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Perturbative Coherence in Field Theory*

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Abstract

The basic field equations of a field theory are not always derivable from a Lagrangian. Lagrangian theories are perturbatively coherent, in the sense that they have well defined vertices. Non-Lagrangian theories are sometimes coherent, sometimes not. Coherent theories are, up to renormalization, quantizable by perturbative methods. The general condition for coherent quantization by perturbative methods is given.

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1.Introduction

A field model is defined by (i) listing all the fields involved. (ii) specifying their response to the transformations of interest and (iii) giving the fundamental field equations and boundary conditions corresponding to the physical situation. The last item fixes the supposed dynamical behavior of the system. More frequently a Lagrangian is given from which the field equations are obtained as the Euler-Lagrange equations. Or the action functional is given, which is global in character and in principle includes all the information concerning the system. This Lagrangian formulation has many advantages: it allows a simple treatment of symmetries and gives a workable way to quantization by Feynman path integral methods. which furthermore provide a convenient means to dispose of nonphysical degrees of freedom. Although most quantization methods suppose a Lagrangian. a field model can also be quantized by a perturbative procedure starting directly from the field equations. the Kallèn-Yang-Feldman[1] method. There is some advantage in working with the field equations, as not all systems of field equations are derivable from a Lagrangian.

It is commonly believed that non-Lagrangian theories are not amenable to quantization. It is one of our objectives here to show that this is not so. Field models are perturbatively quantizable provided they satisfy a general condition which we shall find out. All Lagrangian theories will be seen to satisfy automatically that condition so that, as expected, all such theories can be quantized by the perturbative procedure. The condition is, however, less stringent than the condition for the existence of a Lagrangian, so that some non-Lagrangian theories can also be quantized. Notice that this has nothing to do with renormalizability, a question we shall ignore here. A coherent theory may turn out to be non-renormalizable and consequently non-quantizable in a strict sense, but an incoherent model is defective in a much more primitive way: it has no well-defined vertices. A simple example is the case of two scalar fields φ_1 and φ_2 obeying the equations of motion

$$
\begin{aligned}
(\Box + m^2)\,\varphi_1 &= g(\partial^\mu\varphi_1\partial_\mu\varphi_2)\varphi_2 \\
(\Box + m^2)\,\varphi_2 &= g(\partial^\mu\varphi_1\partial_\mu\varphi_2)\varphi_1 \,.
\end{aligned}
\tag{1}
$$

This is a non-Lagrangian system, as shown below, but there is no difficulty in quantizing it. Power counting tells us that it is non-renormalizable in four-dimensional spacetime but renormalizable in two dimensions. Starting from the field equations, perturbative quantization is performed in a very simple way (we shall have in mind here the operational version of this method as described by [illegible]). Let us recall the procedure for the model above: first, with the help of the Green's function of the differential operator, we write down the formal expressions for the solutions as

$$
\varphi_1 = \varphi_1^{(in)} + (\Box + m^2)^{-1} g(\partial^\mu\varphi_1\partial_\mu\varphi_2)\varphi_2 \tag{2}
$$

$$\varphi_2 = \varphi_2^{(in)} + (\Box + m^2)^{-1} g(\partial^\mu \varphi_1 \partial_\mu \varphi_2)\varphi_1 . \qquad (3)$$

