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NON-INTEGRABILITY OF TIME-DEPENDENT SPHERICALLY SYMMETRIC YANG-MILLS EQUATIONS

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ԺԱՄԱՆԱԿԱՅԻՆ ԿԱԽՈՒՄ ՈՒՆԵՑՈՂ ՅԱՆԳ-ՄԻԼՍԻ ԳՆԴԱՁԵՎ ՀԱՄԱՉԱՓ ՀԱՎԱՍԱՐՈՒՄՆԵՐԻ ԱՆԻՆՏԵԳՐԵԼԻՈՒԹՅՈՒՆԸ

Ուսումնասիրված է ժամանակային կախում ունեցող Յանգ-Միլսի գնդաձև համաչափ հավասարումների ինտեգրելիությունը Ֆերմի-Պաստա-Ուլամի եղանակով: Ցույց է տրված, որ այդ համակարգի փուլային տարածությունում չկա ինտեգրելի կամ դրանց մոտ համակարգերին հատուկ պայմանականորեն պարբերական շարժում: Մասնավորապես, Վու-Յանգի հայտնի անփոփոխ լուծումը անկայուն է, այնպես որ նրա շրջակայքը փուլային տարածությունում ստոխաստիկության տիրույթ է:

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S.G. MATINYAN, E.B. PROKHORENKO, G.K. SAVVIDY

NON-INTEGRABILITY OF TIME-DEPENDENT SPHERICALLY SYMMETRIC YANG-MILLS EQUATIONS

The integrability of time-dependent spherically symmetric Yang-Mills equations is studied using the Fermi-Pasta-Ulam method. The phase space of this system is shown to have no quasi-periodic motion specific for integrable systems. In particular, the well-known Wu-Yang static solution is unstable, so its vicinity in phase space is the stochasticity region.

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НЕИНТЕГРИРУЕМОСТЬ СФЕРИЧЕСКИ СИММЕТРИЧНЫХ УРАВНЕНИЙ ЯНГА-МИЛЛСА, ЗАВИСЯЩИХ ОТ ВРЕМЕНИ

Методом Ферми-Паста-Улама изучен вопрос интегрируемости сферически симметричных уравнений Янга-Миллса, зависящих от времени. Показано, что в фазовом пространстве этой системы отсутствует условно периодическое движение, свойственное интегрируемым или близким к ним системам. В частности, известное статическое решение Ву-Янга неустойчиво, так что его окрестность в фазовом пространстве является областью стохастичности.

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1. Introduction. Formulation of the Problem.

The present work is a continuation of the numerical experiments started in Ref. [1] aimed at studying the important problem of integrability of Yang-Mills (Y.M.) classical equations. In [1] we considered the case when gauge fields depended on time only (homogeneous model). Such a system reduces to a finite-dimensional mechanical system ("Y.M. classical mechanics"), whose non-integrability was demonstrated by studying the instability of periodical solutions [1] and Poincare cross sections [2] as well as by other methods [3-6].

The proof of existence of non-integrable subsystem of Y.M. equations is a crucial serious argument in favor of non-integrability of these equations in the general case. There are some other arguments too, based on the so-called Penleve criterion and making this claim reasonable [7].

All this testifies that it has become extremely necessary to investigate the 3+1 -dimensional Y.M. classical field system from the viewpoint of its integrability.

Here we investigate Y.M. classical equations in case of spherical symmetry, i.e. when non-abelian vector potential A_μ^a depends on $r = |\vec{r}|$ and t. Effectively, this is a classical field theory in 1+1 -dimensional space-time, and the problem of its integrability is clearly defined in the sense. that the behavior of two-dimensional integrable classical systems of the type of equations of sin-Gordon, Korteveg-de Vriez, etc. is studied quite well.

In the problem under consideration one can simultaneously obtain an answer to the question about stability of the well-known spherically symmetric static solutions of Wu-Yang type [8] .

Unfortunately, one cannot answer analytically, with complete definiteness the direct question on integrability of equations of the form:

$$u_{tt} - u_{xx} = F(x, u, u_x, u_t, \dots) \qquad (1.1)$$

(u_t , u_{tt} , etc. are a first, second, etc. derivatives with respect to corresponding argument) the considered system reduces to. The reason is clear: information about the whole (infinite-dimensional) phase space of the system is necessary.

One should bear in mind that as shown in [9] , the necessary condition of integrability by the inverse scattering method of some class of partial differential equations is the belonging of the corresponding reduced ordinary differential equations to those of Penleve type. However this condition is not proved so far with sufficient generality [9,10] .

Another appropriate method to investigate integrability of equations of (1.1) type is the approach suggested in the well-known work of Fermi-Pasta-Ulam [11] . It consists in the replacement of continuous nonlinear string in (1.1) by its discrete analog with finite number of oscillating points - the chain of coupled anharmonic oscillators. Integration of such a chain with modern computers offers no difficulties. The real problem is to find out criteria which would allow us to conclude whether the given system is integrable or not.

First, one could follow the string shape. In their well-known work [12] Zabusky and Kruskal carried out a similar numerical experiment with Korteveg-de Vriez equation and discovered elastic scattering of solitons, i.e. the string oscillations after scattering did not turn into "shapeless ripples". Later on, it became clear that such a behavior is due to total integrability of Korteveg-de Vriez equation, i.e. availability of infinite number of integrals of motion [13] and possibility of transition to variables of "action - angle" type.

