ANL-HEP-CP-87-15
UIC-87-14
March 10, 1987

ANL-HEP-CP--87-15

DE87 011482

UNORTHODOX LATTICE FERMION DERIVATIVES AND THEIR SHORTCOMINGS*†

GEOFFREY T. BODWIN
High Energy Physics Division
Argonne National Laboratory
Argonne, IL 60439

EVE V. KOVACS
Department of Physics
University of Illinois at Chicago
Chicago, Illinois 60680

Abstract

We discuss the DWY (Lagrangian), Quinn-Weinstein, and Rebbi proposals for incorporating fermions into lattice gauge theory and analyze them in the context of weak coupling perturbation theory. We find that none of these proposals leads to a completely satisfactory lattice transcription of fully-interacting gauge theory.

*Talk presented by Eve V. Kovacs at the 1987 DPF Meeting at Salt Lake City, Utah.

†Work supported by the U.S. Department of Energy, Division of High Energy Physics, Contract W-31-109-ENG-38.

1. INTRODUCTION

In recent years, much effort has been devoted to finding lattice fermion formulations that solve the spectrum doubling problem without destroying chiral invariance. Three candidate formulations of current interest are the DWY (SLAC) derivative[1], the Quinn-Weinstein variant of the DWY derivative[2], and the recent proposal of Rebbi.[3] While all of these schemes eliminate the spectrum doubling in the case of free fermions, it is unclear whether they represent satisfactory lattice transcriptions in the case of a fully-interacting gauge theory. In the present work, we attempt to resolve this issue by making use of perturbation theory—in some cases to all orders—to analyze the various proposals.

2. REVIEW OF THE DOUBLING PROBLEM

The basic problem in transcribing any continuum field theory onto the lattice is to replace derivatives by an appropriate finite difference operator. Three local operators that yield the derivative in the continuum limit are

$$\nabla^+_\mu f(x) = \frac{1}{a}[f(x+a_\mu) - f(x)] \tag{1a}$$

$$\nabla^-_\mu f(x) = \frac{1}{a}[f(x) - f(x-a_\mu)] \tag{1b}$$

$$\nabla^\pm_\mu f(x) = \frac{1}{2a}[f(x+a_\mu) - f(x-a_\mu)], \tag{1c}$$

where a_μ is a unit vector in the μ-direction whose length is equal to the lattice spacing a. The operators ∇^+ and ∇^- are the nearest neighbor forward and backward differences, respectively; $\nabla^\pm$ is the second-nearest neighbor difference. If we take bilinears in these operators to form a lattice Laplacian, the only Hermitian combinations are $\sum_\mu(\nabla^\pm_\mu)^2$ and $\sum_\mu \nabla^+_\mu \nabla^-_\mu$. The momentum space representations of these operators are as follows:

$$-\sum_\mu(\nabla^\pm_\mu)^2 \longrightarrow \frac{1}{a^2}\sum_\mu \sin^2(k_\mu a) \tag{2a}$$

$$-\sum_\mu \nabla^+_\mu \nabla^-_\mu \longrightarrow \frac{1}{a^2}\sum_\mu 4\sin^2(\tfrac{1}{2}k_\mu a). \tag{2b}$$

In Fig. 1 we have plotted the square root of these momentum space representations for one component of momentum k_μ. For both of these finite difference Laplacians, the region near $k_\mu = 0$ is linear and therefore yields the usual continuum dispersion relation. However, for $(\nabla^\pm_\mu)^2$ there is an additional zero-crossing at $k_\mu = \pm\pi/a$. In the neighborhood of this extra zero-crossing one also obtains the continuum dispersion relation. Thus, this region of momentum space could potentially give rise to extra particle species in the continuum limit. Of course, $(\nabla^+_\mu\nabla^-_\mu)^{1/2}$ does not suffer from the problem of exta zero crossings, and it provides a satisfactory lattice transcription for the Klein-Gordon operator.

In the case of the continuum Dirac operator, $\sum_\mu -i\gamma_\mu\partial_\mu + m$, one has an expression that is linear in the derivative. (Throughout this paper we use a Euclidean formulation of field theory; the γ's are the anti-Hermitian Dirac matrices satisfying $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$.)

