can be another state, or one or two other sub-states.

4. OUTLINE OF THE DERIVATION

4.1 The Probabilities (r.s)

Formulae are derived for the probabilities (r.s) where r and s are one or other of the four states 1,2,3, or 4, or one of the four sub-states 2D, 3D, 5AB or 5BA. There are 64 such probabilities, 36 of which are zero. The probabilities are listed in **table 1** (see **section 6**), where zero denotes zero probability, and the other figures are the numbers of the relevant formulae in the text below.

From these formulae, another set of probabilities (r.s) is obtained in which r and s represent one or other of the six states 1,2,3,4, 5A, or 5B. This is shown in **table 2** (see **section 6**), in which some of the probabilities are the same as those in **table 1**, whereas others are the sums of two or three of those in **table 1**. The formulae for (r.s) in **table 2** consist of P_r multiplied by a factor which is less than unity. The formulae satisfy the equation

$$P_r = \sum_s (r.s) \quad . \tag{1}$$

For example,

$$P_1 = (1.1) + (1.2) + (1.3) + (1.4) + (1.5A) + (1.5B) \quad .$$

This follows from the fact that if the first disintegration occurs with the detectors in state 1, the second disintegration occurs in one of the states 1,2, 3, 4, 5A or 5B.

4.2 Solving Six Equations for P_1, P_2, P_3, P_4, P_{5A} and P_{5B}

Consider a pair of successive disintegrations. Let r and s denote one or other of the states 1, 2, 3, 4, 5A or 5B. The probability of the second disintegration occurring with the counters in state s is P_s. The probability of the second disintegration occurring with the counters in state s is also equal to the sum

$$(1.s) + (2.s) + (3.s) + (4.s) + (5A.s) + (5B.s)$$

Thus,

$$P_s = \sum_r (r.s) \quad . \tag{2}$$

There are six such equations in which P_1, P_2, P_3, P_4, P_{5A} and P_{5B} are considered as unknowns (see **section 7 - equations 47, 48, 49, 50, 51** and **52**). Only five of these equations are independent. To solve for these unknowns, a further equation is required, namely,

$$P_1 + P_2 + P_3 + P_4 + P_{5A} + P_{5B} = 1 \tag{3}$$

Thus, using **equation 3** and five of the six equations mentioned above, formulae for P_1, P_2, P_3, P_4, P_{5A} and P_{5B} are obtained (see **section 7 - equations 62, 61, 57, 53, 54** and **58**).

4.3 The Equation for the Observed Coincidence Rate

An accidental coincidence is recorded when one counter goes dead as a result of detecting a disintegration, and the other counter goes dead as a result of detecting another disintegration within the resolving time of the system. **Equation 63** gives the observed coincidence rate as the sum of the rate of true coincidences and the rate of accidental coincidences. The formulae for P_1, P_2, P_3, P_{5A} and P_{5B} are substituted into this equation and the disintegration rate (N) is then computed by an iterative method.

4.4 Bryant's Equation for the Disintegration Rate

In **appendix B**, Bryant's equation is obtained by abridging the derivation which has been outlined above.

5. DERIVATION OF THE PROBABILITIES (r.s)

5.1 The Probability (1.1)

The probability (1.1) is the probability that both the first and second disintegrations in a pair of successive disintegrations take place when the system is in state 1. There are two cases to consider:

(a) Both detectors are live for the first disintegration, but the disintegration is not detected by either. The detectors are therefore live for the second disintegration.

(b) The first disintegration is detected by one or both of the detectors, so that one or both detectors go dead for time T, as at instants I'_1, I''_1 or I'''_1 in **figure 1**. For both detectors to be live for the second disintegration, no disintegration must take place during time T. The probability of this occurring is e^{-NT}. This latter probability is derived in **appendix A**.

Thus (1.1) = probability of (a) + probability of (b).

Case (a) The probability that detectors A and B are both live for the first disintegration is P_1. The probability of this disintegration being detected by A is E_A. The probability that the disintegration is not detected by A is $1\text{-}E_A$. The probability that it is not detected by B is $1\text{-}E_B$. Consider S pairs of disintegrations, where S is very large. The number of these for which the first disintegration occurs with the system in state 1 is P_1S. Of these pairs, the number for which the first disintegration is not detected by A is $P_1S\ (1-E_A)$, and of these, the number whose first disintegration is not detected by B is $P_1S\ (1-E_A)(1-E_B)$. Thus the probability that A and B are live for the first disintegration, and that neither detect the first disintegration, is $P_1\ (1-E_A)\ (1-E_B)$.

Case (b) Of the P_1S pairs, the number whose first disintegration is not detected by A or B is $P_1\ S\ (1-E_A)(1-E_B)$. Therefore the first disintegrations which are detected by one detector or both detectors amount to $P_1S - P_1S\ (1-E_A)(1-E_B)$. Of this number of ensuing dead-times, the fraction during which no further disintegration occurs is e^{-NT}. Then the probability of case (b) is $P_1\ [1-(1-E_A)(1-E_B)]\ e^{-NT}$. Thus,

$$\begin{aligned}(1.1) &= P_1\ (1-E_A-E_B+E_AE_B)+P_1\ [1-(1-E_A-E_B+E_AE_B)]e^{-NT}\ ,\\ &= P_1+P_1\ (E_AE_B-E_A-E_B)-P_1\ (E_AE_B-E_A-E_B)e^{-NT}\ ,\\ &= P_1\ \ [1+(E_AE_B-E_A-E_B)(1-e^{-NT})]\ ,\end{aligned} \qquad (4)$$

5.2 The Probability (1.2)

The probability that a disintegration as at instant I'_1 in **figure 1**, will occur when the system is in state 1, and that the disintegration will result in detector B going dead for an interval T is $P_1\ E_B$. The probability that in addition the disintegration is not detected by A, so that A remains live during the interval T, is $P_1\ E_B(1-E_A)$. The probability that one or more disintegrations will occur during T is $(1-e^{-NT})$. Thus the probability that the second disintegration will occur in state 2 is $(1-e^{-NT})$. Therefore

$$(1.2) = P_1\ E_B(1-E_A)(1-e^{-NT})\ . \qquad (5)$$

5.3 The Probability (1.3)

By a derivation similar to that for probability (1.2),

$$(1.3) = P_1\ E_A\ (1-E_B)(1-e^{-NT}). \qquad (6)$$

5.4 The Probability (1.4)

The second of the pair of successive disintegrations occurs with A and B dead as the result of a coincidence. The probability of the first disintegration having been detected by both A and B is $P_1\ E_A\ E_B$. The second disintegration has to occur during the dead-time T. Therefore

$$(1.4) = P_1\ E_A\ E_B\ (1-e^{-NT}). \qquad (7)$$

5.5 The Probability (1.5AB)

In **figure 4**, consider a disintegration which occurs in state 1 at instant I_1 and which is not detected by A or B. The system changes to sub-state 5AB after disintegrations at I'_1 and I'''_3, which are detected by A and B, respectively. Therefore a disintegration at I_{5AB} in sub-state 5AB cannot be the next disintegration after that at I_1.

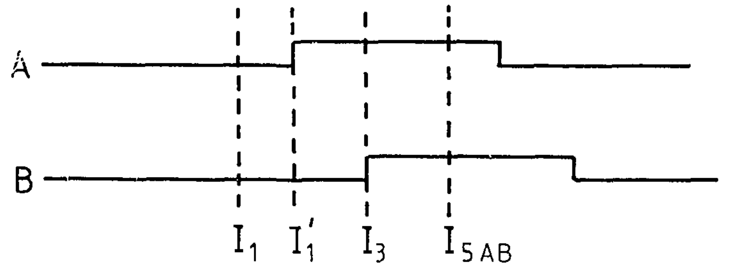

Figure 4 The probability (1.5AB)

Similarly, if we consider the disintegration which occurs in state 1 at I_1', and which is detected by A, the next disintegration cannot be that at I_{5AB} in sub-state 5AB. Thus

$$(1.5AB) = 0. \qquad (8)$$

5.6 The Probability (1.5 BA)

By reasoning similar to that for probability (1.5AB),

$$(1.5BA) = 0. \quad (9)$$

5.7 The Probability (2.1)

There are two cases to consider:

Case (a) In **figure 5(a)**, the first disintegration occurs, for example, at instant I_2 and is detected by A. The second disintegration will take place with the system in state 1 if there is no disintegration during the dead-time of A. The probability of case (a) is $P_2 E_A e^{-NT}$.

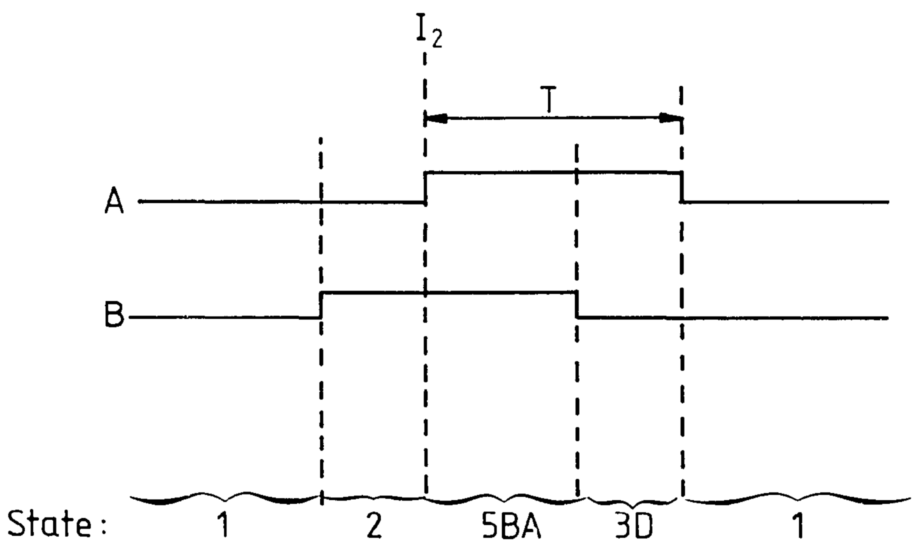

Figure 5(a) The probability (2.1), case (a)

Case (b) In **figure 5(b)**, the first disintegration at I_2 is not detected by A. The probability of the first disintegration occurring in state 2 and not being detected by A is $P_2(1-E_A)$. It is assumed that all values of x are equally probable. Thus the probability that the first disintegration is not detected by A, and that it occurs at a time between x and $x + \delta x$ before the end of the dead-time of B is $P_2(1-E_A)(\delta x/T)$. The probability that, in addition, the second disintegration does not occur during the interval x is $P_2(1-E_A)(\delta x/T)e^{-Nx}$.