The progress achieved in investigation of classical and quantum dynamics of the 1+1 -dimensional integrable field theories is well known [14,15] .

More general and frequent is the situation when the system is non-integrable, and its phase space represents invariant tori intermittent with regions of ergodic (chaotic) motion. (This situation at the case of small perturbations is described by the well-known KAM-theorem [16].) In this case the following of the string shape is not informative. Therefore Fermi, Pasta and Ulam followed the energy distribution in string oscillations harmonics rather than its shape. Such "fourier-analysis" of the string gives a richer information than the above-quoted following of its shape, for it is quite possible that the string shape changes substantially, whereas from the viewpoint of the fourier-harmonic analysis there occurs a periodical energy transfer between several harmonics, this corresponding, in cornection with above-stated, to motion over KAM-tori. The authors of this outstanding work, while observing such a picture, naturally concluded that the system is "thermalized" anomalously slowly. Later on [17] , there were carried out numerical experiments according to the same scheme; however a greater number of harmonics were excited initially, owing to which the system motion took place this time in ergodic layer: this resulted in uniform energy distribution in harmonics. As a result this means that the Fermi-Pasta-Ulam system is non-integrable, and its phase space is described by the KAM-theorem. Exactly this approach we have chosen here to investigate the system of Y.M. equations possessing spherical symmetry.

It is shown that the phase space of this system is ergodic, while the system itself is non-integrable. The result obtained convinces us again how nontrivial is the dynamics of non-abelian gauge field.

In Section 2 of this paper the system of initial equations is formulated and analytically investigated; Section 3 presents the results of numerical analysis; in Section 4 the transition to continuous limit is considered, while Section 5 is devoted to concluding remarks.

2. Spherically Symmetric Y.M. Field.

Let us define the class of fields which will be considered below. Remind with this aim some definitions.

An arbitrary tensor field is invariant under definite group of coordinate transformations if its Lee derivative is zero. In case of gauge fields, it is sufficient to impose a weaker condition with a demand that the Lee derivative is compensated by gauge transformation.

A general form of spherically symmetric gauge field of the SU(2)-group in 3+1 -dimensional space-time is given by the expression [18] :

$$A_j^a = \frac{\varphi_1}{r}\left[\delta_{ja} - n_j n_a\right] + \frac{1+\varphi_2}{r}\varepsilon_{jak} n_k + A_1 n_j n_a \,,$$

$$A_0^a = A_0 n_a \;, \quad \left(n_i = \frac{x_i}{r}\right) \tag{2.1}$$

(upper indices refer to the SU(2)-group, $a = 1,2,3$; lower ones - , $j, k = 1,2,3$ refer to the Lorentz group), where the arbitrary functions $\varphi_{1,2}$ and $A_{0,1}$ depend on $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ and t .

The class of fields (2.1) is invariant under abelian gauge transformations with a matrix $U = \exp\{if(z,t)x_a T^a\}$, where f is an arbitrary function, and T^a is the SU(2)-group generator.

The action for fields (2.1) has the form:

$$S = -\frac{1}{16\pi g^2}\int d^4x\, F^a_{\mu\nu} F^a_{\mu\nu} =$$

$$= \frac{2}{g^2}\int_0^\infty dz \int_{-\infty}^{+\infty} dt \Big[\frac{1}{2}(\mathcal{D}_\mu \varphi_1)^2 + \frac{1}{2}(\mathcal{D}_\mu \varphi_2)^2 - \qquad (2.2)$$

$$- \frac{1}{8} z^2 F^2_{\mu\nu} - \frac{1}{4z^2}(1-\varphi_1^2-\varphi_2^2)^2 \Big] ,$$

where

$$\mathcal{D}_\mu \varphi_i \equiv \partial_\mu \varphi_i + \varepsilon_{i\ell} A_\mu \varphi_\ell$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu ,$$

$$\mu,\nu = 0,1 \;;\; i,\ell = 1,2 \;;\quad \partial_1 \equiv \frac{\partial}{\partial z} \;;\quad \partial_0 \equiv \frac{\partial}{\partial t} .$$

Effectively, (2.2) represents a two-dimensional field theory coinciding with the Higgs model in 1+1 -dimensional space-time. Corresponding to (2.2) equations of motion are:

$$\mathcal{D}_\mu \mathcal{D}^\mu \varphi_i - \frac{1}{z^2}(1-\varphi_1^2-\varphi_2^2)\varphi_i = 0$$

$$\frac{1}{2}\partial_\nu (z^2 F_{\mu\nu}) - \mathcal{D}_\mu \varphi_i \varepsilon_{ij}\varphi_j = 0. \qquad (2.3)$$

In what follows we shall study in detail a special case when $\varphi_1 = A_0 = A_1 = 0$ *). Then Eqs. (2.3) reduce to a single equation ($\varphi \equiv \varphi_2$):

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*) A_0 may always be taken zero owing to abelian gauge transformation mentioned above.

$$(\partial_t^2 - \partial_r^2)\varphi = -\frac{1}{r^2}\varphi(\varphi^2 - 1) \tag{2.4}$$

which has the form of nonlinear string (1.1). The static solution of (2.4) $\varphi = 0$ is the Wu-Yang monopole [8] $A_j^a = \frac{1}{r}\varepsilon_{jaK} n_K$, $\varphi = -1$ is the vacuum solution of $A_j^a = 0$, $\varphi = 1$ gives the field $A_j^a = \frac{2}{r}\varepsilon_{jaK} n_K$ which is gauge-equivalent to vacuum one. Eq.(2.4) can be considered as a Hamiltonian system in infinite-dimensional phase space with coordinates $\varphi(r,t)$ and canonically conjugated momenta $\pi_\varphi(r,t) = \partial\varphi/\partial t$; so our purpose is to describe the motions in this space.