The most obvious lattice transcription is achieved by substituting one of the lattice derivatives of Eq. (1). However, the only Hermitian choice is the second-nearest neighbor operator $\nabla^{\pm}_{\mu}$ of Eq. (1c), which yields

$$\sum_{\mu} -i\gamma_{\mu}\partial_{\mu} + m \longrightarrow \frac{1}{a}\sum_{\mu}\gamma_{\mu}\sin(k_{\mu}a) + m. \tag{3}$$

Since $\nabla^{\pm}_{\mu}(k)$ has zeros at $k_{\mu} = 0$ and $k_{\mu} = \pm\pi/a$, this so-called 'naive' fermion derivative leads to a doubled spectrum, with 2^d species occurring in d dimensions. For example, in two dimensions there are four species, which come from the momentum regions (k_0, k_1) near $(0,0)$, $(0,\pi/a)$, $(\pi/a,0)$, and $(\pi/a,\pi/a)$.

One solution to the spectrum doubling problem was proposed by Wilson[4] and involves adding to the naive derivative an extra momentum dependent term $2/a\sin^2(k_{\mu}a/2)$ as follows:

$$\sum_{\mu} -i\gamma_{\mu}\partial_{\mu} + m \longrightarrow \frac{1}{a}\sum_{\mu}[\gamma_{\mu}\sin(k_{\mu}a) + 2\sin^2(\tfrac{1}{2}k_{\mu}a)] + m. \tag{4}$$

This extra term vanishes in the continuum limit for any finite k, but gives the extra species from k_{μ} near $\pm\pi/a$ a mass proportional to $1/a$. Hence, the extra species decouple in the continuum limit. Unfortunately, the Wilson mass term explicitly breaks the chiral symmetry that is present in the massless continuum theory. Hence, the Wilson fermions are not protected from mass renormalization even in the limit of zero bare mass. Therefore, in order to maintain a vanishing renormalized mass, one must adjust the bare mass parameter as the limit $a \to 0$ is taken . It turns out that in the continuum limit the Wilson derivative yields the correct chiral anomaly.

The Kogut-Susskind[5] derivative is something of a compromise between the Wilson and naive derivatives. It partially eliminates the species doubling by distributing the components of the Dirac spinor across several lattice sites. Some doubling still occurs because certain of the spinor degrees of freedom are reinterpreted as extra flavors. The full chiral symmetry of the theory is now a *flavor* chiral symmetry. In the Kogut-Susskind scheme, only one flavor nonsinglet component of this symmetry is realized. The momentum space representation of the Dirac operator contains a Wilson-like mass term that has been multiplied by γ_5 and a flavor matrix τ_{μ}:

$$\sum_{\mu} -i\gamma_{\mu}\partial_{\mu} + m \longrightarrow \frac{1}{a}\sum_{\mu}[\gamma_{\mu}\sin(k_{\mu}a) - 2\tau_{\mu}\gamma_5\sin^2(\tfrac{1}{2}k_{\mu}a)] + m. \tag{5}$$

Attempts to find a chirally symmetric lattice theory with an undoubled spectrum and the correct chiral anomaly have led to a number of 'no-go' theorems. These theorems fall into two general categories. The first category is typified by the theorems of Nielsen and Ninomiya[6,7] and Karsten and Smit.[8] Loosely speaking, they conclude that any local lattice derivative either breaks chiral invariance or exhibits spectrum doubling. This result can be understood intuitively by trying to construct a momentum space version of the lattice derivative that passes linearly through zero at $k_{\mu} = 0$. Since a derivative that is local in position space must have a continuous Fourier transform, the lattice periodicity implies that the derivative function in momentum space must have at least one additional zero crossing. The second category of 'no-go' theorem is typified by the result of Ninomiya and Tan.[8] They use topological arguments to prove that a lattice theory with a continuous chiral symmetry and the correct anomaly is impossible. The properties of the naive, Wilson, and Kogut-Susskind derivatives are all compatible with the 'no-go' theorems. The proposals for the fermion derivative that are the subject of this paper all circumvent the first category of 'no-go' theorem by virtue of their nonlocality. Even the applicability of the second category of 'no-go' theorem could be in doubt, owing to the singular nature of the continuum limit or the peculiar chiral structure of these lattice schemes.