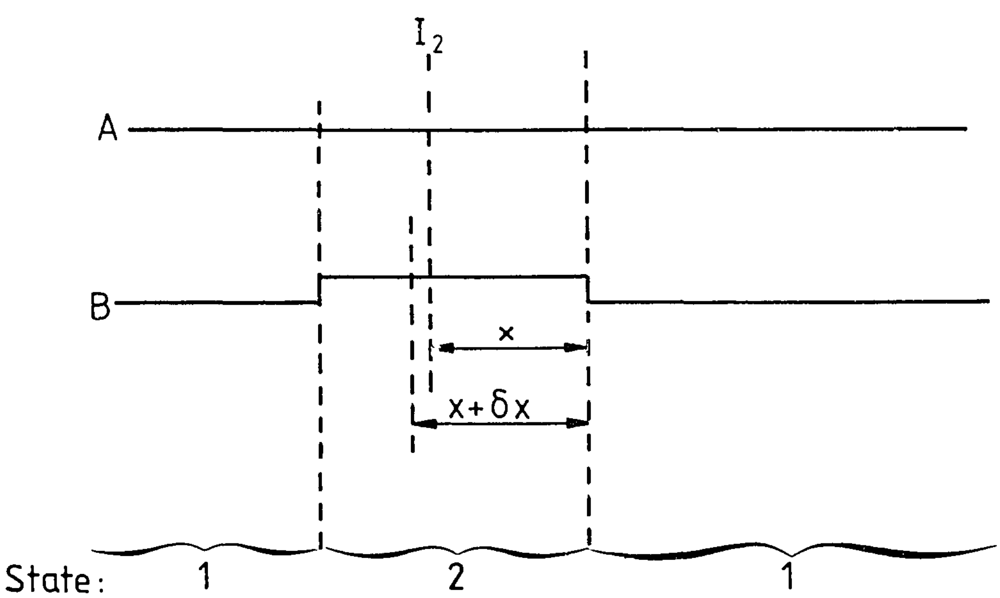

Figure 5(b) The probability (2.1), case (b)

Summing for all the intervals δx, the probability of case (b)

$$= \int_0^T \frac{P_2(1-E_A)e^{-Nx}}{T}\,dx,$$

$$= \frac{P_2(1-E_A)}{T}\left[-\frac{e^{-Nx}}{N}\right]_0^T,$$

$$= \frac{P_2(1-E_A)}{T}\left[\frac{1}{N}-\frac{e^{-NT}}{N}\right],$$

$$= P_2(1-E_A)\,\frac{(1-e^{-NT})}{NT}\ .$$

Thus,

$$(2.1) = P_2\left[E_A\,e^{-NT} + \frac{1}{NT}(1-E_A)(1-e^{-NT})\right]\ . \tag{10}$$

5.8 The Probability (3.1)

The derivation of the probability (3.1) is the same as that for (2.1) except that P_3 replaces P_2, and E_B replaces E_A. Thus

$$(3.1) = P_3\left[E_B\,e^{-NT} + \frac{1}{NT}(1-E_B)(1-e^{-NT})\right]\ . \tag{11}$$

5.9 The Probability (2.2)

In **figure 5 (b)**, the first disintegration at I_2 is not detected by A, and the second disintegration occurs during the interval x. The probability of a disintegration occurring during x is $1-e^{-Nx}$. Thus

$$(2.2) = \frac{P_2(1-E_A)}{T}\int_0^T (1-e^{-Nx})\,dx\ ,$$

$$= \frac{P_2(1-E_A)}{T}\left[x+\frac{e^{-Nx}}{N}\right]_0^T,$$

$$= \frac{P_2(1-E_A)}{T}\left(T+\frac{e^{-NT}}{N}-\frac{1}{N}\right),$$

$$= P_2(1-E_A)\left[1-\frac{1}{NT}(1-e^{-NT})\right]\ . \tag{12}$$

5.10 The Probability (3.3)

By a derivation similar to that for probability (2.2),

$$(3.3) = P_3(1-E_B)\left[1-\frac{1}{NT}(1-e^{-NT})\right]\ . \tag{13}$$

5.11 The Probability (2.3D)

In **figure 6**, the first disintegration occurs at I_2 and A goes dead. The second disintegration occurs after B has become live again, with A still dead. The probability that the first disintegration occurs between x and $x+\delta x$, and is detected by A is $P_2\,E_A\,\delta x/T$.

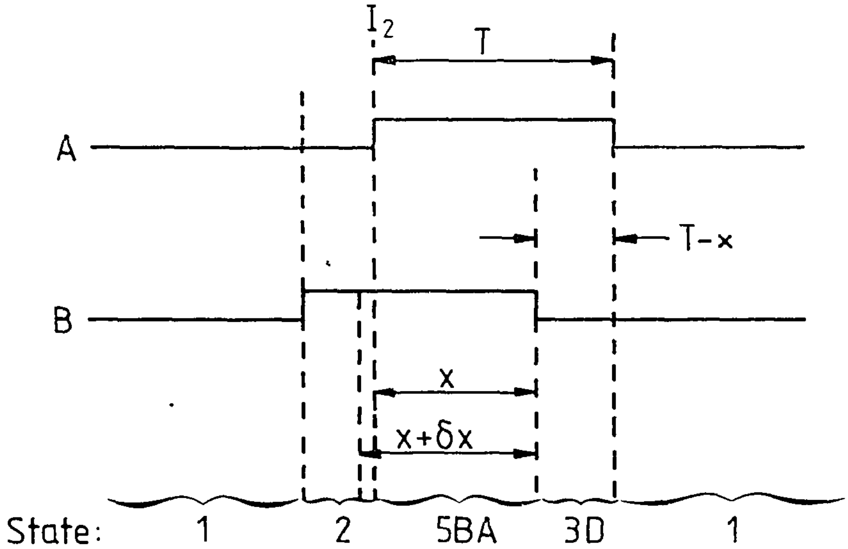

Figure 6 The probability (2.3D)

The probability of the second disintegration taking place after the interval x is e^{-Nx}. The probability of the second disintegration taking place in the interval T – x, *i.e.* in state 3D, is $1 - e^{-N(T-x)}$. Thus,

$$(2.3D) = \int_0^T \frac{P_2E_A}{T} e^{-Nx}(1 - e^{-N(T-x)})\, dx \ ,$$

$$= \frac{P_2E_A}{T} \int_0^T (e^{-Nx} - e^{-NT})\, dx \ ,$$

$$= \frac{P_2E_A}{T} \left[- \frac{e^{-Nx}}{N} - x\, e^{-NT} \right]_0^T ,$$

$$= \frac{P_2E_A}{T} \left[- \frac{e^{-NT}}{N} + \frac{1}{N} - Te^{-NT} \right] ,$$

$$= P_2E_A \left[\frac{1}{NT} (1 - e^{-NT}) - e^{-NT} \right] . \qquad (14)$$

5.12 The Probability (3.2D)

By a derivation similar to that for probability (2.3D),

$$(3.2D) = P_3E_B \left[\frac{1}{NT} (1 - e^{-NT}) - e^{-NT} \right] . \qquad (15)$$

5.13 The Probabilities (2.4) and (3.4)

If the second of two successive disintegrations occurs with the system in state 4, the first disintegration must also occur in state 4, or alternatively in state 1. In the latter case, the disintegration is that which sends A and B dead. Therefore

$$(2.4) = 0 . \qquad (16)$$

Similarly,

$$(3.4) = 0 . \qquad (17)$$

5.14 The Probability (2.5BA)

In **figure 6**, the probability of the first disintegration occurring between x and x + δx, and being detected by A, is $P_2(\delta x/T)E_A$. For the second disintegration to occur while the system is in sub-state 5BA, one or more disintegrations must occur during the interval x, and the probability of the latter occurrence is $1 - e^{-Nx}$.

Therefore

$$(2.5BA) = \frac{P_2E_A}{T}\int_0^T (1 - e^{-Nx})\, dx \;,$$

$$= \frac{P_2E_A}{T}\left[x + \frac{e^{-Nx}}{N} \right]_0^i ,$$

$$= \frac{P_2E_A}{T}\left[T + \frac{e^{-NT}}{N} - \frac{1}{N} \right] ,$$

$$= P_2E_A\left[1 - \frac{1}{NT}\,(1 - e^{-NT}) \right] . \qquad (18)$$

5.15 The Probability (3.5AB)

By a derivation similar to that for probabiltity (2.5BA),

$$(3.5AB) = P_3E_B\left[1 - \frac{1}{NT}\,(1 - e^{-NT}) \right] . \qquad (19)$$

5.16 The Probabilities (4.1) and (4.4)

In **figure 7**, the first disintegration takes place in state 4, for example, at instant I_4. The probability of the first disintegration taking place between x and x + δx is $P_4(\delta x/T)$.

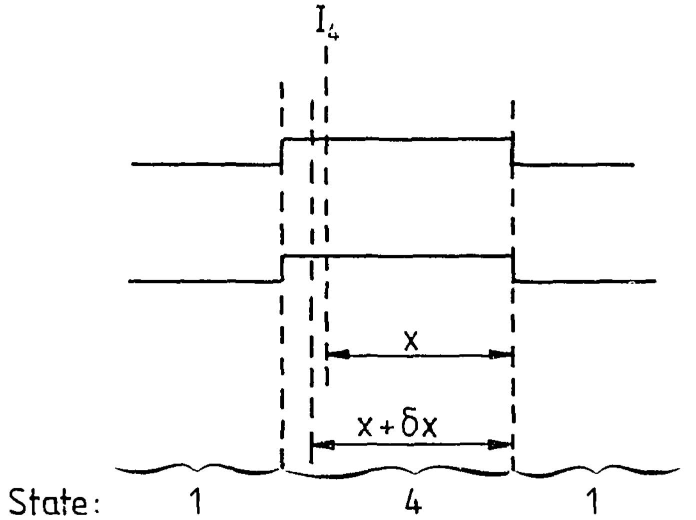

Figure 7 The probabilities (4.1) and (4.4)

The probability of no disintegration taking place during the interval x is e^{-Nx}. Therefore

$$(4.1) = \frac{P_4}{T}\int_0^T e^{-Nx}\, dx \;,$$

$$= \frac{P_4}{T}\left[-\frac{e^{-NT}}{N} + \frac{1}{N} \right] ,$$

$$= P_4\,\frac{1}{NT}\left[1 - e^{-NT} \right] . \qquad (20)$$

Similarly,

$$(4.4) = \frac{P_4}{T}\int_0^T (1 - e^{-Nx})\, dx \;,$$

$$= \frac{P_4}{T}\left[T + \frac{e^{-NT}}{N} - \frac{1}{N} \right] ,$$

$$= P_4\left[1 - \frac{1}{NT}\,(1 - e^{-NT}) \right] . \qquad (21)$$

5.17 The Probabilities (4.2), (4.3), (4.5AB) and (4.5BA)

Because state 4 must be followed by state 1, probabilities (4.2), (4.3), (4.5AB) and (4.5BA) are all zero.

5.18 Introduction to the Probabilities (5BA.s) and (3D. s)

The system is in state 5A when A is dead, and the dead-time of A overlaps an earlier dead-time of B, as shown in **figure 8**. The difference between the start of these dead-times is between zero and T. The probability that the first disintegration occurs between x and x + δx before the end of A's dead-time is $P_{5A}\delta x/T$. It is assumed that all degrees of overlapping of the dead-times are equally probable. Therefore the probability that the end of the dead-time of B occurs between y and y + δy is δy/T. The probability that at instant I_{5BA} detector B is dead, *i.e.* the system is in sub-state 5BA, is

$$\int_0^x \frac{1}{T}\, dy = \frac{x}{T} \ .$$

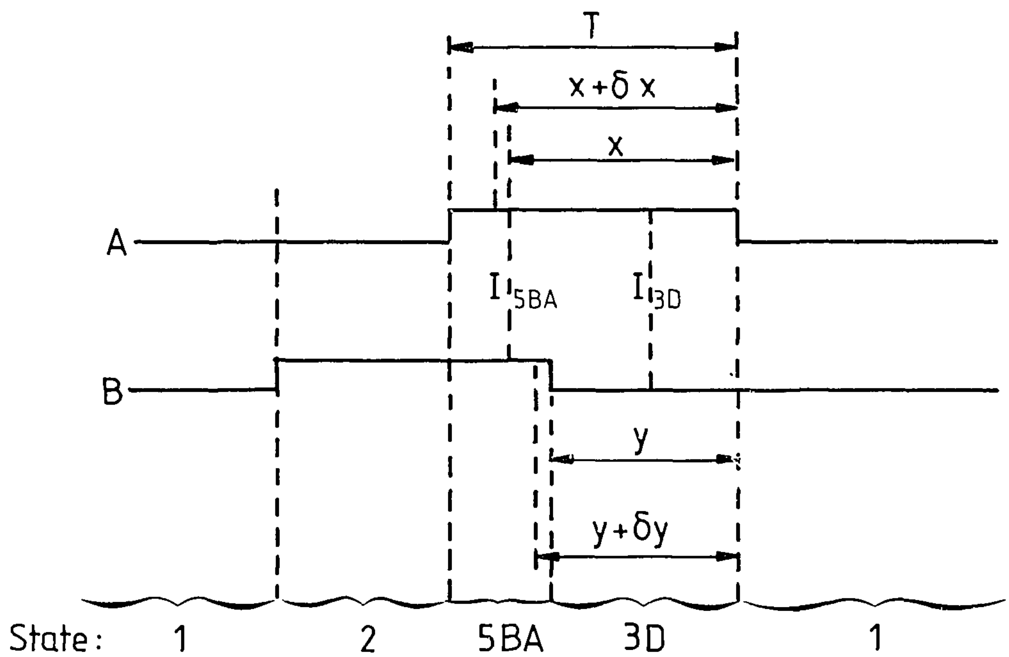

Figure 8 The probabilities (5BA.s) and (3D.s)