First of all we shall describe the equilibrium states of the system ($\pi_\varphi = 0$) which correspond to static solutions of Eq.(2.4). In order to eliminate non-autonomy of Eq.(2.4), we shall pass in it to a new variable σ :

$$r = r_0 \exp(\sigma) \ , \qquad \sigma \in (-\infty, +\infty) \tag{2.5}$$

thus arriving at equation

$$\partial_\sigma^2 \varphi - \partial_\sigma \varphi + \varphi(1 - \varphi^2) = 0 \tag{2.6}$$

which is a particular case of the well-known Duffing equation $y'' + ay' + y + by^3 = 0$ *).

Separatrices of Eq.(2.6) (curves) and the field of its directions (arrows) are shown in Fig.1, whence one can obtain a qualitative information about

*) Two cases of this equation are known well in the literature [19] : $a, b > 0$ ("strong spring") and $a > 0$, $b < 0$ ("weak spring"). Eq.(2.6) corresponds to $a < 0$, $b < 0$. This "antifriction" ($a < 0$) responsible for "instability as a whole" is a characteristic feature of Y.M. equations.

behavior of all solutions. The separatrices divide the plane φ , $\partial_6 \varphi$ into six regions. Solutions in region II come out from the coordinate origin and go away to infinity at finite $\sigma = \sigma_1$. Solutions in region II' (with replacement $\varphi \to -\varphi$) behave analogously.

In terms of $\varphi(\tau)$, for all the solutions we have $\varphi(0) = 0$, $\varphi(\tau_1) = \pm\infty$ ($\tau_1 = \tau_0 e^{\sigma_1}$). An exception are two separatrices dividing regions II and II' when the solutions start at a zero point and tend to ±1 at $\tau \to \infty$ (see Fig.2 for $\varphi_{c_1} \to \pm 1$). In other words, the point $\varphi = 0$ (Wu-Yang "monopole") is an unstable focus.

In regions I and I' the solutions are determined in a finite interval of values $\tau > 0$, $\tau_1 < \tau < \tau_2$; φ increases unlimitedly at boundaries of this interval, whereas inside the latter, φ is always modulo larger than unity ($\varphi^{I}(\tau) \xrightarrow[\tau \to \tau_{1,2}]{} +\infty$, $\varphi^{I'}(\tau) \xrightarrow[\tau \to \tau_{1,2}]{} -\infty$, $\varphi^{I}(\tau) > 1$, $\varphi^{I'}(\tau) < -1$).

Finally, in regions III and III' the solutions are determined again in the finite range $\tau > 0$ $\tau_1' < \tau < \tau_2'$ and

$$\varphi^{III}(\tau_1') = -\varphi^{III}(\tau_2') = +\infty \quad , \quad \varphi^{III'}(\tau_1') = \varphi^{III'}(\tau_2') = -\infty \quad ;$$

note that $\partial_\tau \varphi^{III} < 0$, $\partial_\tau \varphi^{III'} > 0$ (see Appendix 1).

Thus, solutions $\varphi = \pm 1$ are unstable saddle points. Their separatrices (correponding to $\varphi = 1$) are shown in Fig.3 a,b,c.

As a result we see that already static solutions of spherically symmetric Y.M. equations possess instability: small perturbations of initial conditions ($\varphi(\tau)$ and $\partial\varphi/\partial\tau$) sharply change the behavior of solutions, in particular, singularities $\varphi(\tau)$ either appear or change their position. Only five solutions ($\varphi = \pm 1$, $\varphi = 0$ and separatrices $\varphi_{c_{1,2}}(\tau)$ in Fig.2) remain finite for all $\tau \geqslant 0$. It should be noted that solutions $\varphi_{c_{1,2}}(\tau)$, coinciding at $\tau \to 0$ with the Wu-Yang solution and at

$\tau \to \infty$ with vacuum ones, are new solutions.

Now let us proceed to investigation of trajectories in phase space near $\pi_\varphi = 0$ plane, for which one should take account of time dependence of Eq.(2.4) solutions *).

For small perturbations $\delta\varphi(\tau, t)$, near static solutions in linear approximation we have

$$(\partial_t^2 - \partial_\tau^2)\delta\varphi = -\frac{1}{\tau^2}\left(3\varphi^{(0)2}(\tau) - 1\right)\delta\varphi \tag{2.7}$$

where $\varphi^{(0)}(\tau)$ is static solution described above. More interesting are perturbations near the five non-singular static solutions mentioned. Separating variables τ and t in (2.7) we shall obtain:

$$\delta\varphi(\tau, t) = u(\tau)e^{i\omega t} \tag{2.8}$$

where $u(\tau)$ satisfies equation

$$-u_{\tau\tau} + \frac{1}{\tau^2}\left(3\varphi^{(0)2}(\tau) - 1\right)u(\tau) = \omega^2 u(\tau) \tag{2.9}$$

coinciding in its form with one-dimensional stationary Schredinger equation with $E = \omega^2$ and a potential

$$V(\tau) = \frac{1}{\tau^2}\left(3\varphi^{(0)2}(\tau) - 1\right). \tag{2.10}$$

Time instability corresponds to the presence of levels with negative energy ($\omega^2 < 0$). Therefore for $\varphi^{(0)} = \pm 1$ (vacuum solutions), as is seen from (2.10), small perturbations are stable, while for $\varphi^{(0)} = 0$ and $\varphi^{(0)}_{C_{1,2}}(\tau)$ solutions of Fig.2 Eq.(2.9) has negative levels (see Fig.4):

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*) Time-dependent spherically symmetric solutions near Wu-Yang monopole were considered from another viewpoint in Ref.[20] .

for this reason, small perturbations grow exponentially with time for each negative level whose number is infinitely large.