We are using a symbolic notation, omitting integrations. Then, to obtain the S matrix elements for each field, such general solutions, written as a sum of an ingoing free field plus interaction terms, are to be projected on outgoing free fields of the same kind (and again integrated). The perturbative solution to a certain order is obtained by simply iterating the above equations, that is, replacing the fields in the interaction terms by the formal solutions and retaining terms up to the desired order. The Green's operator, once acted on from the left by the outgoing free field, gives again a free field. As a consequence, the vertices are obtained as the first contributions in the series with no remaining Green's functions. For example, projection of (2) on $\varphi_1^{(out)}$ will lead to the vertex $g\varphi_1(\partial^\mu\varphi_1\partial_\mu\varphi_2)\varphi_2$. The same comes out from the projection of expression (3) on $\varphi_2^{(out)}$; the vertex appears the same when seen from both (2) and (3). This seems natural enough and embodies what we shall understand by *perturbative coherence*. It would not be a property of the above model if the source terms in (1) were not carefully chosen. A trivial case of vertex incoherence would show up if in (1) the coupling constants in the two equations were different. *Perturbative incoherence* shows up when a vertex appears different when looked at from different channels. This would seem to be contrary to intuition, but our intuition is based on a familiarity acquired in Lagrangian models which are, as seen below, always coherent. Of course, in the case of incoherent models the very notion of vertex looses its meaning, but we shall keep using the word "vertex" for simplicity of language. The model (1) will be incoherent if taken with two different coupling constants but coherent (although non-Lagrangian) with a unique coupling constant. In more involved incoherent models, it may even happen that a vertex which is seen in one channel is simply absent when looked at in another channel. This is for instance the case for the Poincaré gauge model[3].

2. The coherence conditions

Suppose we have a set (φ^a) of relativistic fields $(a=1,2,\ldots,N)$ submitted to a set of N equations of which we shall write only two:

$$K^a[\varphi] = J^a[\varphi] \tag{4}$$

$$K^b[\varphi] = J^b[\varphi]. \tag{5}$$

K^a is the kinematical operator (Klein-Gordon, Dirac, etc) acting on φ^a, K^b that acting on φ^b and the J's are the source currents. The general form of a current functional involving j_1 fields of kind φ_1, j_2 fields of type φ_2, ... j_p fields of type φ_p, with a total of $k=\sum_i j_i$ fields, will be

$$J^a[\varphi] = \int d^4x_1 d^4x_2 \ldots d^4x_k\, \varphi_1(x_1)\varphi_1(x_2)\ldots\varphi_1(x_{j_1})\varphi_2(x_{j_1+1})\ldots\varphi_2(x_{j_1+j_2}) \varphi_3(x_{j_1+j_2+1})\ldots\varphi_3(x_{j_1+j_2+j_3})\ldots\varphi_p(x_k)\; C^a_{j_1 j_2 \ldots j_p}(x_1,x_2,\ldots x_k). \tag{6}$$

If the current is a simple monomial in the fields, the coefficient $C^a{}_{j_1 j_2 \ldots j_p}$ is a product of Dirac deltas. If the current involves derivatives of the fields, the coefficient will be a product of deltas and derivatives of deltas. When the current is a sum of terms involving different numbers of fields, it will be necessary for our purposes to examine each term separately, as they would correspond to distinct vertices. The general expression for the coefficient is, formally,

$$C^a{}_{j_1 j_2 \ldots j_p} = \left. \frac{\delta^k J^a}{\delta\varphi_1(x_1)\ldots\delta\varphi_2(x_{j_1+1})\ldots\delta\varphi_p(x_k)} \right|_{\varphi=0} \qquad (7)$$

We shall again omit the integrations and put together fields of the same type, so as to rewrite (6) simbolically as

$$J^a[\varphi] = C^a{}_{j_1 j_2 \ldots j_p} \varphi_1{}^{j_1} \varphi_2{}^{j_2} \ldots \varphi_a{}^{j_a} \ldots \varphi_b{}^{j_b} \ldots \varphi_p{}^{j_p} \ , \qquad (8)$$

and analogously for $J^b[\varphi]$. Here no summation on repeated indices is implied. We have intentionally signaled the presence of the fields φ_a and φ_b. The S matrix element is obtained by projecting equation (4) on an outgoing field of type φ_a, or by projecting equation (5) on an outgoing field φ_b. The coherence condition will be, in the above compact notation,

$$J^a[\varphi]\varphi_a = J^b[\varphi]\varphi_b \qquad (9)$$

(no summation on a,b) . Comparing with (8) and using the purely multiplicative character of the coefficients (7), the condition may be put into the form

$$C^a{}_{j_1 j_2 \ldots j_a \ldots j_b \ldots j_p} = C^b{}_{j_1 j_2 \ldots (j_a+1) \ldots (j_b-1) \ldots j_p} \quad . \tag{10}$$