Therefore the probability that the first disintegration occurs between x and x + δx, and that the system is in sub-state 5BA is

$$P_{5A}\ \frac{x}{T^2}\ \delta x \ . \tag{22}$$

The probability that at instant I_{3D} detector B is live, *i.e.* the system is in sub-state 3D, is

$$\int_x^T \frac{1}{T}\, dy = \frac{T-x}{T} \ .$$

Therefore the probability that the first disintegration occurs between x and x + δx, and that the system is in sub-state 3D, is

$$P_{5A}\ \frac{(T-x)}{T^2}\ \delta x \ . \tag{23}$$

5.19 The Probability (5BA.1)

With reference to **figure 8**, the second disintegration occurs after A becomes live again. The probability of no disintegration occurring during x is e^{-Nx}. Combining this with formula 22 and summing for all the intervals δx,

$$(5BA.1) = \frac{P_{5A}}{T^2}\int_0^T x\,e^{-Nx}\,dx\ ,$$

$$= -\frac{P_{5A}}{NT^2}\left[(x+\frac{1}{N})\,e^{-Nx}\right]_0^T ,$$

$$= -\frac{P_{5A}}{NT^2}\left[Te^{-NT}+\frac{e^{-NT}}{N}-\frac{1}{N}\right] ,$$

$$= P_{5A}\left[\frac{1}{N^2T^2}(1-e^{-NT})-\frac{1}{NT}\,e^{-NT}\right] . \qquad (24)$$

5.20 Introduction to the Probabilities (5AB.s) and (2D.s)

In **figure 8**, if A and B are interchanged, it can be shown, as in **section 5.18**, that the probability that the first disintegration occurs between x and x + δx, and that the system is in sub-state 5AB, is

$$P_{5B}\,\frac{x}{T^2}\,\delta x\ . \qquad (25)$$

The probability that the first disintegration occurs between x and x + δx, and that the system is in sub-state 2D, is

$$P_{5B}\,\frac{(T-x)}{T^2}\,\delta x\ . \qquad (26)$$

5.21 The Probability (5AB.1)

It can be shown, as in **section 5.19**, that

$$(5AB.1) = P_{5B}\left[\frac{1}{N^2T^2}\,(1-e^{-NT})-\frac{1}{NT}\,e^{-NT}\right] . \qquad (27)$$

5.22 The Probability (5BA.3D)

In **figure 8** and **formula 22**, the probability that the first disintegration occurs between x and x + δx, and that the system is in sub-state 5BA, is $P_{5A}\,(x/T^2)\,\delta x$. The probability that no further disintegration occurs during the interval (x − y) is $e^{-N(x-y)}$. For one particular value of x, all values of y between 0 and x are equally probable. Thus, for one particular value of x, the probability that the end of the dead-time of B occurs between y and y + δy is δy/x.

During the interval y, when the system is in the sub-state 3D, one or more disintegrations must occur. The probability of that occurrence is $1-e^{-Ny}$.

The probability that the first disintegration occurs between x and x + δx, and the second disintegration occurs when the system is in sub-state 3D, is equal to

$$\int_0^x P_{5A}\,\frac{x}{T^2}\,\delta x\,e^{-N(x-y)}\,\frac{1}{x}\,(1-e^{-Ny})\,dy$$

$$= \frac{P_{5A}\,\delta x\,e^{-Nx}}{T^2}\int_0^x (e^{Ny}-1)\,dy\ ,$$

$$= \frac{P_{5A}\,\delta x\,e^{-Nx}}{T^2}\left[\frac{e^{Nx}}{N}-\frac{1}{N}-x\right] ,$$

$$= \frac{P_{5A}}{T^2}\left[\frac{1}{N}-\frac{e^{-Nx}}{N}-x\,e^{-Nx}\right]\delta x\ .$$

Thus

$$(5BA.3D) = \frac{P_{5A}}{T^2}\int_0^T (\frac{1}{N}-\frac{e^{-Nx}}{N}-xe^{-Nx})\,dx\ ,$$

$$= \frac{P_{5A}}{T^2}\left[\frac{x}{N}+\frac{e^{-Nx}}{N^2}+\frac{xe^{-Nx}}{N}+\frac{e^{-Nx}}{N^2}\right]_0^T ,$$

$$= \frac{P_{5A}}{T^2}\left[\frac{T}{N}+\frac{2e^{-NT}}{N^2}-\frac{2}{N^2}+\frac{Te^{-NT}}{N}\right] ,$$

$$= P_{5A}\left[\frac{1}{NT}(1+e^{-NT})-\frac{2}{N^2T^2}(1-e^{-NT})\right] . \qquad (28)$$

5.23 The Probability (5AB.2D)

A derivation similar to that for **equation** 28 gives the result

$$(5AB.2D) = P_{5B}\left[\frac{1}{NT}(1+e^{-NT})-\frac{2}{N^2T^2}(1-e^{-NT})\right] . \qquad (29)$$

5.24 The Probability (5BA.5BA)

Again referring to **figure 8**, if the first disintegration occurs between x and x + δx before the end of the dead-time of A, then one or more disintegrations must occur before B becomes live again. The probability of the latter occurrence is $1 - e^{-N(x-y)}$. The probability that the first disintegration occurs between x and x + δx, that B becomes live between y and y + δy, and that the second disintegration also occurs with the system in sub-state 5BA, is

$$P_{5A}\,\frac{\delta x}{T}\,\frac{x}{T}\,\frac{\delta y}{x}\,(1-e^{-Nx}e^{Ny}) .$$

$$(5BA.5BA) = \int_0^T\left[\int_0^x \frac{P_{5A}}{T^2}(1-e^{-Nx}e^{Ny})\,dy\right]dx ,$$

$$= \frac{P_{5A}}{T^2}\int_0^T\left[y - e^{-Nx}\frac{e^{Ny}}{N}\right]_0^x dx ,$$

$$= \frac{P_{5A}}{T^2}\int_0^T\left[x-\frac{1}{N}+\frac{e^{-Nx}}{N}\right]dx ,$$

$$= \frac{P_{5A}}{T^2}\left[\frac{x^2}{2}-\frac{x}{N}-\frac{e^{-Nx}}{N^2}\right]_0^T ,$$

$$= \frac{P_{5A}}{T^2}\left[\frac{T^2}{2}-\frac{T}{N}-\frac{e^{-NT}}{N^2}+\frac{1}{N^2}\right] ,$$

$$= P_{5A}\left[\frac{1}{2}-\frac{1}{NT}+\frac{1}{N^2T^2}(1-e^{-NT})\right] . \qquad (30)$$

5.25 The Probability (5AB.5AB)

If A and B are interchanged in **figure 8**, then a derivation similar to that in **section 5.24** gives the result

$$(5AB.5AB) = P_{5B}\left[\frac{1}{2}-\frac{1}{NT}+\frac{1}{N^2T^2}(1-e^{-NT})\right] . \qquad (31)$$

5.26 The Probability (3D.1)

In **figure 9(a)**, suppose that the first disintegration takes place at instant I_{3D}. From **formula 23**, the probability that the first disintegration occurs between x and x + δx, and that the system is in sub-state 3D, is

$$P_{5A}\,\frac{(T-x)}{T^2}\,\delta x .$$

There are two cases to consider:

Case (a) In **figure 9(a)**, the first disintegration is detected by B with a probability E_B, and B goes dead. The second disintegration occurs after an interval T has elapsed. The probability of case (a) is

$$\int_0^T P_{5A}\,\frac{(T-x)}{T^2}\,E_B\,e^{-NT}\,dx ,$$

$$= \frac{P_{5A}\, E_B\, e^{-NT}}{T^2} \left[Tx - \frac{x^2}{2} \right]_0^T ,$$

$$= \frac{1}{2} P_{5A}\, E_B\, e^{-NT} . \qquad (32)$$

Case (b) In **figure 9(b)**, the first disintegration at instant I_{3D} is not detected by B, and the second disintegration occurs after the interval x has elapsed. The probability of case (b) is

$$\int_0^T P_{5A} \frac{(T-x)}{T^2} (1 - E_B)\, e^{-Nx}\, dx$$

$$= \frac{P_{5A}(1 - E_B)}{T^2} \left[- \frac{Te^{-Nx}}{N} + \frac{xe^{-Nx}}{N} + \frac{e^{-Nx}}{N^2} \right]_0^T ,$$

$$= \frac{P_{5A}(1 - E_B)}{T^2} \left[- \frac{Te^{-NT}}{N} + \frac{T}{N} + \frac{Te^{-NT}}{N} + \frac{e^{-NT}}{N^2} - \frac{1}{N^2} \right] ,$$

$$= P_{5A}(1 - E_B) \left[\frac{1}{NT} - \frac{1}{N^2T^2} (1 - e^{-NT}) \right] .$$

Thus,

$$(3D.1) = \frac{1}{2} P_{5A}\, E_B\, e^{-NT} + P_{5A}(1 - E_B) \left[\frac{1}{NT} - \frac{1}{N^2T^2} (1 - e^{-NT}) \right] . \qquad (33)$$

5.27 The Probability (2D.1)

If A and B in the preceding derivation are interchanged, it can be shown that

$$(2D.1) = \frac{1}{2} P_{5B}\, E_A\, e^{-NT} + P_{5B}(1 - E_A) \left[\frac{1}{NT} - \frac{1}{N^2T^2} (1 - e^{-NT}) \right] . \qquad (34)$$

5.28 The probability (3D.3D)

In **figure 9(b)**, the first disintegration, which occurs for example at instant I_{3D}, is not detected by B, and the second disintegration occurs during the interval x.

Thus,

$$(3D.3D) = \int_0^T P_{5A} \frac{(T-x)}{T^2} (1 - E_B)(1 - e^{-Nx})\, dx ,$$

$$= \frac{P_{5A}(1 - E_B)}{T^2} \int_0^T (T - Te^{-Nx} - x + xe^{-Nx})\, dx ,$$

$$= \frac{P_{5A}(1 - E_B)}{T^2} \left[Tx + \frac{Te^{-Nx}}{N} - \frac{x^2}{2} - \frac{xe^{-Nx}}{N} - \frac{e^{-Nx}}{N^2} \right]_0^T ,$$

$$= \frac{P_{5A}(1 - E_B)}{T^2} \left[T^2 + \frac{Te^{-NT}}{N} - \frac{T}{N} - \frac{T^2}{2} - \frac{Te^{-NT}}{N} - \frac{e^{-NT}}{N^2} + \frac{1}{N^2} \right] ,$$

$$= P_{5A}(1 - E_B) \left[\frac{1}{2} - \frac{1}{NT} + \frac{1}{N^2T^2} (1 - e^{-NT}) \right] . \qquad (35)$$

5.29 The Probability (2D.2D)

A derivation similar to that in **section 5.28** shows that

$$(2D.2D) = P_{5B}(1 - E_A) \left[\frac{1}{2} - \frac{1}{NT} + \frac{1}{N^2T^2} (1 - e^{-NT}) \right] . \qquad (36)$$