These levels exist for any $\omega^2 < 0$, and one can demonstrate that their wave functions are normalizable (see Appendix 2).

Similarly one can qualitatively describe non-static perturbations of singular solutions as well, for which in regions where $\varphi^{(0)^2}(\tau) < 1/3$, there arise, generally speaking, negative levels and hence exponential instability. This also means that the one-loop correction to classical action near solutions $\varphi^{(0)}(\tau)$ has an imaginary part.

Evidently this analysis is impossible for finite perturbations. Here, as mentioned in Introduction, an appropriate method of investigation is numerical simulation of Fermi-Pasta-Ulam type, which is just carried out in Section 3.

3. Numerical Analysis of Equation (2.4) Phase Trajectories.

We carried out a numerical analysis for solutions of Eq.(2.4) in the vicinity of static solutions described in Section 2.

The continuous string of (2.4) was approximated by a set of nonlinear coupled oscillators $\varphi(i)$ (i = 1,2,...,N), whose number N in our numerical experiments was taken equal to 64 and 128.

A numerically integrated discrete analog of Eq.(2.4) was written in the form:

$$\ddot{\varphi}(i,t) = \frac{\varphi(i+1,t) - 2\varphi(i,t) + \varphi(i-1,t)}{(\Delta\tau)^2} - \frac{\varphi(i,t)\,(\varphi^2(i,t) - 1)}{(i\Delta\tau)^2} \qquad (3.1)$$

where $\Delta\tau$ is the string discretization step which we took equal to 0.1 (dots over φ denote differentiation with respect to (continuous) time).

The result of integration of (3.1) was expanded in harmonics:

$$\varphi(i,t) = \sqrt{2/N} \sum_{j=1}^{N-1} \varphi(j,t) \sin\left(\frac{\pi i j}{N}\right) \tag{3.2}$$

so that energy concentrated in all harmonics when neglecting the right-hand side of Eq.(2.4) (or the second term in the r.h.s. of Eq.(3.1)) is

$$E_o = \frac{\Delta\tau}{2} \sum_{j=1}^{N-1} \left[\dot{\psi}^2(j,t) + \Omega_j^2 \psi^2(j,t) \right]$$
$$\Omega_j = \frac{2}{\Delta\tau} \sin\frac{\pi j}{2N} , \tag{3.3}$$

while total energy of string (2.4) discrete analog is given by the expression

$$E_{tot} = E_o + \frac{1}{4} \sum_{i=1}^{N-1} \frac{(1-\varphi^2(i,t))^2}{(i\Delta\tau)^2} \Delta\tau . \tag{3.4}$$

Let us choose initial position of string $\varphi(i,0)$ and boundary conditions $\varphi(0,t)$, $\varphi(N,t)$. While considering perturbations of static solution $\varphi = 0$, it is natural to put

$$\varphi(i,0)=0, \quad \varphi(0,t) = \varphi(N,t) = 0 \tag{3.5}$$

whereas initial velocity $\partial\varphi(i,t)/\partial t\big|_{t=0}$ not equal to zero, this corresponding to non-deformed "string" at t = 0.

The other choice of initial and boundary conditions corresponds to deformed but initially resting string:

$$\varphi(i,0) = \sqrt{\frac{2}{N}} \sum_{\{j_o\}} \psi(j_o,0) \sin\left(\frac{\pi i j_o}{N}\right) \tag{3.6}$$
$$\varphi(0,t) = \varphi(N,t) = 0 \qquad \frac{\partial\varphi(i,t)}{\partial t}\bigg|_{t=0} = 0 ,$$

where $\{j_o\}$ stands for a set of modes excited at t = 0; summation in the first equality in (3.6) is carried out over this set.

The problem is formulated analogously near solutions $\varphi = \pm 1$ and separatrices $\varphi_{c_{1,2}}(z)$.

If after primary excitation of some set of harmonics $\{j_o\}$ all the rest ones become excited too, then as mentioned in Introduction, we have all grounds to regard the system thermalizable, and hence we deal with a non-integrable system.

In case there takes place periodical energy transfer between some modes as occurred in the experiment of Fermi, Pasta, Ulam, then the system is in stable region of phase space (KAM-tori).

Fig.5 gives examples of time dependence for energies of three modes j = 8,9,10 at primary excitation of five modes (amplitude A = 0.1) j_o =6-10. Here we took N = 64, E_{tot} = 4.8 ($\Delta E_{tot}/E_{tot} < 1$ %). Modes = 6,7 not presented in the figure behave analogously, while all the rest ones are practically non-excited (mean energy per one such mode does not exceed 1 %).

We can see that at small energy the system has approximately quasi-periodical motion.

The picture changes essentially with increasing string energy.