3. RESULTS

3.1 The DWY (SLAC) Derivative

The momentum space representation of the DWY derivative, $D^{DWY}(k)$, is shown in Fig. 2. It duplicates the continuum derivative within the boundaries of the Brillouin zone, but, owing to the requirement of lattice periodicity, it is discontinuous at the zone boundaries. These discontinuities imply that the DWY derivative is non-local in position space. Hence, it evades the first category of 'no-go' theorem. The DWY derivative was originally proposed in the context of a Hamiltonian formulation. Its extension to the Lagrangian formulation of lattice field theory has been discussed by Rabin.[10] All of the results that we report here are derived from the Lagrangian formulation.

The discontinuous nature of the DWY derivative is a potential source of difficulty when one takes the limit $a \to 0$. Indeed, Karsten and Smit[11,12], found that the one loop contributions to the vacuum polarization and the anomaly in weak-coupling perturbation theory contain non-covariant and non-renormalizable ultraviolet divergences that arise from the discontinuities. The peculiar nature of the continuum limit has cast some doubt on the applicability of the second category of 'no-go' theorem.

Rabin[10] has proposed that the difficulties encountered by Karsten and Smit are not fundamental to the DWY Lagrangian theory. Rather, he suggests, they are artifacts of the perturbation expansion that can be removed through a partial resummation of that expansion. The resummation has the effect of softening the discontinuities in the DWY derivative, as indicated, for example, by the dashed curve in Fig. 3. We denote this 'smeared' DWY derivative by $\not{D}^{DWY}(k)$. Because $\not{D}^{DWY}(k)$ is a continuous function, it has an extra zero-crossing at $k_\mu = \pm\pi/a$, and thus could lead to spectrum doubling. However, in the limit $a \to 0$, the slope at $k_\mu = \pm\pi/a$ can, in general, tend to infinity, so the final outcome is not obvious.

Our approach[13] is to analyze the DWY derivative in the context of a model that is completely soluable in the continuum, namely the Schwinger model[14] (two-dimensional quantum electrodynamics). We find, as expected, that the lattice version of this model is also completely soluable in the continuum limit. Our analysis makes use of Rabin's resummation procedure as a technical device to control singularities that would otherwise appear in some of the amplitudes. In this study we examine the mass gap, the anomaly, and the chiral order parameter $\langle\overline{\psi}\psi\rangle$. The lattice diagrams that one needs to consider in order to evaluate these quantities are shown in Figs. 4-6, respectively. In Figs. 4(a) and 4(b) we present the nonseagull and seagull graphs, respectively, that contribute to the vacuum polarization and the anomaly in two dimensions. The anomaly graphs are obtained by inserting a factor γ_5 to the right of one of the vertices in the vacuum polarization graphs. This insertion is indicated by a 5 in the figures. In Fig. 5 we show the complete photon propagator, which, as usual, is obtained from a geometric series involving the vacuum polarization. This series gives rise to the Schwinger model mass gap. Fig. 6 contains the graphs corresponding to the four-point function $\langle\overline{\psi}(x)\psi(x)\overline{\psi}(0)\psi(0)\rangle$, from which one can determine $\langle\overline{\psi}\psi\rangle$. The blobs on the photon lines denote complete photon propagators. For the details of these calculations we refer the reader to Ref. 13. Here we merely present a few of the essential features.

In the Schwinger model, the 'smeared' DWY derivative $D^{DWY}(k)$ takes on precisely the form shown in Fig. 3. The singularity at $k_\mu = \pm\pi/a$ is smoothed out over a region of width $e\ln^{1/2}(1/\mu a)$, where e is the coupling constant and μ is the mass gap. In the limit $a \to 0$, the width of the smeared region tends to infinity, but not as rapidly as the width of the Brillouin zone, which grows like $1/a$. Hence, the slope of $D^{DWY}(k)$ at $k_\mu = \pm\pi/a$ tends to infinity in the continuum limit.