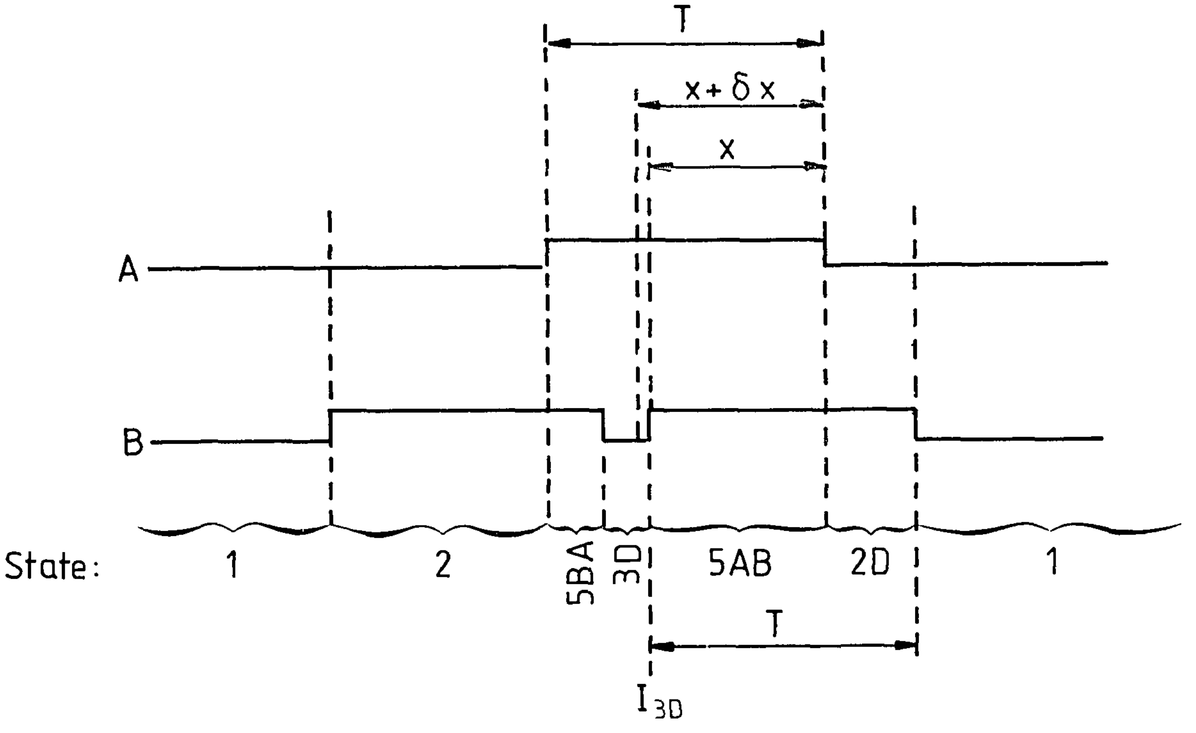

Figure 9(a) The probability (3D.1), case (a)

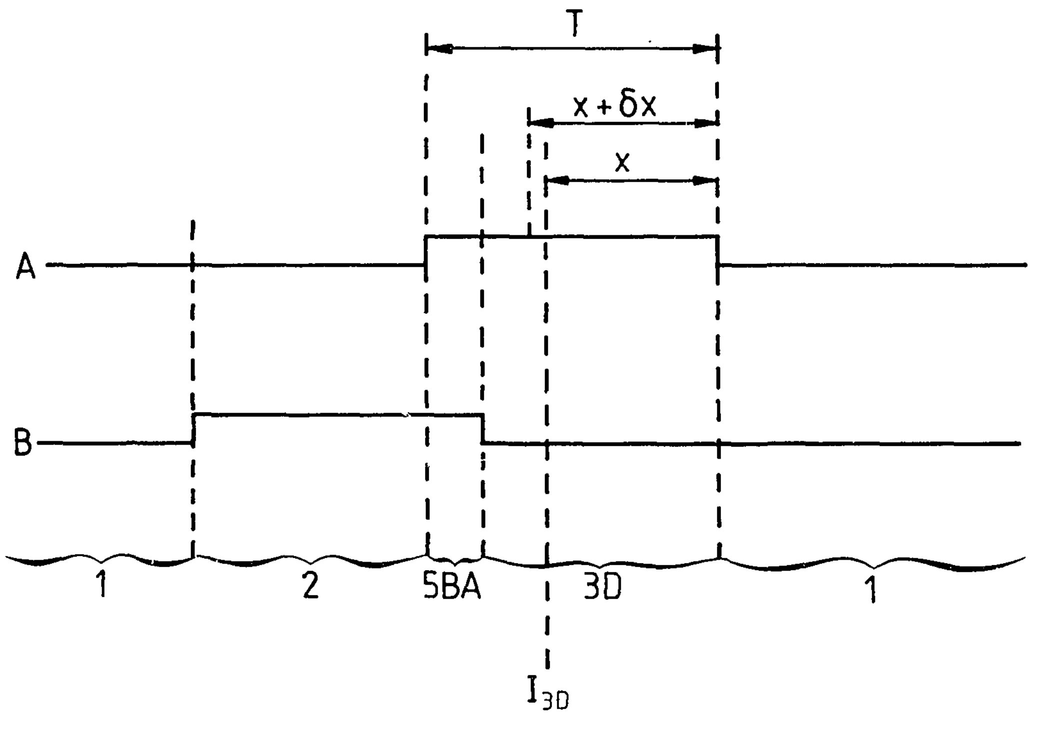

Figure 9(b) The probability (3D.1), case (b)

5.30 The Probability (3D.2D)

In **figure 10**, for the second disintegration to occur with the system in sub-state 2D, the first disintegration at I_{3D} will be detected by B, there will be no disintegration during the interval x, and one or more disintegrations will occur in the interval T–x.

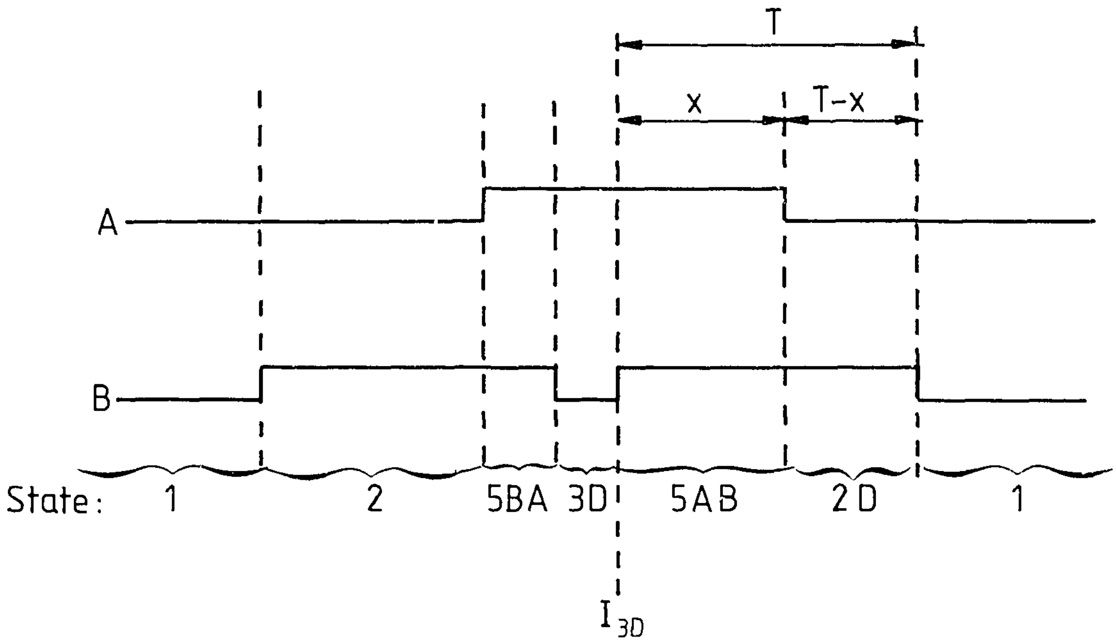

Figure 10 The probability (3D.2D)

Thus,

$$(3D.2D) = \int_0^T P_{5A} \frac{(T-x)}{T^2} E_B e^{-Nx}(1 - e^{-N(T-x)})\, dx \ ,$$

$$= \frac{P_{5A}E_B}{T^2}\int_0^T (T-x)(e^{-Nx} - e^{-NT})\, dx \ ,$$

$$= \frac{P_{5A}E_B}{T^2}\int_0^T (Te^{-Nx} - Te^{-NT} - xe^{-Nx} + xe^{-NT})\, dx \ ,$$

$$= \frac{P_{5A}\, E_B}{T^2}\left[-\frac{Te^{-Nx}}{N} - xTe^{-NT} + \frac{xe^{-Nx}}{N} + \frac{e^{-Nx}}{N^2} + \frac{x^2e^{-NT}}{2}\right]_0^T ,$$

$$= \frac{P_{5A}\, E_B}{T^2}\left[-\frac{Te^{-NT}}{N} + \frac{T}{N} - T^2e^{-NT} + \frac{Te^{-NT}}{N} + \frac{e^{-NT}}{N^2} - \frac{1}{N^2} + \frac{T^2e^{-NT}}{2}\right] ,$$

$$= P_{5A}E_B\left[\frac{1}{NT} - \frac{1}{2}e^{-NT} - \frac{1}{N^2T^2}(1 - e^{-NT})\right] . \qquad (37)$$

5.31 The Probability (2D.3D)

A derivation similar to that in **section 5.30** shows that

$$(2D.3D) = P_{5B}E_A\left[\frac{1}{NT} - \frac{1}{2}e^{-NT} - \frac{1}{N^2T^2}(1 - e^{-NT})\right] . \qquad (38)$$

5.32 The Probability (3D.5AB)

In **figure 10**, the first disintegration is detected at I_{3D} by detector B, and the second occurs during the interval x. Thus,

$$(3D.5AB) = \int_0^T P_{5A} \frac{(T-x)}{T^2} E_B (1 - e^{-Nx})\, dx.$$

The above integral is the same as that for probability (3D.3D) in **section** 5.28, except that E_B replaces $1 - E_B$. Therefore,

$$(3D.5AB) = P_{5A}\, E_B \left[\frac{1}{2} - \frac{1}{NT} + \frac{1}{N^2T^2}\,(1-e^{-NT}) \right] . \qquad (39)$$

5.33 The Probability (2D.5BA)

By a derivation similar to that in **section 5.32**,

$$(2D.5BA) = P_{5B}E_A \left[\frac{1}{2} - \frac{1}{NT} + \frac{1}{N^2T^2}\,(1 - e^{-NT}) \right] . \qquad (40)$$

6. TABULATION OF THE PROBABILITIES (r.s)

The (r.s) probabilities which have not been dealt with above are zero. In **table 1** the zero probabilities are denoted by 0. The other probabilities are represented by their equation numbers.

TABLE 1

The zeros indicate (r.s) probabilities of zero, and the numbers are those of the relevant (r.s) formulae in the text. The symbol (r.s) denotes the probability that the first disintegration occurs in state or sub-state r, and the second disintegration occurs in the state or sub-state s.

		s ; Second State or Sub-state							
		1	2	2D	3	3D	4	5AB	5BA
r; First State or Sub-state	1	(4)	(5)	0	(6)	0	(7)	0	0
	2	(10)	(12)	0	0	(14)	0	0	(18)
	2D	(34)	0	(36)	0	(38)	0	0	(40)
	3	(11)	0	(15)	(13)	0	0	(19)	0
	3D	(33)	0	(37)	0	(35)	0	(39)	0
	4	(20)	0	0	0	0	(21)	0	0
	5AB	(27)	0	(29)	0	0	0	(31)	0
	5BA	(24)	0	0	0	(28)	0	0	(30)

Another set of (r.s) probabilities can now be obtained in which r and s represent one or other of the six states 1,2,3,4, 5A or 5B. From **figure 6**, it can be seen that since

$$P_{5A} = P_{5BA} + P_{3D} , \qquad (41)$$

then $(2.5A) = (2.5BA) + (2.3D)$. (42)

In **figure 9(b)**,

$$(5A.1) = (5BA.1) + (3D.1), \qquad (43)$$

$$(5A.5A) = (5BA.5BA) + (5BA.3D) + (3D.5BA) + (3D.3D) .$$

But since, from **table 1**,

$$(3D.5BA) = 0, \text{ then}$$

$$(5A.5A) = (5BA.5BA) + (5BA.3D) + (3D.3D) . \qquad (44)$$

From **figure 10**, since

$$P_{5B} = P_{5AB} + P_{2D} , \qquad (45)$$

then

(5A.5B) = (5BA.5AB) + (5BA.2D) + (3D.5AB) + (3D.2D).

But since, from **table 1**,

(5BA.5AB) = 0 and (5BA.2D) = 0 ,

then

$$(5A.5B) = (3D.5AB) + (3D.2D) \ . \qquad (46)$$

In **equations 42, 43, 44,** and 46, if A and B are interchanged, and 2 and 3 are interchanged, similar formulae are obtained for (3.5B), (5B.1), (5B.5B) and (5B.5A). The probabilities (r.s) for the six states are given in **table 2**. Some of the probabilities are represented by their (r.s) symbols, the zero probabilities are denoted by 0, and the remainder are shown as sums of two or three of the probabilities given in table 1.