In Fig.6 (N = 64, E_{tot} = 1100, $\Delta E_{tot}/E_{tot}$ = 0.1 %, j_o = 30,31,32) one can clearly see the process of equal distribution of energies for primarily excited (amplitude A = 1, j_o = 30,31,32) and the rest of modes (Fig.6 presents energies of some modes, j = 29,33,34,57). The picture remains qualitatively the same at further thrice-enhancement of energy and corresponding increase of amplitude for the same modes j_o = 30,31,32.

In this case the system thermalization is already explicitly evident.

The same is pointed to by Fig.7, where we have given a distribution of energy $\bar{E}_j$ (j = 1,2,...,64) averaged over large time (range of averaging

780 with a step 0.04) with respect to modes. The figure shows that on the average all modes became excited.

4. Continuous String Limit.

One important circumstance should be emphasized, which is connected with transition from a continuous to a discrete case necessary in numerical experiments. Such discretization introduces scales to a system not containing dimensional parameters like which the Y.M. system is. Therefore the problem of inverse transition to continuous limit of above-considered discrete model is highly important.

In the continuous case (2.4) under transformations

$$\tau \to \beta \tau , \qquad t \to \beta t \tag{4.1}$$

the system with energy E and coupling constant g transforms into a system with E' and g' :

$$E' g'^2 = \beta^{-1} E g^2$$

which means similarity of phase trajectories of systems with different energies and coupling constants, just as it was the case with Y.M. classical mechanics [1]. In other words, to study the system, it is enough to investigate trajectories with only one energy E and coupling constant g .

At discretization (which represents a regularization analog in classical field theory [21]), there arise additional parameters N and $\Delta\tau$ so that modes Ω_j in (3.3) lie in the range $\Omega_{min} \sim 1/N\Delta\tau$, $\Omega_{max} \sim 1/\Delta\tau$.

To continuous limit there correspond $\Delta\tau \to 0$, $N \to \infty$ but such that $L \equiv N \cdot \Delta\tau \to \infty$.

This limit can be realized as follows: first, by decreasing $\Delta\tau$ at fixed N, then, by increasing N at fixed $\Delta\tau$ and so on. Here in the first case (N is fixed) the system motion is determined only by dimensionless parameter $\pi \equiv g^2 E \cdot \Delta\tau$, in which one may be convinced by means of (4.1) type transformation $\tau \to \beta\tau$, $t \to \beta t$ and expression for energy (3.3), (3.4) in the discrete case:

$$\Delta\tau' \longrightarrow \beta\,\Delta\tau, \qquad E' g'^2 = \beta^{-1} E g^2 .$$

This means that phase trajectories with different $\Delta\tau$ but the same π are equivalent. In such conditions the continuous limit is achieved by tending of only N to infinity.

Following this remark, we have investigated what happens with our system at increasing N.

Fig.8 gives energies of modes for a string with doubled N (N=128) with the same initial configuration φ as in the case with N=64.

One can see from the figure that with increasing N (i.e. at approaching to continuous limit) the string is "thermalized" faster, the region covered by invariant tori is narrowing. It should be thought that in real continuous limit ($N = \infty$) this region is absent at all, exactly as in Yang-Mills-Higgs classical mechanics [2] at tending of Higgs field mean expectation value to zero the invariant tori vanished thus leading to total stochasticity of Y.M. mechanics [6].

5. Conclusion.

As a result of the present investigation, we have all grounds to claim that not only Yang-Mills classical mechanics [1] but also Yang-Mills classical field theory describing a system with infinite number of degrees of freedom is non-integrable, i.e. possesses dynamical stochasticity.

In other words, one may claim that the real QCD is a field theory which in classical limit manifests dynamical chaos.

A reasonable question arises here: whether the traces of this stochasticity remain in the real QCD ?

If yes, then the confinement problem finds as is well known (see, e.g. [22]) a natural explanation.

However the question on the relation between classical and quantum theories from this viewpoint is far from being simple.

There are known e.g. consequences for quantum systems possessing in classical limit dynamical stochasticity, which are related with their spectrum structure and properties of wave functions (see, e.g. [23,24,22,6])*).

This allows to think that real spectrum of hadrons must carry traces of irregularity corresponding to here proved dynamical chaos of Y.M. classical field theory.

It is quite possible that also other characteristics of hadronic phenomena can be revealed, which reflect dynamical stochasticity of non-abelian gauge fields governing the world of hadrons.

A real progress is possible in this direction. Here first one should note Monte-Carlo calculations on lattice [27] of spectral distribution $\rho_c(\alpha)$ of expectation values of Wilson loops $W(C) = \int_{-\pi}^{+\pi} d\alpha\, \rho_c(\alpha)\, e^{i\alpha}$.

As follows from these calculations, for distances (loop sizes) τ less than confinement radius τ_c spectral density $\rho_c(\alpha)$ has a peak at $\alpha = 0$,

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*) The recent investigation of Yang-Mills-Higgs quantum mechanics carried out in some works [25,26] has shown that distribution of distances between neighbour energy levels of this problem reflects stochasticity of Yang-Mills-Higgs classical mechanics [2] .

which means strong correlation of fields at small distances. However for loops with $\tau \gtrsim \tau_c$ the distribution $\rho_c(\alpha)$ turns out to be practically uniform, i.e. fields are non-correlated and the distribution of expectation values of $W(C)$ corresponds to disordered configurations.

