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2ω - DIMENSIONAL LIGHT-CONE INTEGRALS WITH MOMENTUM SHIFT

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Summary:

I consider a class of light-cone integrals typical to one-loop calculations in the two-component formalism. For the particular cases considered, convergence is verified though the results cannot be expressed as a finite sum of elementary functions.

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1. Introduction

In the evaluation of one-loop diagrams (such as the "swordfish" diagrams) in the two-component formalism of the light-cone gauge, one finds integrals of the type

$$K(p,q) = \int \frac{dr}{r^2 (r-q)^2} \frac{1}{(p^+ + r^+)} \tag{1.1}$$

and

$$K^l(p,q) = \int \frac{dr}{r^2 (r-q)^2} \frac{(p^l + r^l)}{(p^+ + r^+)} \quad , \quad l = 1,2 \tag{1.2}$$

where the measure dr according to the dimensional regularization technique is defined over an analytically continued space-time of 2w dimensions.

Right from the start one should be aware that naive shifts of integration variable are not permissible in light-cone integrals which are linearly divergent by power counting assessment [1]. Happily none of the above integrals falls into this category and for convenience I consider the shifted versions

$$\tilde{K}(p,q) = \int \frac{dr}{(r-p)^2 (r-p-q)^2} \frac{1}{r^+} \tag{1.1'}$$

and

$$\tilde{K}^l(p,q) = \int \frac{dr}{(r-p)^2 (r-p-q)^2} \frac{r^l}{r^+} \tag{1.2'}$$

instead of equations (1.1) and (1.2). Here I treat the singularity at $r^+ = 0$ according to the prescription first suggested by S.Mandelstam [2], namely

$$\frac{1}{r^+} = \lim_{\epsilon \to 0^+} \frac{1}{r^+ + i\epsilon r^-} \tag{1.3}$$

2. Evaluation of $\tilde{K}(p,q)$

Using the standard procedure of exponentiating propagators, equation (1.1') becomes

$$\tilde{K}(p,q) = -\int_0^\infty d\alpha\, d\beta\, e^{i(\alpha p^2+\beta q^2)} \int \frac{dr}{r^+}\, e^{ix(r^2+2r\cdot R)} \quad , \tag{2.1}$$

where

$$x = \alpha + \beta \tag{2.2}$$

$$xR = \beta q - \alpha p \ . \tag{2.3}$$

Resolving the momentum integral through the employment of Mandelstam prescription, equation(1.3), one obtains [3]

$$\tilde{K}(p,q) = \frac{(-\pi)^{\omega}\Gamma(2-\omega)}{(p^+ + q^+)} \left\{ (q^2)^{\omega-2} \int_0^1 dy\, \frac{y^{\omega-2}(1-y)^{\omega-2}}{(1-\sigma y)} - (\hat{q}^2)^{\omega-2} \int_0^1 dy\, \frac{(y-\xi)^{\omega-2}(y-\bar{\xi})^{\omega-2}}{(1-\sigma y)} \right\} \tag{2.4}$$

where

$$y = \frac{\alpha}{\alpha+\beta} \tag{2.5}$$

$$\sigma = \frac{q^+}{p^+ + q^+} \tag{2.6}$$

$$\hat{q}^2 = q^1 q^1 + q^2 q^2 = 2q^+ q^- - q^2 \tag{2.7}$$

$$\xi_1 = \frac{(1+v-\rho)+\sqrt{(1+v-\rho)^2-4v}}{2} \qquad (2.8)$$

$$\bar{\xi}_1 = \frac{(1+v-\rho)-\sqrt{(1+v-\rho)^2-4v}}{2} \qquad (2.9)$$

$$v = \frac{2(p^+ + q^+)(p^- + q^-)}{\hat{q}^2} \qquad (2.10)$$

$$\rho = \frac{2p^+p^-}{\hat{q}^2} \qquad (2.11)$$

$$p^\pm = \frac{p^0 \pm p^3}{\sqrt{2}} \qquad (2.12)$$

In order to carry out the y-integration, first I expand the denominator of the integrands in power series

$$\frac{1}{(1-\sigma y)} = \sum_{k=1}^{\infty} (\sigma y)^{k-1} \qquad (2.13)$$

so that now, the y-integrations can be expressed in terms of beta functions and hypergeometric functions of two variables [4]

$$\tilde{K}(p,q) = \frac{(-\pi)^\omega \Gamma(2-\omega)}{(p^+ + q^+)} \sum_{k=1}^{\infty} \sigma^{k-1} \Big[(q^2)^{\omega-2} B(k+\omega-2,\omega-1) - (v\hat{q}^2)^{\omega-2} B(1,k) F_1(k,2-\omega,2-\omega;k+1;\xi^{-1},\bar{\xi}^{-1}) \Big] . \qquad (2.14)$$