TABLE 2

The probability that the first of two successive disintegrations takes place when the detectors are in the first state, and the second disintegration takes place when the detectors are in the second state, is shown in this table either as a probability from **table 1**, or as a sum of such probabilities.

First State \ Second State	1	2	3	4	5A	5B
1	(1.1)	(1.2)	(1.3)	(1.4)	0	0
2	(2.1)	(2.2)	0	0	(2.5BA)+ (2.3D)	0
3	(3.1)	0	(3.3)	0	0	(3.5AB)+ (3.2D)
4	(4.1)	0	0	(4.4)	0	0
5A	(5BA.1)+ (3D.1)	0	0	0	(5BA.5BA)+ (5BA.3D)+ (3D.3D)	(3D.5AB)+ (3D.2D)
5B	(5AB.1)+ (2D.1)	0	0	0	(2D.5BA)+ (2D.3D)	(5AB.5AB)+ (5AB.2D)+ 2D.2D

As a check on the correctness of the formulae for the probabilities (r.s), where r and s are 1,2,3,4,5A or 5B, it can be shown that the formulae satisfy **equation 1**. Thus, if the probabilities in the top row of **table 2** are added together, it will be found that the sum is equal to P_1, and likewise for the other five rows.

7. DERIVATION OF FORMULAE FOR P_1, P_2, P_3, P_4, P_{5A}, AND P_{5B}

Six simultaneous equations containing P_1, P_2, P_3, P_4, P_{5A} and P_{5B} as unknowns can now be constructed. From **equation 2**, it can be seen that P_1 equals the sum of the formulae in column 1 of **table 2**:

$$\text{i.e. } P_1 = (1.1) + (2.1) + (3.1) + (4.1) + (5A.1) + (5B.1) \ ;$$

$$\text{i.e. } P_1 = (1.1) + (2.1) + (3.1) + (4.1) + (5BA.1) + (3D.1) + (5AB.1) + (2D.1) \ .$$

Substituting into the above equation the formulae for the probabilities (r.1) written in an abbreviated form,

$$P_1 = P_1 F_{11} + P_2 F_{21} + P_3 F_{31} + P_4 F_{41} + P_{5A} (F_{51} + F_{3D1}) + P_{5B} (F_{51} + F_{2D1}) , \qquad (47)$$

where $F_{11} = 1 + (E_A E_B - E_A - E_B)(1 - e^{-NT})$,

$$F_{21} = E_A e^{-NT} + \frac{1}{NT} (1 - E_A)(1 - e^{-NT}) ,$$

$$F_{31} = E_B e^{-NT} + \frac{1}{NT} (1 - E_B)(1 - e^{-NT}) ,$$

$$F_{41} = \frac{1}{NT} (1 - e^{-NT}) ,$$

$$F_{51} = \frac{1}{N^2T^2} (1 - e^{-NT}) - \frac{1}{NT} e^{-NT} ,$$

$$F_{3D1} = \frac{1}{2} E_B e^{-NT} + (1 - E_B) \left[\frac{1}{NT} - \frac{1}{N^2T^2} (1 - e^{-NT}) \right] , \text{ and}$$

$$F_{2D1} = \frac{1}{2} E_A e^{-NT} + (1 - E_A) \left[\frac{1}{NT} - \frac{1}{N^2T^2} (1 - e^{-NT}) \right] .$$

Adding the formulae in column 2 of **table 2**, the equation $P_2 = \sum_r (r.2)$ becomes

$$P_2 = P_1 F_{12} + P_2 F_{22} , \qquad (48)$$

where $F_{12} = E_B (1 - E_A)(1 - e^{-NT})$, and

$$F_{22} = (1 - E_A) \left[1 - \frac{1}{NT} (1 - e^{-NT}) \right] .$$

Similarly,

$$P_3 = P_1 F_{13} + P_3 F_{33} , \qquad (49)$$

where $F_{13} = E_A (1 - E_B)(1 - e^{-NT})$, and

$$F_{33} = (1 - E_B) \left[1 - \frac{1}{NT} (1 - e^{-NT}) \right] ;$$

$$P_4 = P_1 F_{14} + P_4 F_{44} , \qquad (50)$$

where $F_{14} = E_A E_B (1 - e^{-NT})$, and

$$F_{44} = 1 - \frac{1}{NT} (1 - e^{-NT}) ;$$

$$P_{5A} = P_2 (F_{2BA} + F_{23D}) + P_{5A} (F_{BABA} + F_{BA3D} + F_{3D3D}) + P_{5B} (F_{2DBA} + F_{2D3D}), \qquad (51)$$

where $F_{2BA} = E_A \left[1 - \frac{1}{NT} (1 - e^{-NT}) \right]$,

$$F_{23D} = E_A \left[\frac{1}{NT} (1 - e^{-NT}) - e^{-NT} \right] ,$$

$$F_{BABA} = \frac{1}{2} - \frac{1}{NT} + \frac{1}{N^2T^2} (1 - e^{-NT}) ,$$

$$F_{BA3D} = \frac{1}{NT} (1 + e^{-NT}) - \frac{2}{N^2T^2} (1 - e^{-NT}) ,$$

$$F_{3D3D} = (1 - E_B) \left[\frac{1}{2} - \frac{1}{NT} + \frac{1}{N^2T^2} (1 - e^{-NT}) \right] ,$$

$$F_{2DBA} = E_A\left[\frac{1}{2} - \frac{1}{NT} + \frac{1}{N^2T^2}(1-e^{-NT})\right] \text{, and}$$

$$F_{2D3D} = E_A\left[\frac{1}{NT} - \frac{1}{2}e^{-NT} - \frac{1}{N^2T^2}(1-e^{-NT})\right] ;$$

$$P_{5B} = P_3(F_{3AB}+F_{32D}) + P_{5A}(F_{3DAB}+F_{3D2D}) + P_{5B}(F_{ABAB}+F_{AB2D}+F_{2D2D}) , \tag{52}$$

where $F_{3AB} = E_B\left[1-\frac{1}{NT}(1-e^{-NT})\right]$,

$$F_{32D} = E_B\left[\frac{1}{NT}(1-e^{-NT}) - e^{-NT}\right] ,$$

$$F_{3DAB} = E_B\left[\frac{1}{2} - \frac{1}{NT} + \frac{1}{N^2T^2}(1-e^{-NT})\right] ,$$

$$F_{3D2D} = E_B\left[\frac{1}{NT} - \frac{1}{2}e^{-NT} - \frac{1}{N^2T^2}(1-e^{-NT})\right] ,$$

$$F_{ABAB} = \frac{1}{2} - \frac{1}{NT} + \frac{1}{N^2T^2}(1-e^{-NT}) ,$$

$$F_{AB2D} = \frac{1}{NT}(1+e^{-NT}) - \frac{2}{N^2T^2}(1-e^{-NT}) \text{, and}$$

$$F_{2D2D} = (1-E_A)\left[\frac{1}{2} - \frac{1}{NT} + \frac{1}{N^2T^2}(1-e^{-NT})\right] .$$

Of the six equations, only five are independent. These are solved, together with **equation 3**, as shown below, to obtain P_1, P_2, P_3, P_4, P_{5A} and P_{5B}. **Equation 49** is not used in the following solution.

Rearranging **equation 48**,

$$P_2(1-F_{22}) = P_1 F_{12} ,$$

which can be written in the abbreviated form

$$P_2 = P_1 V_1 , \tag{53}$$

where

$$V_1 = F_{12} + (1-F_{22}) . \tag{48A}$$

Rearranging **equation 50**,

$$P_4(1-F_{44}) = P_1 F_{14} ,$$

i.e. $P_4 \frac{1}{NT}(1-e^{-NT}) = P_1 E_A E_B (1-e^{-NT})$,

$$P_4 = P_1 V_2 , \tag{54}$$

where

$$V_2 = E_A E_B NT . \tag{55}$$

Writing **equation 51** in abbreviated form and substituting from **equation 53**,

$$P_{5A} = P_1 V_1 V_3 + P_{5A} V_4 + P_{5B} V_5 , \tag{56}$$

$$\left.\begin{aligned} \text{where } V_3 &= F_{2BA} + F_{23D}, \\ V_4 &= F_{BABA} + F_{BA3D} + F_{3D3D}, \text{ and} \\ V_5 &= F_{2DBA} + F_{2D3D} . \end{aligned}\right\} \tag{51A}$$

Rearranging **equation 56** and abbreviating further,

$$P_1 V_1 V_3 = P_{5A}(1 - V_4) - P_{5B} V_5 ,$$

$$P_1 = P_{5A} V_6 - P_{5B} V_7 , \tag{57}$$

$$\text{where } V_6 = \frac{1 - V_4}{V_1 V_3} ,$$

$$\text{and } V_7 = \frac{V_5}{V_1 V_3} .$$

Writing **equation 52** in abbreviated form,

$$P_{5B} = P_3 V_8 + P_{5A} V_9 + P_{5B} V_{10}$$

$$\left.\begin{aligned} \text{where } V_8 &= F_{3AB} + F_{32D} , \\ V_9 &= F_{3DAB} + F_{3D2D} , \text{ and} \\ V_{10} &= F_{ABAB} + F_{AB2D} + F_{2D2D} \end{aligned}\right\} . \tag{52A}$$

Rearranging,

$$P_3 V_8 = P_{5B}(1 - V_{10}) - P_{5A} V_9 ,$$

$$P_3 = P_{5B} V_{11} - P_{5A} V_{12} , \tag{58}$$

$$\text{where } V_{11} = \frac{1 - V_{10}}{V_8} ,$$

$$\text{and } V_{12} = \frac{V_9}{V_8} .$$

In **equation 3**, P_2, P_3 and P_4 are replaced by the expressions given in **equations 53, 56 and 54**, respectively;

$$P_1 + P_1 V_1 + P_{5B} V_{11} - P_{5A} V_{12} + P_1 V_2 + P_{5A} + P_{5B} = 1 .$$

Collecting the terms together, and replacing P_1 by the expression given in **equation 57**,

$$(P_{5A} V_6 - P_{5B} V_7)(1 + V_1 + V_2) + P_{5A}(1 - V_{12}) + P_{5B}(1 + V_{11}) = 1.$$

Rearranging,

$$P_{5A}\Big[V_6(1 + V_1 + V_2) + 1 - V_{12} \Big] + P_{5B}\Big[1 + V_{11} - V_7(1 + V_1 + V_2) \Big] = 1.$$

Rearranging and writing in an abbreviated form,

$$P_{5B} = \frac{1 - P_{5A} V_{13}}{V_{14}} , \tag{59}$$

$$\text{where } V_{13} = V_6(1 + V_1 + V_2) + 1 - V_{12} , \text{ and}$$

$$V_{14} = 1 + V_{11} - V_7(1 + V_1 + V_2) .$$

In **equation 47**, P_2, P_3, and P_4 are replaced by the expressions given in **equations 53, 58 and 54**, respectively.

$$P_1 = P_1 F_{11} + P_1 V_1 F_{21} + (P_{5B} V_{11} - P_{5A} V_{12}) F_{31}$$

$$+ P_1 V_2 F_{41} + P_{5A}(F_{51} + F_{3D1}) + P_{5B}(F_{51} + F_{2D1}) .$$

Replacing P_1 by the expression given in **equation 57** and rearranging,

$$(P_{5A} V_6 - P_{5B} V_7)(F_{11} + V_1 F_{21} + V_2 F_{41} - 1) +$$
$$P_{5A}(F_{51} + F_{3D1} - V_{12} F_{31}) + P_{5B}(F_{51} + F_{2D1} + V_{11} F_{31}) = 0 \ .$$

This can be written in an abbreviated form :

$$(P_{5A} V_6 - P_{5B} V_7) V_{15} + P_{5A} V_{16} + P_{5B} V_{17} = 0 \ , \qquad (60)$$

where $V_{15} = F_{11} + V_1 F_{21} + V_2 F_{41} - 1$,

$V_{16} = F_{51} + F_{3D1} - V_{12} F_{31}$, and

$V_{17} = F_{51} + F_{2D1} + V_{11} F_{31}$.