This argument if not being an artefact of Monte-Carlo lattice calculations points out that some amount of stochastic component is really present in QCD.

Of course, the qiestion on whether this component is really manifestation and "relict" of classical stochasticity remains open so far; so further investigations are needed here.

It should be mentioned also some recent works [28] , in which multiple production is considered from the viewpoint of those stochastic regularities which are typical of quite various phenomena of Nature, from developed turbulence up to galaxies distribution in clusters.

Quantitative generality of these at first sight absolutely not related phenomena is expressed by universal relationship between essential parameters of these phenomena including fractal dimension [29] .

Apparently, the essence of this generality contains yet a classical aspect rather than quantum one.

Note in conclusion that the observed instability of Wu-Yang solution and new solutions $\varphi_{C_{1,2}}(\tau)$ makes extremely important to examine the similar question for a more realistic case - T'Hooft-Polyakov monopole. In other words, the study of spherically symmetric Yang-Mills-Higgs equations is necessary. The results will be published elsewhere.

Appendix 1

Let is present asymptotics of Eq.(2.6) solutions.

At $|y-1| \ll 1$ we have:

$$y \approx 1 + C_1 e^{-\sigma} + C_2 e^{2\sigma},$$

At $|y| \ll 1$ we have:

$$y \approx e^{\sigma/2}\left[A\cos\left(\frac{\sqrt{3}}{2}\sigma\right) + B\sin\left(\frac{\sqrt{3}}{2}\sigma\right)\right].$$

Finally at $|y| \gg 1$ we have:

$$y \approx \pm\left(\frac{\sqrt{2}}{\sigma-\sigma_1} - \frac{1}{3\sqrt{2}}\right).$$

These expressions are obtained via analysis of formal power series for Eq.(2.6) solutions in corresponding regions.

Appendix 2

At $\varphi^{(0)} = \pm 1$ we have $V(\tau) = 2/\tau^2$ - singular repulsive potential. Non-degenerate continuous spectrum exists at $\omega^2 \geqslant 0$. At $\tau \to 0$ $V(\tau) \sim \tau^2$. Just the second solution $\sim 1/\tau$ does not belong to continuous spectrum. In region $\omega^2 < 0$ there are no quadratically integrable solutions, i.e. discrete spectrum. So, the spectrum occupies a semiaxis $\omega^2 > 0$.

Consider now the cases $\varphi^{(0)} = 0$ or $\varphi^{(0)} = \varphi_{c_{1,2}}$. At $\tau \to \infty$ $V(\tau) \to g/\tau^2$, where g = -1 for the former case, and g = 2 for the latter one. Examine the differential operator

$$H = -\partial_\tau^2 + V(\tau) \qquad \text{(A 2.1)}$$

from Eq.(2.9). It is symmetric for functions from dense subset $L^2(0,\infty)$ and has defect indices (1.1). This means that it will be self-conjugated for the definition regions $\{\mathcal{D}(H)\}$: $u \in L^2(0,\infty)$, $Hu \in L^2(0,\infty)$, $\cos(\alpha u(0)) + \sin(\alpha u'(0))$, where α is real, $|\alpha| < \pi/2$, and existence of $\lim_{z\to 0} u(z)$ and $\lim_{z\to 0} u'(z)$ is assumed.

At $\alpha = 0$ we have a condition $u(0) = 0$. Then for $\omega^2 \geqslant 0$ we have double-degenerate continuous spectrum, while for $\omega^2 < 0$ we have single discrete spectrum occupying the whole semiaxis. This can be seen from asymptotics of Eq.(2.9) solutions:

$$u(z) \to \sqrt{z}\left[A\cos\left(\frac{\sqrt{3}}{2}\ln z\right) + B\sin\left(\frac{\sqrt{3}}{2}\ln z\right)\right] \tag{A 2.2}$$

at $z \to 0$

$$u(z) \to Ae^{i\omega z} + Be^{-i\omega z}$$ at $z \to \infty$ $(\omega \neq 0)$

$$u(z) \to Az^{(1+\sqrt{1+4g})/2} + Bz^{(1-\sqrt{1-4g})/2}$$ at $z \to \infty$ $(\omega = 0)$

Usually only the case $\alpha = 0$ is considered. At $\alpha \neq 0$ spectrum does not exist and hence is absent at $\omega^2 < 0$. However if considering the interval $(\varepsilon, +\infty)$, $\varepsilon > 0$, then at any α we shall have a countable set of discrete spectrum levels at $\omega^2 < 0$. The spectrum is unlimited from below. At $\varepsilon \to 0$ the upper discrete level tends to $-\infty$, and the distance between the levels increases as ε^{-1} .

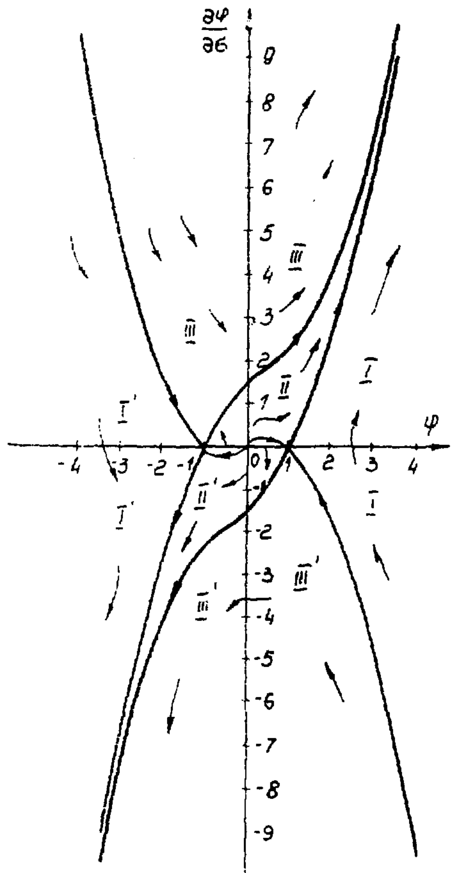

Fig.1. Separatrices and field of directions of Eq.(2.6).