As explained in detail in Ref. 13, the only superficially divergent graphs in the continuum Schwinger model are the $O(e^2)$ fermion loops. In a lattice model these graphs are potentially logarithmically divergent in the limit $a \to 0$. However, the graphs of interest contain at least one vector vertex, and if they are evaluated at zero external momentum, these graphs vanish by virtue of gauge invariance. We can, therefore, subtract from the potentially divergent graphs their values at zero external momentum without changing the final result. In the subtracted graphs the potential for logarithmic divergences is removed. Then, for all the graphs in the theory, detailed power counting arguments show that, for any smooth lattice derivative D_μ, the leading contributions in the limit $a \to 0$ come from the linear regions in the neighborhoods of the zero-crossings of the D_μ. In particular, near the ith zero-crossing, located at $k_\mu = \overline{k}_{i\mu}$ we may approximate the fermion propagator as follows:

$$\frac{1}{\sum_\mu \gamma_\mu D_\mu(k) + m} = \frac{1}{\sum_\mu c_{i\mu}\gamma_\mu(k_\mu - \overline{k}_{i\mu}) + m}, \tag{6a}$$

where

$$c_{i\mu} = D'_\mu(k)\big|_{k=\overline{k}_i} \tag{6b}$$

is the slope of the lattice derivative function at the zero-crossing. In particular, for $D^{DWY}(k)$ in the limit $a \to 0$, $c_{i\mu} = 1$ for $\overline{k}_{i\mu} = 0$ and $c_{i\mu} \to -\infty$ for $\overline{k}_{i\mu} = \pm\pi/a$. One might think that this infinite coefficient in the denominator for k_μ near $\pm\pi/a$ would suppress the contributions from the extra species. However, the coupling to the gauge field can also become large near $k_\mu = \pm\pi/a$. To see this, note that, in general, gauge invariance requires that the $O(e)$ fermion-photon vertex be of the form

$$V_\mu^{(1)}(k,l) = -e\gamma_\mu \frac{D_\mu(k+l) - D_\mu(k)}{(2/a)\sin(\frac{1}{2}l_\mu a)}, \tag{7}$$

where, as shown in Fig. 4(a), k is the fermion momentum, l is the photon momentum, and μ is the index that carries the photon's polarization. For k_μ near $\overline{k}_{i\mu}$, this vertex becomes

$$V_\mu^{(1)}(k,l) \approx -e\gamma_\mu c_{i\mu}. \tag{8}$$

We see that in the case of the 'smeared' DWY derivative, for k_μ near $\pm\pi/a$, the large coefficients cancel between the propagators and the vertices. Thus, each of the species associated with (k_0,k_1) near $(0,0)$, $(0,\pi/a)$, $(\pi/a,0)$, and $(\pi/a,\pi/a)$ could, in principle, contribute to the amplitudes.

In Table 1 we show the results of our Schwinger model analysis for the DWY derivative, along with some results obtained from similar analyses of the Wilson, naive, and Kogut-Susskind derivatives. The quantities given in the table are the constants of proportionality between the lattice expressions and the continuum ones. It turns out that the value of the vacuum polarization $\Pi_{\mu\nu}$ controls both the mass gap and $\langle\overline{\psi}\psi\rangle$, with the mass gap being proportional to the square root of the vacuum polarization and $\langle\overline{\psi}\psi\rangle$ vanishing unless the vacuum polarization takes on the continuum value.

Table 1. The values of $\Pi_{\mu\nu}$, the mass gap, the anomaly, and the chiral order parameter $\langle\overline{\psi}\psi\rangle$ obtained from the continuum limits of the various lattice versions of the Schwinger model.

Derivative	$\Pi_{\mu\nu}$	Mass Gap	Anomaly	$\langle\overline{\psi}\psi\rangle$
Wilson	1	1	1	1
Naive	4	2	0	0
Kogut-Susskind	2	$\sqrt{2}$	2	0
DWY	2	$\sqrt{2}$	0	0

As expected, the Wilson theory correctly reproduces the continuum results.