Rearranging **equation 60**,

$$P_{5B}\left[V_7 V_{15} - V_{17} \right] = P_{5A}\left[V_6 V_{15} + V_{16} \right] \ .$$

This can be rearranged and abbreviated further:

$$P_{5B} = P_{5A} V_{18} \ , \qquad (61)$$

where $V_{18} = \left[V_6 V_{15} + V_{16} \right] \div \left[V_7 V_{15} - V_{17} \right]$.

Combining **equations 59** and **61** to eliminate P_{5B},

$$1 - P_{5A} V_{13} = P_{5A} V_{18} V_{14} \ .$$

Rearranging,

$$P_{5A} = \frac{1}{V_{18} V_{14} + V_{13}} \ . \qquad (62)$$

Thus the formula for P_{5A} is given by **equation 62**. The formula for P_{5B} is obtained by substituting **equation 62** into **equation 61**. The formulae for P_1 and P_3 are obtained by substituting the formulae for P_{5A} and P_{5B} into **equations 57** and **58**, respectively. The formulae for P_2 and P_4 are obtained by substituting the formula for P_1 into **equations 53** and **54**, respectively.

8. THE EQUATION FOR THE OBSERVED COINCIDENCE RATE

8.1 True Coincidences

For a true coincidence to be recorded, both detectors must be live at the time of the disintegration. The number of disintegrations per second which occur when both detectors are live is NP_1. The number of true coincidences per second is $NP_1 E_A E_B$.

8.2 Accidental Coincidences

Consider disintegrations which occur when the detectors are in state 2, and which are detected by A. The number of such disintegrations per second is $N P_2 E_A$. Suppose such a disintegration occurs at instant I_2 in **figure 11**. Because I_2 is within the interval T_R, which denotes the resolving time of the system, the disintegration is recorded as a coincidence.

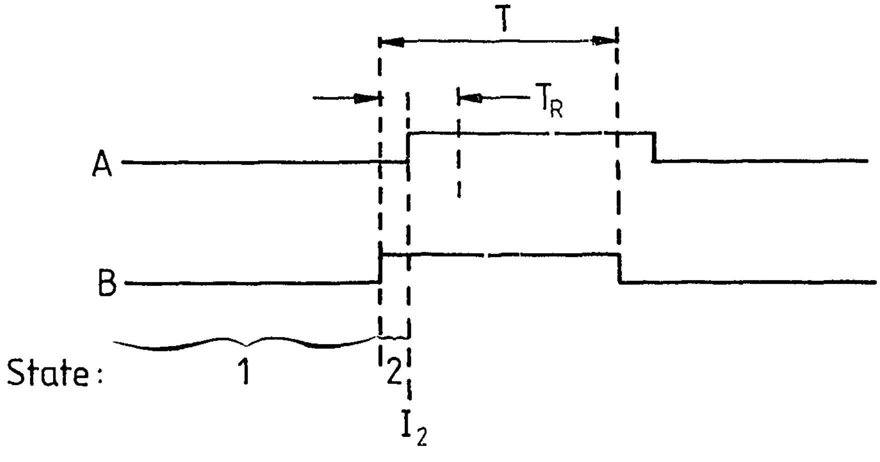

Figure 11 Accidental coincidence, first type

It is assumed that all degrees of overlapping of dead-times are equally probable. Therefore the number of such accidental coincidences per second will be $N P_2 E_A T_R \div T$. If the reverse case is considered, where A goes dead before B, the rate of accidental coincidences is $N P_3 E_B T_R \div T$.

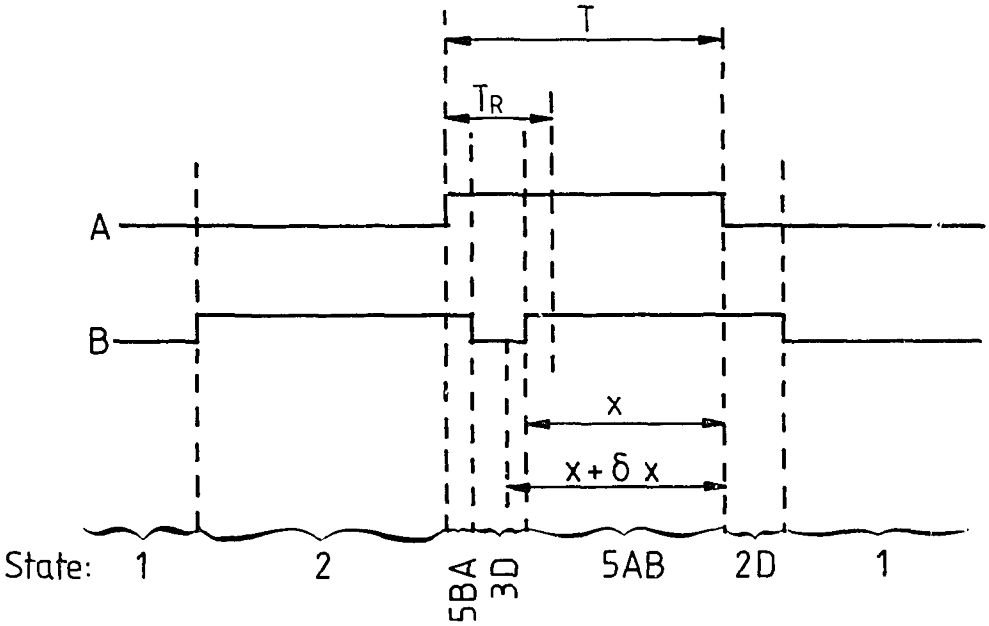

Figure 12 Accidental coincidence, second type

In **figure 12**, consider a disintegration which occurs when the system is in sub-state 3D. Suppose that the instant of the disintegration is between $(x + \delta x)$ before A becomes live again and x before A becomes live again. The probability that such a disintegration is detected by B is

$$P_{5A} \times \frac{T-x}{T} \times \frac{\delta x}{T} \times E_B \ .$$

Because the disintegration occurs in the interval T_R, it will be recorded as a coincidence. The number of such accidental coincidences per second is

$$\frac{N P_{5A} E_B}{T^2} \int_{T-T_R}^{T} (T-x)\, dx$$

$$= \frac{N P_{5A} E_B}{T^2} \left[Tx - \frac{x^2}{2} \right]_{T-T_R}^{T} ,$$

$$= \frac{N P_{5A} E_B}{T^2} \left[T^2 - T^2 + T T_R - \frac{T^2}{2} + \frac{T^2 - 2T T_R + T_R^2}{2} \right] ,$$

$$= \frac{N P_{5A} E_B T_R^2}{2T^2} \ .$$

If the case is considered in which A and B are interchanged, then by a similar derivation the accidental coincidence rate is

$$\frac{N P_{5B} E_A T_R^2}{2 T^2} \ .$$

8.3 The Observed Coincidence Rate

The observed coincidence rate with the background subtracted is denoted by C', and is the sum of the true coincidence rate and the four accidental coincidence rates whose formulae have been derived above. Thus,

$$C' = N \left[P_1 E_A E_B + (P_2 E_A + P_3 E_B) \frac{T_R}{T} + (P_{5A} E_B + P_{5B} E_A) \frac{T_R^2}{2T^2} \right] . \qquad (63)$$

9. FORMULAE FOR THE DETECTOR EFFICIENCIES

The efficiency, E_A, of detector A is defined as the probability that a disintegration which occurs while A is live will be detected. The observed count rate in detector A, including background, is denoted by A″. Detector A is dead for the fraction A″T of each second. Thus the live time per second is 1–A″T, and the number of disintegrations taking place during this live-time is N (1 – A″T). Of these disintegrations, the number actually detected will be equal to A′, where A′ is defined as the observed count rate in A with the background rate subtracted. Thus,

$$E_A = \frac{A'}{N(1 - A''T)} \quad . \tag{64}$$

Similarly, if the observed count rate in detector B, including background, is denoted by B″, and the observed count rate with the background rate subtracted is denoted by B′,

$$E_B = \frac{B'}{N\,(1 - B''T)} \quad . \tag{65}$$

10. THE COMPUTATION OF THE DISINTEGRATION RATE

By substituting into **equation 63** the formulae for P_1, P_2, P_3, P_{5A}, P_{5B}, E_A and E_B obtained from **equations 53, 57, 58, 61, 62, 64** and **65**, the observed coincidence rate, C′, is given as a function of A′, A″, B′, B″, T, T_R and N. The disintegration rate can now be computed from experimental data by an iterative method.

As a check on the solution of the simultaneous equations which gives the formulae for P_1, P_2, P_3, P_4, P_{5A} and P_{5B}, it can be shown by computation that these latter formulae satisfy

(a) the six equations represented by **equation 2**, namely

$P_s = \sum_r (r.s)$, and

(b) **equation 3**, namely

$P_1 + P_2 + P_3 + P_4 + P_{5A} + P_{5B} = 1.$

11. EXPERIMENTAL VERIFICATION

The method of computing N, which is described in **section 10**, was tested experimentally in the following manner. Four ^{60}Co sources were prepared with strengths ranging from 20 to 70 kBq. The solution from which the depositions were made was diluted once, and three sources in the range 1 to 2 kBq were prepared from the diluted solution. The depositions were made from pycnometers which were weighed before and after each deposition on a Mettler M3 electronic microbalance. The source mounts were VYNS foils (20 μg cm^{-2}) coated underneath with a layer of gold-palladium (15 μg cm^{-2}). Each source contained about 10 μg of Catanac spreading agent. After the deposited ^{60}Co and Catanac solutions had evaporated to dryness on the foil, the radioactive residue was covered with a similar foil, the gold-palladium layer being uppermost.

The sources were counted in 10% methane in argon, with a resolving time of 1.1 μ s. The dead-times were 9.1 and 5.0 μ s, respectively. The radioactivity of the sources was calculated by using Bryant's equation, and the method described in **section 10**. The radioactivity concentration of the original ^{60}Co solution was calculated from the source radioactivities, then plotted against the source radioactivities calculated by the method outlined in **section 10**. The results for the seven sources and the two dead-times are shown in **figure 13**. For the sources in the 1 to 2 kBq range, the two correction methods give results which differ only minutely. The radioactivity concentrations calculated from these sources are assumed to be adequately corrected for dead-time losses and accidental coincidences, and the mean of these results is taken as the radioactivity concentration of the original solution .

As can be seen from **figure 13**, the radioactivity concentrations calculated from the results on the stronger sources are too high. The concentrations calculated by using Bryant's equation are significantly higher than those obtained by using the method described in **section 10**.

If the radioactivity concentrations which were calculated from the results of the strongest source are considered, it can be seen that the concentration which was calculated using the method of **section 10** is 0.5 per cent too high. The concentration which was calculated using Bryant's equation is 1.3 per cent too high, for a dead-time of 5.0 μ s, and 2.1 per cent too high, for a dead-time of 9.1 μ s.