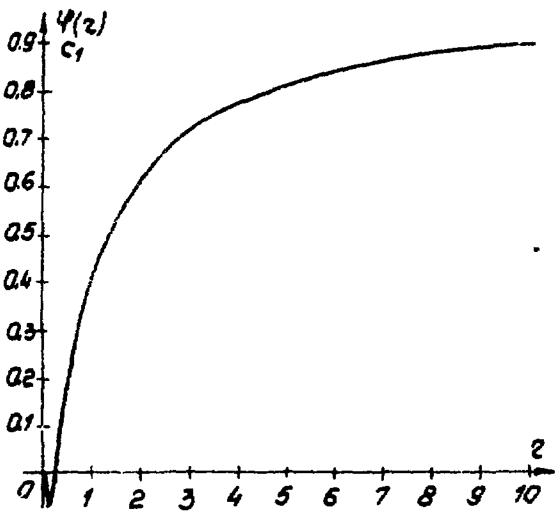

Fig.2. Non-singular static solution (separatrix) of Eq.(2.6) φ_{C_1}

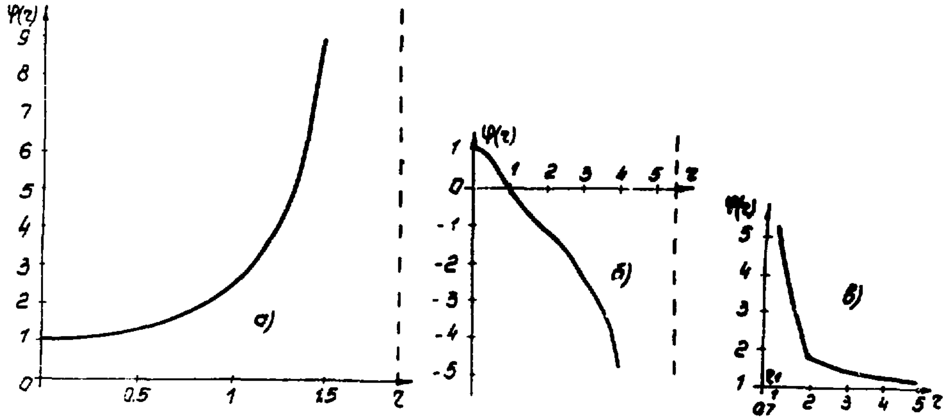

Fig.3. Singular solutions (separatrices) of Eq.(2.6) corresponding to a saddle point $\varphi = 1$.

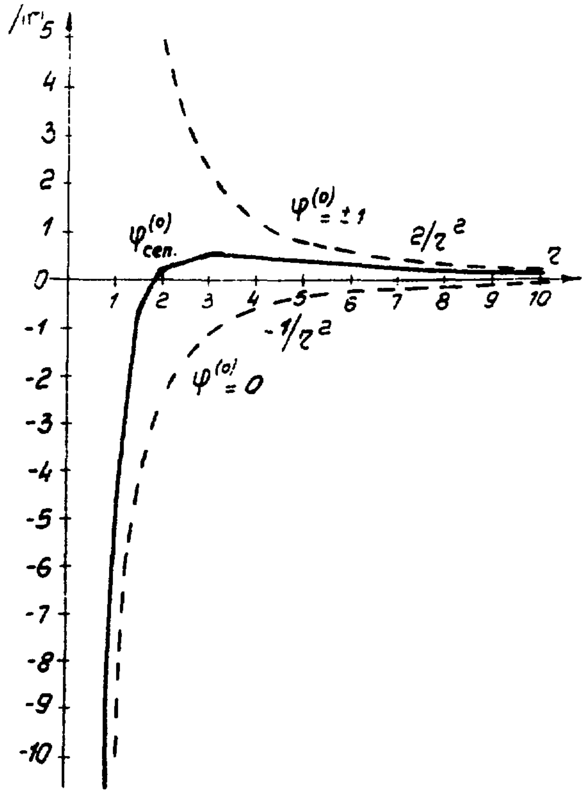

Fig.4. A "potential" of Eq.(2.9) corresponding to static solutions $\varphi(r) = 0, \pm 1$ and $\varphi_{c_{1,2}}(r)$.