Isolating the $k = 1$ term of the sum and employing the following functional relations for the hypergeometric functions [5]

$$
\begin{aligned}
F_1(\alpha,\beta,\beta';\gamma;x,y) &= F_1(\alpha+1,\beta,\beta';\gamma;x,y) \\
&\quad - \frac{\beta}{\gamma} x \, F_1(\alpha+1,\beta+1,\beta';\gamma+1;x,y) \\
&\quad - \frac{\beta'}{\gamma} y \, F_1(\alpha+1,\beta,\beta'+1;\gamma+1;x,y)
\end{aligned}
\qquad (2.15)
$$

$$F_1(\alpha,\beta,\beta';\alpha;x,y) = (1-x)^{-\beta}(1-y)^{-\beta'} \qquad (2.16)$$

$$F_1(\alpha,\beta,0;\gamma;x,y) = {}_2F_1(\alpha,\beta;\gamma;x) \qquad (2.17)$$

$$F_1(\alpha,0,\beta';\gamma;x,y) = {}_2F_1(\alpha,\beta';\gamma;y) \qquad (2.18)$$

and expanding, wherever necessary, for $w \to 2$ one gets

$$
\begin{aligned}
\widetilde{K}(p,q) &= \frac{\pi^2}{(p^+ + q^+)} \left\{ \ln\left(\frac{2p^+p^-}{q^2}\right) - \xi \ln\left(\frac{\xi-1}{\xi}\right) - \bar{\xi} \ln\left(\frac{\bar{\xi}-1}{\bar{\xi}}\right) \right\} \\
&\quad + \frac{\pi^2}{(p^+ + q^+)} \sum_{k=1}^{\infty} \frac{\sigma^k}{(k+1)} \left\{ \frac{2}{(k+1)} + \ln\left(\frac{2p^+p^-}{q^2}\right) + \sum_{m=1}^{k} \frac{1}{m} \right. \\
&\quad \left. + \frac{{}_2F_1(1,k+2;k+3;\xi^{-1})}{(k+2)\,\xi} + \frac{{}_2F_1(1,k+2;k+3;\bar{\xi}^{-1})}{(k+2)\,\bar{\xi}} \right\} + O(2-\omega).
\end{aligned}
\qquad (2.19)
$$

Furthermore, using the expansion [4]

$$\frac{{}_2F_1(1,k+2;k+3;z)}{(k+2)} = -\frac{\ln(1-z)}{z^{(k+2)}} - \sum_{m=0}^{k} \frac{z^{m-k-1}}{(1+m)} \qquad (2.20)$$

the final expression for $\widetilde{K}$ is written as

$$
\begin{aligned}
\tilde{K}(pq) = {} & \frac{\pi^2}{(p^+ + q^+)} \left\{ \ln\left(\frac{2p^+p^-}{q^2}\right) - \xi \ln\left(\frac{\xi - 1}{\xi}\right) - \bar{\xi} \ln\left(\frac{\bar{\xi} - 1}{\bar{\xi}}\right) \right\} \\
& + \frac{\pi^2}{(p^+ + q^+)} \sum_{k=1}^{\infty} \frac{\sigma^k}{(k+1)} \left\{ \frac{2}{(k+1)} + \ln\left(\frac{2p^+p^-}{q^2}\right) - \xi^{k+1} \ln\left(\frac{\xi - 1}{\xi}\right) - \bar{\xi}^{k+1} \ln\left(\frac{\bar{\xi} - 1}{\bar{\xi}}\right) \right. \\
& \left. + \sum_{m=1}^{k} \frac{1}{m} - \sum_{m=0}^{k} \frac{(\xi^{k-m} + \bar{\xi}^{k-m})}{(1+m)} \right\} + O(2-\omega) \qquad (2.21)
\end{aligned}
$$

3. Evaluation of $\tilde{K}^{l}(p,q)$

The evaluation of $\tilde{K}^{l}(p,q)$ follows in a completely analogous way. After the y-integrations are performed the result is

$$\tilde{K}^{l}(p,q) = (p^{l}+q^{l})\tilde{K}(p,q) - \pi^{2}\frac{q^{l}}{(p^{+}+q^{+})}\Gamma(2-\omega)\sum_{k=1}^{\infty}\sigma^{k-1}\Big[(-\pi q^{2})^{\omega-2}B(k+\omega-1,\omega-1)$$

$$-(-\pi v q^{2})^{\omega-2}B(1,k+1)F_{1}(k+1,2-\omega,2-\omega;k+2;\xi^{-1},\bar{\xi}^{-1})\Big] \qquad (2.22)$$

which yields

$$\tilde{K}^{l}(p,q) = (p^{l}+q^{l})\tilde{K}(p,q) - \pi^{2}\frac{q^{l}}{(p^{+}+q^{+})}\sum_{k=1}^{\infty}\frac{\sigma^{k-1}}{(k+1)}\Bigg\{\frac{2}{(k+1)}+\ln\left(\frac{2p^{+}p^{-}}{q^{2}}\right)-\xi^{k+1}\ln\left(\frac{\xi-1}{\xi}\right)$$

$$-\bar{\xi}\ln\left(\frac{\bar{\xi}-1}{\bar{\xi}}\right)+\sum_{m=1}^{k}\frac{1}{m}-\sum_{m=0}^{k}\frac{(\xi^{k-m}+\bar{\xi}^{k-m})}{(1+m)}\Bigg\}+O(2-\omega). \qquad (2.23)$$

4. Concluding Remarks

The explicit calculation showed that the integral defined in equation (1.2) has its pole part cancelled out. It suggests that in the Mandelstam prescription the transversal components of the vector momentum, $r^{\perp}$, over the longitudinal component, r^{+}, yield $n < 0$ as far as power counting is concerned. There remains to be seen whether the lower limit for n can be determined accurately. In any case, it is interesting to note that this pattern of convergence for momentum integrals in the light-cone gauge "à la" Mandelstam is characteristic of this gauge.

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