An analogous reasoning holds for every pair of the N indices a,b,c , etc, so that we have in reality a whole series of N!/[(N-2)!2!] conditions like (10). Use of (7) puts (10) into the form

$$\frac{\delta^{k-1}}{\delta\varphi_1{}^{j_1}\delta\varphi_2{}^{j_2}\ldots\delta\varphi_a{}^{j_a}\ldots\delta\varphi_b{}^{j_b-1}\ldots\delta\varphi_p{}^{j_p}}\left[\frac{\delta J^a}{\delta\varphi_b} - \frac{\delta J^b}{\delta\varphi_a}\right]\Bigg|_{\varphi=0} = 0 \tag{11}$$

for each pair of indices a,b, with a=b.The "derivatives" in this expression are to be taken as functional (Fréchet) derivatives and, as such, as linear operators . When no derivatives on the fields are present in the J's, they will have the same algebra of usual derivatives. However, when derivatives are present, integrations by parts are to be carefully considered. Instead of going into these details here, we shall use another, simpler and more powerful formalism, which will allow the whole set of conditions (11) to be put in a simple form.

3. The calculus of variational forms

Let us recall some properties of functional differential forms (called *Forms* from now on), of which a less incomplete presentation has been given elsewhere.[4] Consider an action functional $S[\varphi]$, dependent on the fields $\varphi_1, \varphi_2, \ldots \varphi_N$. Its variation δS will be a variational form of first degree, which can be written as

$$E = E_a \delta\varphi^a \tag{12}$$

with

$$E_a = \frac{\delta}{\delta\varphi_a} S[\varphi] \,. \tag{13}$$

This is analogous to the differential $df = (\partial_i f)dx^i$ of a function f. The Euler-Lagrange equations coming from the action S, or from its integrand, the Lagrangian density, are, of course, $E_a = 0$. Expressions like (12) will be called *1-Forms*, in analogy with the usual differential 1-forms of calculus. In special, 1-Forms related to differential equations will be called *Euler Forms*. The analogy with differential calculus goes, in reality, much further. Just as a general 1-form $\omega = \omega_i dx^i$ is not necessarily the differential of a function (is not necessarily *exact*), a general 1-Form as (12) is not necessarily the variation of a functional. In this case, the corresponding equations $E_a = 0$ are not related to an action functional and are said to be *non-Lagrangian*. When does a Lagrangian exist for the equations ? Once more the analogy with calculus is perfect: for the form ω to be locally the differential of a function, it is necessary and sufficient that $d\,\omega = 0$. For E to be locally an exact 1-Form, it is necessary and sufficient that $\delta E = 0$. The algebra of the exterior variations δ is formally the same algebra of the exterior differentials in calculus, the 2-Form δE being written as

$$\delta E = (1/2)\left[\, \delta E_a / \delta\varphi^b - \delta E_b / \delta\varphi^a \,\right]\ \delta\varphi^b \wedge \delta\varphi^a \qquad . \qquad (14)$$

The formal analogy is complete indeed, provided the derivatives are interpreted as Fréchet derivatives. Acting on typical actions, which are 0-Forms, such derivatives reduce to the usual Lagrangian derivatives. This analogy leads, in particular, to the boundary-has-no-boundary property $\delta^2 = 0$.