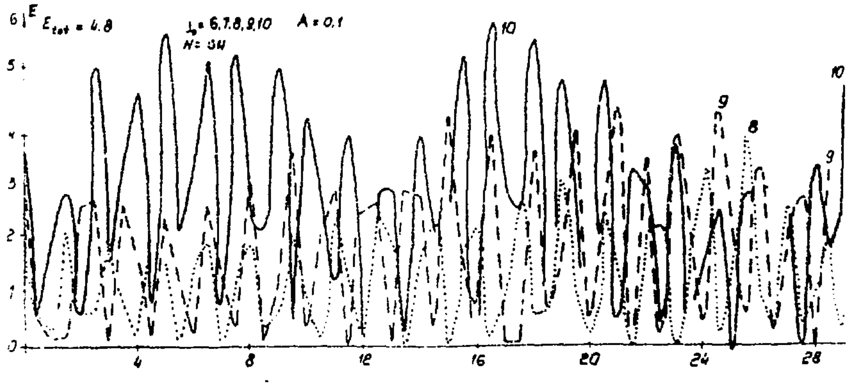

Fig.5. Quasi-periodical motion at small amplitude of initial perturbation

($A = 0.1$; $j_o = 6,7,8,9,10$; $N = 64$).

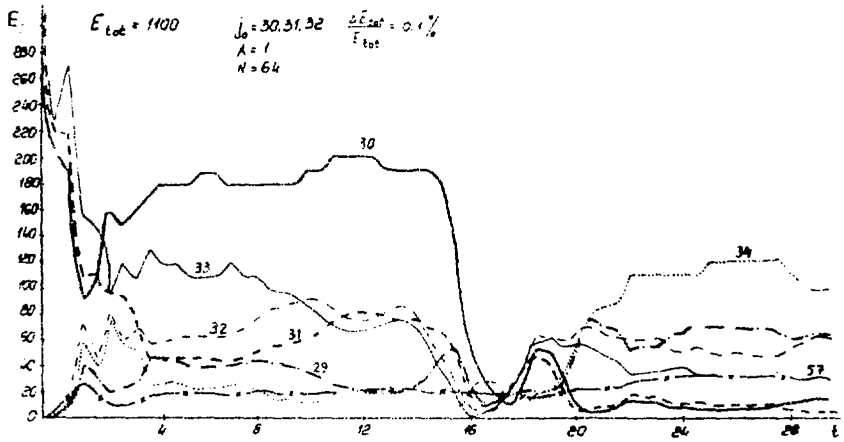

Fig.6. Energy of modes $j = 29,33,34,57$ as a function of time

at large amplitude of initial perturbation ($A = 1$; $j_o = 30,31,32$;

$N = 64$).

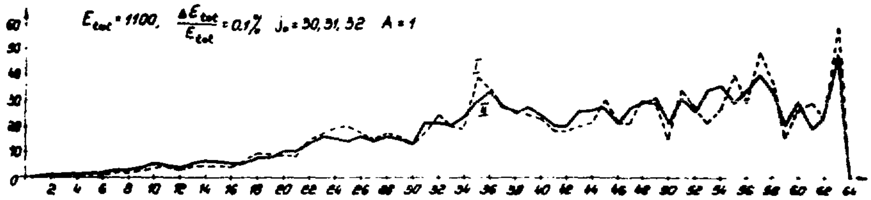

Fig.7. Distribution of averaged over large time interval (total interval of integration 780, step 0.04) energies $\bar{E}_j$ (j = 1,2,...,64) with respect to modes.

Curve I - averaging interval t_i = 40, t_f = 410.

Curve II - averaging interval t_i = 40, t_f = 780.

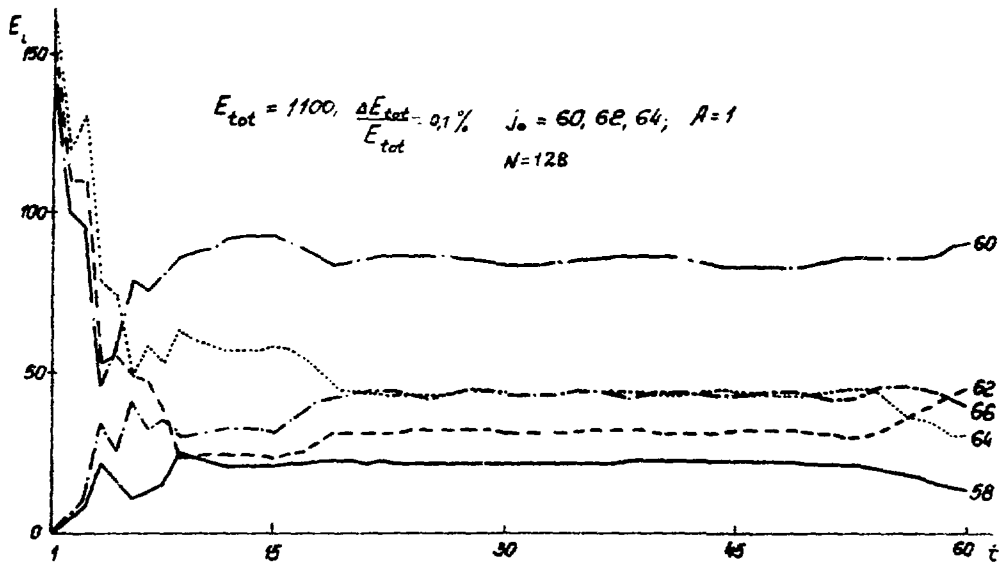

Fig.8. Time-dependent energies of modes j = 58, 66 at large amplitude of initial perturbation (A = 1; j_0 = 60,62,64) for doubled string length (N = 128).

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НЕИНТЕГРИРУЕМОСТЬ СФЕРИЧЕСКИ СИММЕТРИЧНЫХ УРАВНЕНИЙ ЯНГА-МИЛЛСА, ЗАВИСЯЩИХ ОТ ВРЕМЕНИ

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