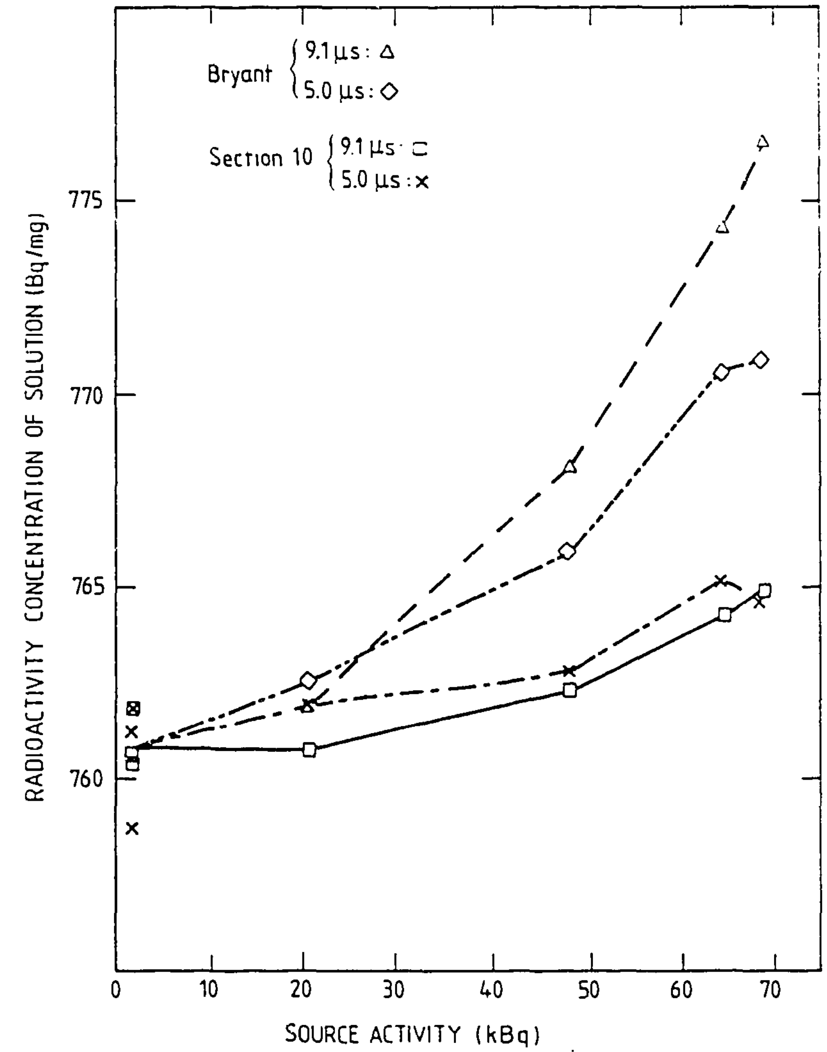

Figure 13 The radioactivity concentration of a ^{60}Co solution, calculated by Bryant's formula and by the method of section 10, and shown as a function of counting-source activity and dead-time.

12. CONCLUSIONS

The equation derived here is more effective than Bryant's [1963] equation, in making corrections for dead-time losses and accidental coincidences when determining the disintegration rate of a radioactive source by coincidence counting. Further improvement will require consideration of other factors, such as the variability in the delay of pulses in the detectors.

13. ACKNOWLEDGEMENT

The author is indebted to Mr A.A. Williams for his assistance with the computing.

14. REFERENCES

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APPENDIX A

PROBABILITY, e^{-Nt}, OF NO DISINTEGRATIONS IN TIME t

It is shown that if the radioactivity of a source is N disintegrations per second, the probability of no disintegrations occurring during an interval t is e^{-Nt}.

Consider a die with L faces. The die could be shaped like a pencil whose cross-section is a polygon with L sides. Suppose the L faces are denoted by $F_1, F_2, \cdots F_l \cdots F_L$. It is assumed that when the die is rolled G times, the number of times the face F_l appears uppermost will approach G/L as G becomes very large. Thus, when the die is rolled, the probability of F_l appearing uppermost is 1/L. Consider the probability, denoted by P, that F_l will not appear uppermost. Any one of the (L-1) other faces may appear, and it is assumed that

$$P = \frac{L-1}{L} .$$

Consider a number of such dice, the number being denoted by Y_o. It will be shown that when the Y_o dice are rolled, the probability that no face F_1 appears uppermost is equal to P^{Y_o}. The total number of possible arrangements when Y_o dice are rolled is L^{Y_o}. It will first be shown that the number of possible arrangements in which F_1 does not appear uppermost is $(L-1)^{Y_o}$.

If the first die to be considered has F_2 uppermost, the second can have (L-1) faces uppermost, viz $F_2 \cdots F_l \cdots F_L$. If the first die has F_3 uppermost, the second can have (L-1) faces uppermost, and so on up to F_L. Thus for two dice there are $(L-1)^2$ possible arrangements, and for Y_o dice there are $(L-1)^{Y_o}$ possible arrangements. Of the total number of possible arrangements of Y_o dice, the fraction with no F_1 uppermost is

$$\frac{(L-1)^{Y_o}}{L^{Y_o}} = P^{Y_o} .$$

If Y_o dice are rolled G times, it is assumed that as G becomes very large, the number of times on which F_1 does not appear will approach GP^{Y_o}. Thus P^{Y_o} is the probability that when Y_o dice are rolled F_1 does not appear uppermost.

Consider a radioactive sample which contains Y_o radioactive atoms at time zero, and Y radioactive atoms at time t. Let N_o and N be the disintegration rates in the sample at times zero and t, respectively. Treating Y as a continuous function of t,

$$N = -\frac{dY}{dt} .$$

The experimentally determined decay equation is

$$N = N_o\, e^{-\lambda t} , \qquad \text{(A1)}$$

where λ is a constant. Combining the last two equations,

$$\frac{dY}{dt} = -N_o\, e^{-\lambda t} .$$

Integrating with respect to t,

$$Y = \frac{N_o}{\lambda}\, e^{-\lambda t} + C , \qquad \text{(A2)}$$

where C is a constant.

$$\underset{t\to\infty}{\text{Lt}}\ Y = \underset{t\to\infty}{\text{Lt}}\ \frac{N_o}{\lambda}\ e^{-\lambda t} + C . \qquad \text{(A3)}$$

From **equation A1**, $\underset{t\to\infty}{\text{Lt}}\ N = 0$. If N is zero, there are no radioactive atoms present. Therefore,

$$\underset{t\to\infty}{\text{Lt}}\ Y = 0 . \qquad \text{(A4)}$$

Furthermore,

$$\underset{t\to\infty}{\mathrm{Lt}} \frac{N_0}{\lambda} e^{-\lambda t} = 0 \ . \qquad \text{(A5)}$$

Substituting **equations A4** and **A5** into **A3**, C = 0, and **equation A2** becomes

$$Y = \frac{N_0}{\lambda} e^{-\lambda t} \ . \qquad \text{(A6)}$$

Putting t = 0,

$$Y_0 = \frac{N_0}{\lambda} \ . \qquad \text{(A7)}$$

Combining **equations A6** and **A7**,

$$\frac{Y}{Y_0} = e^{-\lambda t} \ . \qquad \text{(A8)}$$

Y/Y_0 is the fraction of radioactive atoms which remain unchanged. Because of the random nature of radioactive decay, **equation A1** gives the average value of N. Thus $e^{-\lambda t}$ is the average value of N/N_0 and of Y/Y_0 for the period t. The time of disintegration of a particular atom is unpredictable. Let Q be the probability of an atom remaining unchanged during the period t. The probability Q is equal to the average fraction of atoms which remain unchanged. Thus, from **equation A8**,

$$Q = e^{-\lambda t} \ . \qquad \text{(A9)}$$

The probability that in the period t none of the Y_0 atoms disintegrate will now be considered. It is assumed that the probability of an atom not disintegrating during the period t is analogous to the probability that when a die is rolled a particular face, *e.g.* F_1, does not appear uppermost. Just as P^{Y_0} is the probability that F_1 does not appear uppermost when Y_0 dice are rolled, so also is it assumed the Q^{Y_0} is the probability that the Y_0 atoms all remain unchanged during the period t. Substituting from **equations A9** and **A7**,

$$Q^{Y_0} = \left[e^{-\lambda t}\right]^{\frac{N_0}{\lambda}} = e^{-N_0 t} \ .$$

If the disintegration rate at the start of the period t is denoted by N rather than N_0, the probability of no disintegrations occurring during t is given by the formula:

$$e^{-Nt} \ .$$

APPENDIX B
DERIVATION OF BRYANT'S EQUATION

Bryant's equation is now derived by abridging the method for the computation of N which has been given above. The probabilities P_{5A} and P_{5B} are assumed to be equal ; they are now denoted by P_5. The symbols 2D and 3D are eliminated as follows:

(a) The probabilities (2D.s) and (3D.s) are not used.
(b) The probabilities (r.2D) are called (r.2), and the probabilities (r.3D) are called (r.3).

The states 5AB and 5BA are now regarded as sub-states of state 5. Thus, letting $P_{5A} = P_{5B} = P_5$,

(5 AB.1) + (5BA.1) = (5.1) ,
(2.5BA) becomes (2.5) ,
(3.5AB) becomes (3.5) ,
(5AB.5AB) + (5BA.5BA) = (5.5) , and
$P_1 + P_2 + P_3 + P_4 + P_5 = 1$. (B1)

The effect of these changes is to reduce **table 2** to the array shown in **table B1**, in which the formulae are identical with those in **table 1** of Bryant [1963].

TABLE B1

These are the (r.s) probabilities from **table 2** which are necessary for the derivation of Bryant's equation. They are grouped together into 25 probabilities which are identical with those in table 1 of Bryant [1963] after P_5 has been substituted for P_{5A} and P_{5B}.

First state \ Second state	1	2	3	4	5
1	(1.1)	(1.2)	(1.3)	(1.4)	0
2	(2.1)	2.2)	(2.3D)	0	(2.5BA)
3	(3.1)	(3.2D)	(3.3)	0	(3.5AB)
4	(4.1)	0	0	(4.4)	0
5	(5BA.1)* +(5AB.1)*	(5AB.2D)*	(5BA.3D)*	0	(5BA.5BA)* +(5AB.5AB)*

* In the formulae for these probabilities, P_5 replaces P_{5A} and P_{5B}.

Bryant's equation can be derived as follows. Referring to **table 1** of Bryant, the sum of the formulae in columns 1 and 2 is equal to $P_1 + P_2$. Thus

$$P_1 + P_2 = P_1 - P_1 E_A (1 - e^{-NT}) + P_2 E_A e^{-NT} + P_2 - P_2 E_A + P_3 \frac{1}{NT} (1 - e^{-NT}) + P_4 \frac{1}{NT} (1 - e^{-NT}) + P_5 \frac{1}{NT} (1 - e^{-NT}) .$$

Subtracting $(P_1 + P_2)$ from both sides and rearranging,

$$P_1 E_A (1-e^{-NT}) + P_2 E_A (1 - e^{-NT}) = (P_3 + P_4 + P_5) \frac{1}{NT} (1 - e^{-NT}) .$$

Dividing by $(1 - e^{-NT})$ and substituting from **equation B1**,

$$(P_1 + P_2)\,E_A = (1 - P_1 - P_2)\,\frac{1}{NT}\ ,$$

$$(P_1 + P_2)\,E_A\,NT + P_1 + P_2 = 1\ ,$$

$$P_1 + P_2 = \frac{1}{1+E_A\,NT}\ \text{, and}$$

$$P_2 = \frac{1}{1 + E_A NT} - P_1\ . \qquad \text{(B2)}$$

Similarly,

$$P_1+P_3 = P_1 - P_1\,E_B\,(1 - e^{-NT}) + P_2\,\frac{1}{NT}\,(1 - e^{-NT})$$
$$+ P_3\,E_B e^{-NT} + P_3 - P_3\,E_B$$
$$+ P_4\,\frac{1}{NT}\,(1 - e^{-NT}) + P_5\,\frac{1}{NT}\,(1 - e^{-NT})\ ,$$

$$P_1\,E_B\,(1 - e^{-NT}) + P_3\,E_B\,(1 - e^{-NT}) = (P_2 + P_4 + P_5)\,\frac{1}{NT}\,(1 - e^{-NT})\ ,$$

$$(P_1 + P_3)E_B = (1 - P_1 - P_3)\,\frac{1}{NT}\ ,$$

$$(P_1 + P_3)\,E_B\,NT + P_1 + P_3 = 1\ ,$$

$$P_1 + P_3 = \frac{1}{1+E_B\,NT}\ \text{, and}$$

$$P_3 = \frac{1}{1+E_B\,NT} - P_1\ . \qquad \text{(B3)}$$

Consider the fourth column of **table 1** of Bryant. The sum of the formulae in this column is equal to P_4. Thus, combining **equations 54** and **55** which appear in **section 7**,

$$P_4 = P_1\,E_A\,E_B\,NT\ . \qquad \text{(B4)}$$

Rearranging **equation B1** and substituting for P_2, P_3 and P_4 from **equations B2** to **B4**,

$$P_5 = 1 - P_1 - \left[\frac{1}{1 + E_A\,NT} - P_1\right] - \left[\frac{1}{1 + E_B\,NT} - P_1\right] - P_1\,E_A\,E_B\,NT\ ;$$

$$P_5 = 1 - \frac{1}{1 + E_A\,NT} - \frac{1}{1 + E_B\,NT} + P_1 - P_1\,E_A\,E_B\,NT\ .$$