The condition $\delta E = 0$ for the existence of a Lagrangian for the equations $E_a = 0$ becomes, in view of (14), just the vanishing of the bracketed term. This is a new version of Vainberg's theorem,[5] which gives the conditions for a functional to be the functional derivative of another functional. Applied to the Euler Form

$$J = J_a\, \delta\varphi^a \qquad (15)$$

associated to the currents considered in S 2 , we see that the Lagrangian condition is just the vanishing of the bracketed term in

(11). As a consequence, every Lagrangian model satisfies (11) automatically and can be quantized in a coherent way. More properties of differential forms can be adapted to Forms. One of them is the Poincaré lemma, which includes the above considerations about the existence of Lagrangians as a special case. Let W be any p-Form

$$W[\varphi] = W_{a_1 a_2 \cdots a_p}[\varphi]\, \delta\varphi^{a_1} \wedge \delta\varphi^{a_2} \ldots \wedge \delta\varphi^{a_p}$$

and define its transformed TW as the (p-1)-Form

$$TW[\varphi] = \sum_{j=1}^{p} (-)^{j-1} \int_0^1 dt\, t^{p-1} W_{a_1 a_2 \cdots a_p}[t\varphi]\, \varphi^{a_j} \delta\varphi^{a_1} \wedge \delta\varphi^{a_2} \ldots \wedge \delta\varphi^{a_{j-1}} \wedge \delta\varphi^{a_{j+1}} \ldots$$

$$\ldots \wedge \delta\varphi^{a_p} . \qquad (16)$$

Then the lemma says that W can always be written locally as

$$W[\varphi] = \delta\, TW + T\, \delta W \ . \qquad (17)$$

We see that, when W is an Euler Form E, then $\delta E = 0$ implies the existence of a Lagrangian $\Lambda = TW$. Another notion from differential calculus that can be implemented in the calculus of Forms is that of a Lie derivative. On the space of the φ's, the components φ^{a} may be used as "functional coordinates". Fields (in the geometrical sense of the word) can be introduced, and the set of derivatives $\{e_a = \delta/\delta\varphi^a\}$

may be used as a "natural" local basis for them. A general Field X will be written $X = X^a e_a = X^a \delta/\delta\varphi^a$. The Lie derivative L_X, acting on Forms, will have properties analogous to those found in differential calculus. For example, suppose X to represent a transformation generator on the φ-space. On Forms, the transformation will be given by the Lie derivative L_X. As Lie derivatives commute with differentials, we have

$$L_X E = L_X \delta\Lambda = \delta L_X \Lambda . \tag{18}$$

Consequently, a symmetry of the Lagrangian ($L_X\Lambda = 0$) is a symmetry of the equation ($L_X E = 0$), but the equation may have symmetries which are not symmetries of the Lagrangian, a well known fact. Other notions of differential calculus translate easily to Forms, keeping furthermore analogous properties. Such is the case, for example, of the interior product $i_X W$ of a Field X by a Form W, which has the usual relation to the Lie derivative, $L_X W = i_X(\delta W) + \delta(i_X W)$.

In many of the considerations above some kind of metric is supposed. From (12) on we have been raising and lowering indices. Unless some special metric is at work in the model under consideration (such as the Killing-Cartan form in gauge models for semisimple groups), we shall simply suppose a metric of euclidean type, which identifies components with higher and lower indices.

4. A unified coherence condition

The coherence conditions acquire a simple form in the language of Forms. For each set of indices $(j_1, j_2, \ldots, j_p)$ in (11), define the 1 Form

$$N_{(j_1,j_2,\ldots,j_p)} = \sum_a \sum_{b \neq a} \frac{\delta^{k-1} J^a}{\delta\varphi_1{}^{j_1}\delta\varphi_2{}^{j_2}\ldots\delta\varphi_a{}^{j_a}\ldots\delta\varphi_b{}^{j_b-1}\ldots\delta\varphi_p{}^{j_p}}\,\delta\varphi_a\,. \qquad (19)$$

The coherence condition becomes then

$$\delta N_{(j_1,j_2,\ldots,j_p)} = 0\,. \qquad (20)$$

For each set $(j_1, j_2, \ldots, j_p)$ in the model, the corresponding *coherence Form* N must be closed. Notice that, by its very definition, the coefficients of N are linear in the fields (or some of its derivatives) and condition (20) requires N to be derivable from a certain 0-Form bilinear in the fields (which, by the way, is just the transformed TN calculated by using (16)). It is not difficult to check that N is a multiple Lie derivative of J with respect to the Fields e_a constituting the natural Field basis on the functional space:

$$N_{(j_1,j_2,\ldots,j_p)} = \sum_a \sum_{b \neq a} \left[(L_{e_1})^{j_1}(L_{e_2})^{j_2}\ldots(L_{e_a})^{j_a}\ldots(L_{e_b})^{j_b-1}\ldots(L_{e_p})^{j_p}\right](J) \qquad (21)$$