The naive lattice theory exhibits the anticipated spectrum doubling: each of the four species contribute to the vacuum polarization an amount equal to the continuum vacuum polarization, thus doubling the mass gap; the species from (k_0,k_1) near $(0,0)$ and $(\pi/a,\pi/a)$, which have positive chiral charge, each contribute to the anomaly graph an amount that is equal to the continuum contribution, while the species from (k_0,k_1) near $(0,\pi/a)$ and $(\pi/a,0)$, which have negative chiral charge, each contribute an amount that is equal in magnitude and opposite in sign to the continuum contribution.

Owing to its two flavors, the Kogut-Susskind derivative also displays spectrum doubling, but with only half as many species as the naive derivative. Somewhat surprisingly, the flavor singlet anomaly is also doubled, but $\langle\bar{\psi}\psi\rangle$ vanishes. This is a consequence of the fact that the remnant of chiral symmetry in the Kogut-Susskind theory is a continuous flavor *nonsinglet* chiral symmetry. Thus, the flavor *nonsinglet* axial current, rather than the flavor *singlet* axial current, is conserved in the Kogut-Susskind theory. On the other hand, since $\langle\bar{\psi}\psi\rangle$ is rotated by transformations under a continuous symmetry (i.e. the flavor nonsinglet symmetry), it cannot develop spontaneously a vacuum expectation value in two dimensions.

The DWY derivative also exhibits spectrum doubling, but it turns out that only the species from (k_0, k_1) near $(0,0)$ and $(\pi/a, \pi/a)$ contribute to the vacuum polarization—each yielding a contribution equal to the continuum one. In the case of the anomaly graph, all four species contribute to give the following result:

$$A_{\mu\nu}^{DWY}(l) = \frac{2e^2}{\pi}\left(\epsilon_{\mu\nu} - \frac{\sum_\alpha \epsilon_{\mu\alpha} l_\alpha l_\nu}{l^2}\right) - \frac{2e^2}{\pi}\left(\epsilon_{\mu\nu} - \frac{\sum_\alpha \epsilon_{\mu\alpha} l_\alpha l_\nu}{l_\nu^2}\right). \tag{9}$$

Here μ is the index associated with the axial vector current and ν is the index associated with the vector current. The first term in parentheses comes from the species with (k_0, k_1) near $(0,0)$ and $(\pi/a, \pi/a)$. Each gives a contribution that is equal to the continuum one. The second term in parentheses is noncovariant, even in the continuum limit, and arises from the species with (k_0, k_1) near $(0, \pi/a)$ and $(\pi/a, 0)$. The noncovariance is a reflection of the fact that these species correspond to particles for which the velocity of light is either infinite or infinitessimal in the limit $a \to 0$. Note, however, that

$$\sum_\mu k_\mu A_{\mu\nu}^{DWY} = 0, \tag{10}$$

with the contributions of the noncovariant species cancelling the contributions of the covariant ones. Thus the axial vector current has no anomaly. This cancellation of the anomaly by infinite velocity species was anticipated by Ninomiya and Tan.[8] Our results confirm that their 'no-go' theorem holds even in the case of the DWY derivative with its tricky continuum limit.

3.2 The Quinn-Weinstein Proposal

Now let us discuss the recent proposal of Quinn and Weinstein[2] for eliminating fermion species doubling in lattice gauge theory. The Quinn-Weinstein proposal is based on a Hamiltonian formulation in which the gauge has been fixed according to the condition $A_0 = 0$. In the case of a non-Abelian gauge theory, the Quinn-Weinstein Hamiltonian is quite complicated, involving line integrals of the gauge field over all possible paths between the lattice points on which the fermion fields reside. In order to avoid these complications, we discuss here the Abelian theory, which is much simpler.

One can think of the Hamiltonian formulation of lattice gauge theory as a Lagrangian formulation in which time is continuous. Thus, the time components of the loop momenta range from $-\infty$ to ∞ and the time components of the vertices and the terms in the propagators that depend on the time components of momentum are give by the continuum expressions.