Collecting together the first three terms on the right hand side of the above equation,

$$P_5 = \frac{1 + E_A\,NT + E_B\,NT + E_A\,E_B\,N^2T^2 - 1 - E_A\,NT - 1 - E_B\,NT}{(1 + E_A\,NT)(1 + E_B\,NT)}$$
$$+ P_1 - P_1\,E_A\,E_B\,NT\ ;$$

i.e. $$P_5 = \frac{E_A\,E_B\,N^2\,T^2 - 1}{(1+E_A\,NT)(1+E_B\,NT)} + P_1 - P_1\,E_A\,E_B\,NT\ . \qquad \text{(B5)}$$

Summing the formulae in the first column of the table,

$$P_1 = (1.1) + (2.1) + (3.1) + (4.1) + (5.1)\ .$$

Inserting the formulae of the first column in the above equation, and substituting for P_2, P_3, P_4 and P_5 from **equations B2 - B5**,

$$P_1 = P_1\left\{1 + (E_A\,E_B - E_A - E_B)(1 - e^{-NT})\right\} +$$

$$\left[\frac{1}{1+E_A NT} - P_1\right]\left\{E_A e^{-NT} + \frac{1}{NT}(1-E_A)(1-e^{-NT})\right\} +$$

$$\left[\frac{1}{1+E_B NT} - P_1\right]\left\{E_B e^{-NT} + \frac{1}{NT}(1-E_B)(1-e^{-NT})\right\} +$$

$$P_1 E_A E_B NT\left\{\frac{1}{NT}(1-e^{-NT})\right\} +$$

$$\left[\frac{E_A E_B N^2 T^2 - 1}{(1+E_A NT)(1+E_B NT)} + P_1 - P_1 E_A E_B NT\right]\left\{\frac{2}{N^2 T^2}(1-e^{-NT}) - \frac{2}{NT} e^{-NT}\right\} .$$

Multiplying, and collecting in the first part of the equation the terms containing P_1,

$$0 = P_1 \Big[E_A E_B - E_A - E_B - E_A E_B e^{-NT} + E_A e^{-NT} + E_B e^{-NT} -$$

$$E_A e^{-NT} - \frac{1}{NT} + \frac{E_A}{NT} + \frac{1}{NT} e^{-NT} - \frac{E_A}{NT} e^{-NT} -$$

$$E_B e^{-NT} - \frac{1}{NT} + \frac{E_B}{NT} + \frac{1}{NT} e^{-NT} - \frac{E_B}{NT} e^{-NT} +$$

$$E_A E_B - E_A E_B e^{-NT} + \frac{2}{N^2T^2} - \frac{2}{N^2T^2} e^{-NT} - \frac{2}{NT} e^{-NT} -$$

$$\frac{2E_A E_B}{NT} + \frac{2 E_A E_B}{NT} e^{-NT} + 2 E_A E_B e^{-NT} \Big] +$$

$$\Big[\frac{1}{(1+E_A NT)(1+E_B NT)} \times$$

$$(E_A e^{-NT} + \frac{1}{NT} - \frac{E_A}{NT} - \frac{1}{NT} e^{-NT} + \frac{E_A}{NT} e^{-NT} +$$

$$E_A E_B NT e^{-NT} + E_B - E_A E_B - E_B e^{-NT} + E_A E_B e^{-NT} +$$

$$E_B e^{-NT} + \frac{1}{NT} - \frac{E_B}{NT} - \frac{1}{NT} e^{-NT} + \frac{E_B}{NT} e^{-NT} +$$

$$E_A E_B NT e^{-NT} + E_A - E_A E_B - E_A e^{-NT} + E_A E_B e^{-NT} +$$

$$2 E_A E_B - 2 E_A E_B e^{-NT} - 2 E_A E_B NT e^{-NT} -$$

$$\frac{2}{N^2 T^2} + \frac{2}{N^2 T^2} e^{-NT} + \frac{2}{NT} e^{-NT}) \Big] .$$

Collecting the terms together, and transferring those containing P_1 to the left-hand side of the equation,

$$P_1 \Big[-2 E_A E_B + E_A + E_B + \frac{2}{NT} - \frac{E_A}{NT} + \frac{E_A}{NT} e^{-NT} - \frac{E_B}{NT} +$$

$$\frac{E_B}{NT} e^{-NT} - \frac{2}{N^2T^2} + \frac{2}{N^2T^2} e^{-NT} + \frac{2 E_A E_B}{NT} -$$

$$\frac{2 E_A E_B}{NT} e^{-NT} \Big] =$$

$$\frac{1}{(1+E_A NT)(1+E_B NT)} \Big[\frac{2}{NT} - \frac{E_A}{NT} + \frac{E_A}{NT} e^{-NT} + E_B -$$

$$\frac{E_B}{NT} + \frac{E_B}{NT} e^{-NT} + E_A - \frac{2}{N^2T^2} + \frac{2}{N^2T^2} e^{-NT} \Big] .$$

Factorising,

$$P_1\left[\frac{1}{NT}\ (-2\,E_A\,E_B\,NT + E_A\,NT + E_B\,NT + 2) - \frac{1}{N^2T^2}\ (E_A\,NT + E_B\,NT + 2 - 2\,E_A\,E_B\,NT) + \frac{e^{-NT}}{N^2T^2}\ (E_A\,NT + E_B\,NT + 2 - 2\,E_A\,E_B\,NT)\right] =$$

$$\frac{1}{(1 + E_A\,NT)(1 + E_B\,NT)}\left[\frac{1}{NT}\ (2 + E_A\,NT + E_B\,NT) - \frac{1}{N^2T^2}\ (2 + E_A\,NT + E_B\,NT) + \frac{e^{-NT}}{N^2T^2}\ (E_A\,NT + E_B\,NT + 2)\right] .$$

Rearranging

$$P_1\left[\frac{1}{NT} - \frac{1}{N^2T^2} + \frac{e^{-NT}}{N^2T^2}\right]\left[2 + (E_A + E_B - 2\,E_A\,E_B)\,NT\right] =$$

$$\frac{1}{(1 + E_A\,NT)(1 + E_B\,NT)}\left[\frac{1}{NT} - \frac{1}{N^2T^2} + \frac{e^{-NT}}{N^2T^2}\right]\left[2 + (E_A + E_B)NT\right] .$$

Dividing by the common factor and rearranging,

$$P_1 = \frac{2 + (E_A + E_B)\,NT}{(1 + E_A\,NT)(1 + E_B\,NT)\left[2 + (E_A + E_B - 2\,E_A\,E_B)\,NT\right]} . \tag{B6}$$

The equations for the detector efficiencies which are given in **section 9**, namely **equations 64** and **65**, are modified as follows.

$$E_A = \frac{A'}{N\,(1 - A'T)} . \tag{B7}$$

$$E_B = \frac{B'}{N(1 - B'T)} . \tag{B8}$$

Using equations **B7** and **B8**, four more expressions are obtained which, when substituted into **equation B6**, give P_1 as a function of A′, B′, N and T:

$$E_A + E_B = \frac{1}{N}\left[\frac{A' + B' - 2\,A'\,B'\,T}{(1 - A'T)(1 - B'T)}\right] , \tag{B9}$$

$$E_A\,E_B = \frac{A'B'}{N^2(1-A'T)(1-B'T)} , \tag{B10}$$

$$\frac{1}{1 + E_A NT} = \frac{1}{1 + \dfrac{A'T}{1-A'T}} ,$$

$$\textit{i.e.}\ \frac{1}{1 + E_A NT} = 1 - A'T . \tag{B11}$$

Similarly,

$$\frac{1}{1 + E_B\,NT} = 1 - B'T . \tag{B12}$$

Substituting from **equation B9** into the numerator of the right hand side of **equation B6**,

$$2 + (E_A + E_B)\,NT =$$

$$\frac{2 - 2A'T - 2B'T + 2A'B'T^2 + A'T + B'T - 2A'B'T^2}{(1 - A'T)(1 - B'T)} ,$$

$$\textit{i.e.}\ 2 + (E_A + E_B)\,NT = \frac{2 - A'T - B'T}{(1 - A'T)(1 - B'T)} . \tag{B13}$$

Substituting **equations B11, B12, B13** and **B10** into **B6**,

$$P_1 = \frac{2 - A'T - B'T}{\dfrac{2 - A'T - B'T}{(1 - A'T)(1 - B'T)} - \dfrac{2A'B'T}{N(1 - A'T)(1 - B'T)}} ;$$

$$\textit{i.e. } P_1 = \frac{N(1 - A'T)(1 - B'T)(2 - A'T - B'T)}{N(2 - A'T - B'T) - 2A'B'T} . \qquad \text{(B14)}$$

An equation for the observed coincidence rate is obtained by truncating **equation 63** from **section 8**:

$$C' = N\left[P_1 E_A E_B + (P_2 E_A + P_3 E_B) \frac{T_R}{T} \right] . \qquad \text{(B15)}$$

Substituting **equation B11** into **B2**, and **B12** into **B3**,

$$P_2 = 1 - A'T - P_1 , \qquad \text{(B16)}$$

$$P_3 = 1 - B'T - P_1 . \qquad \text{(B17)}$$

In **equation B15**, E_A, E_B, P_2 and P_3 are replaced by the expressions given in **equations B7, B8, B16** and **B17**, respectively.

$$C' = N\left[P_1 \frac{A'B'}{N^2(1 - A'T)(1 - B'T)} \right] +$$

$$N\left[\frac{(1-A'T-P_1)A'}{N(1 - A'T)} + \frac{(1 - B'T - P_1) B'}{N(1 - B'T)} \right] \frac{T_R}{T} ,$$

$$= P_1 \left[\frac{A'B'}{N(1 - A'T)(1 - B'T)} - \left\{ \frac{A'}{1 - A'T} + \frac{B'}{1 - B'T} \right\} \frac{T_R}{T} \right] + (A' + B') \frac{T_R}{T} .$$

$$\textit{i.e. } C' = \frac{P_1}{(1 - A'T)(1-B'T)} \left[\frac{A'B'}{N} - (A' + B' - 2A'B'T) \frac{T_R}{T} \right] + (A' + B') \frac{T_R}{T}.$$

Replacing $\dfrac{P_1}{(1 - A'T)(1 - B'T)}$ by an expression obtained from **equation B14**,

$$C' = \frac{N(2 - A'T - B'T)}{N(2 - A'T - B'T) - 2A'B'T} \left[\frac{A'B'}{N} - (A' + B' - 2A'B'T) \frac{T_R}{T} \right] + (A' + B') \frac{T_R}{T} .$$

Rearranging,

$$\left[C' - (A' + B') \frac{T_R}{T} \right] \left[N(2 - A'T - B'T) - 2A'B'T \right] =$$

$$(2 - A'T - B'T)A'B' - N(2 - A'T - B'T)(A' + B' - 2A'B'T) \frac{T_R}{T} .$$

Collecting terms containing N on the left hand side,

$$N (2 - A'T - B'T) \left[C' - (A' + B') \frac{T_R}{T} + (A' + B' - 2A'B'T) \frac{T_R}{T} \right]$$

$$= A'B' \left\{ (2 - A'T - B'T) + 2\left[C'T - (A' + B') T_R \right] \right\} .$$

Dividing each side by $(2 - A'T - B'T)$,

$$N\left[C' - 2A'B'T_R \right] = A'B' \left\{ 1 + \frac{2C'T-2(A'+B')T_R}{2 - A'T - B'T} \right\} .$$

Dividing each side by $\left[C' - 2A'B'T_R \right]$,

$$N = \frac{A'B'}{C' - 2A'B'T_R} \left\{ 1 + \frac{2C'T-2(A'+B')\ T_R}{2-A'T-B'T} \right\} . \qquad \text{(B18)}$$

Equation B18 is Bryant's correction formula which appears as **equation 4** in his paper [Bryant, 1963].