So, although J is not necessarily derivable from a Lagrangian for the theory to be coherent, some of its multiple Lie derivatives must be. It is easier to interprete N in terms of Feynman graphs: given the vertex graph, to take a Lie derivative L_{e_j} corresponds to extracting a leg of type "j". Each term in (19) or (21) does the following: (i) fix a leg a; (ii) extract all other legs up to one of type b; (iii) exchange the two remaning legs; (iv) impose invariance under such exchange. The summation then spans all possible channels. This is a compact formalization of the "view-from-two-channels" discussion presented in §2.

Consider the model given by equation (1). The current Euler Form will be

$$J = g(\partial^\mu\varphi_1\partial_\mu\varphi_2)\left[\varphi_2\,\delta\varphi_1 + \varphi_1\,\delta\varphi_2\right] \qquad (22)$$

or

$$J = g(\partial^\mu\varphi_1\partial_\mu\varphi_2)\delta\left[\varphi_1\varphi_2\right] \ .$$

Its variation will be

$$\delta J = g\delta(\partial^\mu\varphi_1\partial_\mu\varphi_2)\wedge\delta\left[\varphi_1\varphi_2\right] \neq 0 \ .$$

Consequently, there is no Lagrangian for equations (1). Coherence is to be examined from the only set of fields of interest, $(j_1 - 1, j_2 - 1)$, to which corresponds the Form

$$N_{(1,1)} = N^i_{(1,1)}\delta\varphi_i \ .$$

with

$$N^{j}_{(1,1)} = \frac{\delta^2 J^j}{\delta\varphi_1\,\delta\varphi_2} \; .$$

We find that

$$N_{(1,1)} = 2g\,\delta(\partial^\mu\varphi_1\partial_\mu\varphi_2),$$

so that $\delta N_{(1,1)} = 0$. The model can be coherently quantized, despite its non-Lagrangian character.

5. The case of gauge fields

The Euler Form corresponding to the Yang-Mills equations is

$$E = [\,\partial_\mu F^{a\mu\nu} + f^{a}{}_{bc}A^{b}{}_{\mu}F^{c\mu\nu}]\;\delta A_{a\nu} \; , \tag{23}$$

where

$$F^{a\mu\nu} = \partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + f^{a}{}_{bc}A^{b\mu}A^{c\nu},$$

the f's being the structure constants of the Lie algebra of the gauge group, the generators T_a satisfying $[T_a, T_b] = f^{c}{}_{ab}T_c$. We shall rewrite the Euler Form as

$$E = [\,K^{a\nu} - J_Y{}^{a\nu} - J_X{}^{a\nu}\,]\;\delta A_{a\nu} \; , \tag{24}$$

where

$$K^{a\nu} = \partial_\mu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})$$

is the kinetic term,

$$J_1{}^{a\nu} = - f^a{}_{bc}\left[\partial_\mu(A^{b\mu}A^{c\nu}) + A^b{}_\mu(\partial^\mu A^{c\nu} - \partial^\nu A^{c\mu})\right] \quad (25)$$

is the current leading to three-legged vertices and

$$J_2{}^{a\nu} = - f^a{}_{bc} f^c{}_{de} A^b{}_\mu A^{d\mu} A^{e\nu} \quad (26)$$

is the current related to four-legged vertices. For each kind of vertex a coherence Form must be defined. For the three-legged case,

$$N^Y{}_{(c\sigma)} = \frac{\delta J^{a\nu}}{\delta A^{c\sigma}}\, \delta A_{a\nu} \quad (27)$$