For the terms in the fermion propagators that depend on the spatial components of momentum. Quinn and Weinstein suggest a modified version of the DWY derivative function. In momentum space, this function reproduces the linear behavior of the continuum derivative at the origin and also crosses zero at $k_\mu = \pm\pi/a$ with a slope that tends to $-\infty$ in the continuum limit. It is very similar in form to the 'smeared' DWY derivative function that we discussed in conjunction with Rabin's procedure in Section 3.1.

In the Quinn-Weinstein Hamiltonian, the spatial components of the fermion-photon vertices couple only to a transverse photon. Quinn and Weinstein assert that the forms of these spatial vertices can be chosen independently of the form of the fermion propagator. That is, they claim that the interactions need not necessarily derive from the simplest gauge invariant combinations of fermion bilinears and link variables that one obtains from the lattice generalization of minimal substitution. In particular, Quinn and Weinstein choose for the spatial vertices the nearest neighbor form, i.e., in $O(e)$,

$$V_\mu^{(1)}(k,l) = -e\gamma_\mu \cos(k_\mu + \tfrac{1}{2}l_\mu)a. \tag{11}$$

This choice is motivated by the desire to eliminate the infinite vertex coefficient for k_μ near $\pm\pi/a$ that one usually encounters in the DWY formulation (see Section 3.1). In the absence of an infinite vertex coefficient for k_μ near $\pm\pi/a$, one would expect the infinite coefficient in the propagator denominator to suppress the extra fermion species.

Since, in the Quinn-Weinstein scheme, the current to which the transverse photon couples is itself transverse, the Hamiltonian is manifestly invariant with respect to the residual set of gauge transformations that respect the condition $A_0 = 0$. However, the Quinn-Weinstein vertices do not satisfy the usual vector Ward identities that derive from the lattice generalization of minimal substitution. For example, the Feynman identity, Eq. 7, does not hold. Recall that it is these Ward identities that allow one to see that the actual degree of divergence of the vacuum polarization is not as high as the superficial degree of divergence that one obtains by simple power-counting. For example, by application of the Ward identities it follows that the actual degree of divergence of the vacuum polarization is only logarithmic, not quadratic, in four dimensions and that the vacuum polarization is convergent, not logarithmically divergent, in two dimensions. In the Quinn-Weinstein formulation one might expect, then, that the vacuum polarization would show the superficial degree of divergence that one obtains by simple power counting.

We have investigated the leading divergence of the vacuum polarization for the Quinn-Weinstein formulation by direct calculation. In two dimensions, because of a special property of the Hamiltonian formulation, the vacuum polarization actually turns out to be finite, rather than logarithmically divergent. The time components, of course, reproduce the continuum result. For the spatial components, the term proportional to δ_{ij} appears with a coefficient that is twice the continuum one. However, this is of no consequence, since in two dimensions there is no transverse photon to couple to the spatial components of the current. In four dimensions, we have computed the most divergent part of the spatial components of the vacuum polarization and find that it is given by

$$\Pi_{ij} \sim \delta_{ij} a^{-2} \tag{12}$$

in the limit $a \to 0$.

One can interpret this divergent quantity as a transverse photon mass renormalization. It could, in principle be eliminated by adding to the Hamiltonian a Pauli-Villars fermion, whose mass would tend to infinity in the continuum limit. Of course, this would defeat the original intent of the Quinn-Weinstein proposal by explicitly breaking chiral invariance. Alternatively, one could introduce a photon mass counterterm, which would be adjusted in the limit $a \to 0$ so as to keep the photon massless. (Presumably, one would need a similar counterterm in order to deal with a logarithmic divergence in the light-by-light scattering graph.) In the case of numerical simulations, one could conceivably implement such a program for an Abelian theory. It is difficult to see how a criterion for adjusting the counterterms could be set up in the non-Abelian case. In fact, the usual continuum renormalization program would probably break down in the presence of an explicit gluon mass.