can be put, after a tedious but direct calculation, in the explicit form

$$N^Y{}_{(c\sigma)} = (1/2)\left\{ \left[f_{(bc)a} - f_{(ac)b} + f_{(ab)c}\right]\left(\partial_\nu A^b{}_\sigma \delta A^{a\nu} - \partial_\nu A^{b\nu} \delta A^a{}_\sigma\right) + \right.$$

$$+f_{(ac)b}\,[\partial_\sigma A^b{}_\nu - \partial_\nu A^b{}_\sigma]\,\delta A^{a\nu}\,\Big\}\ . \qquad (28)$$

Here the symbol (ab) stands for symmetrization. The noticeable fact is that the structure constants appear always symmetrized in the first two indices. As a consequence, the above Form will vanish identically for semisimple gauge groups and the coherence condition for three-legged vertices will be satisfied, in agreement with the fact that gauge models for semisimple groups are Lagrangian theories. Such is not the case for nonsemisimple groups[6], for which the three-legged vertices are well defined only if the above Form is closed indeed. The explicit expression of δN is

$$\delta N^Y{}_{(c\sigma)} = (1/2)\Big\{\,[2f_{(bc)a} - f_{(ac)b}]\,\delta A^a{}_\sigma \wedge \partial_\nu \delta A^{b\nu}$$

$$-(1/2) f_{(ab)c}\,\partial_\sigma \delta A^{b\nu} \wedge \delta A^a{}_\nu \Big\}\ . \qquad (29)$$

From (25), the coherence Form for the four-legged vertices is obtained as

$$N^X{}_{(g\rho)(h\sigma)} = (1/2)\frac{\delta}{\delta A^{g\rho}\,\delta A^{h\sigma}}\Big\{ f_{ac(b}f^c{}_{d)e}A^b{}_\mu A^{d\mu} A^{e\nu}\Big\}\,\delta A^a{}_\nu =$$

$$=\delta_{\rho\sigma}\, f_{ac(h}f^c{}_{g)d}\,A^{d\nu}\,\delta A^a{}_\nu + f_{ac(h}f^c{}_{d)g}\,A^d{}_\sigma\,\delta A^a{}_\rho + f_{dc(g}f^c{}_{a)h}\,A^a{}_\rho\,\delta A^d{}_\sigma .$$

$$(30)$$

Again indices between parenthesis () are symmetrized. The four-legged vertices are coherent if the above Form is closed, which corresponds to the vanishing of

$$\delta N^{x}{}_{(g\rho)(h\sigma)} = (1/2)\delta_{\rho\sigma}\left[f_{(ac)(h}f^{c}{}_{g)d} - f_{(cd)(g}f^{c}{}_{h)a}\right]\delta A^{d\nu}\wedge\delta A^{a}{}_{\nu} +$$

$$\left[f_{(ac)h}f^{c}{}_{dg} - f^{c}{}_{hg}f_{(ad)c} - f^{c}{}_{ah}f_{(dc)g}\right]\delta A^{d}{}_{\sigma}\wedge\delta A^{a}{}_{\rho}\ .$$

We see once more the coherence of the semisimple case: each term is proportional to a structure constant symmetrized in the two first indices.

In general, gauge models for nonsemisimple groups, besides being non-Lagrangian, cannot be coherently quantized by the perturbative method. An example has been found, by other means, in a model involving the Poincaré group.[7]

6. Final comments

We have seen that non-Lagrangian models may have well defined vertices, provided they satisfy what we called the *coherence condition*. We have been rather strict in our language: incoherent models cannot be quantized by the usual techniques of perturbation theory because their vertices are not symmetric under the interchange of identical external legs and consequently the usual Feynman rules do not apply. Such asymmetry, however, is not a novelty in Physics: it is a well known property of vertices in dual models[8], for which specially modified Feynman rules must be introduced[9]. It is a curious point that gauge models for nonsemisimple groups exhibit it. Whether or not they have some relation to dual models is left for future consideration.

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