3.3 The Rebbi Proposal

Finally, let us discuss the recent proposal of Rebbi[3)] for lattice gauge theory. The basis for Rebbi's proposal is that, in the rationalized fermion propagator

$$\frac{1}{\not k + m} = \frac{-\not k + m}{k^2 + m^2}, \tag{13}$$

one need not use the same lattice derivative operator for the numerator and denominator. In particular, Rebbi chooses for the numerator the second-nearest neighbor derivative and for the denominator the nearest neighbor Laplacian:

$$\frac{-\not k + m}{k^2 + m^2} \longrightarrow \frac{\sum_\mu i\gamma_\mu \nabla^{\pm}_\mu + m_{1/2}}{-\sum_\mu \nabla^{+}_\mu \nabla^{-}_\mu + m_0^2}. \tag{14}$$

For reasons which will become apparent later, we have generalized the propagator by allowing for the possibility of different masses in the numerator and denominator. We have denoted the mass in the spinor-like numerator with a subscript 1/2 and the mass in the scalar-like denominator with a subscript 0. Since the nearest neighbor Laplacian in the denominator has a zero-crossing only at $k = 0$ (see Fig. 1), one might hope that Rebbi's formulation would not exhibit spectrum doubling.

Couplings to a gauge field can be introduced by the lattice generalization of minimal substitution. For the numerator derivative,

$$\nabla^{\pm}_\mu \psi(x) \longrightarrow \nabla_\mu \psi(x) = \frac{1}{2a}[U^\dagger_\mu(x)\psi(x + a_\mu) - U_\mu(x - a_\mu)\psi(x - a_\mu)]. \tag{15a}$$

Here $U_\mu(x)$ is the usual lattice gauge field link operator that starts at site x and ends at site $x + a_\mu$. For the denominator Laplacian,

$$\nabla_\mu^+ \nabla_\mu \psi(x) \longrightarrow (\Delta - e \sum_{\mu\nu} \sigma_{\mu\nu} F_{\mu\nu}) \psi(x), \tag{15b}$$

where

$$\sigma_{\mu\nu} = (i/4)[\gamma_\mu, \gamma_\nu], \tag{15c}$$

$$\Delta\psi(x) = \frac{1}{a^2} \sum_\mu [U_\mu^\dagger(x)\psi(x + a_\mu) + U_\mu(x - a_\mu)\psi(x - a_\mu) - 2\psi(x)], \tag{15d}$$

and $eF_{\mu\nu}$ is given by any expression with the correct continuum limit. We take the second-nearest neighbor form

$$eF_{\mu\nu} = -i[\nabla_\mu, \nabla_\nu]. \tag{15e}$$

In order to preserve chiral symmetry, we must have a vanishing mass $m_{1/2}$ in the spinor numerator of Rebbi's propagator (Eq. (14)). On the other hand, the mass m_0 in the scalar denominator is on the same footing with respect to chiral symmetry as the Laplacian, and it does not break chiral invariance. Thus, if we start with a massless bare theory ($m_{1/2} = m_0 = 0$), there is no symmetry to prevent quantum corrections from dynamically generating a mass m_0.

In order to test for this possibility, we have calculated the leading behavior in four dimensions of the $O(e^2)$ fermion self-energy in the limit of vanishing external momentum p (see Fig. 7). In the continuum theory, the leading behavior of the self-energy with external legs is given by

$$\sim \frac{\delta m}{p^2}, \tag{16}$$

where δm is the logarithmically divergent fermion mass correction, which vanishes in the limit of zero bare fermion mass. That is, the fermion mass is protected from dynamical generation by the chiral symmetry. The two powers of $1/p$ come from the external fermion legs, as shown in Fig. 7(b). In the case of the Rebbi propagator, the leading contributions to the fermion self-energy with external legs come from the graphs shown in Fig. 8(a). The leading behavior of these graphs is given by

$$\sim \not{p} \frac{1}{p^2} \delta m^2 \frac{1}{p^2} \sim \frac{\delta m^2}{p^3}, \tag{17a}$$

where

$$\delta m^2 \sim 1/a^2. \tag{17b}$$

We can interpret the leading behavior as being due to a quadratically divergent correction to the scalar mass m_0. (See Fig. 8(b).) As a check, we have verified that if one replaces the denominator of Eq. (14) with the second-nearest neighbor form, so that one has, in effect, the 'naive' derivative, then δm^2 vanishes.

Our calculation is similar to one carried out by Longhitano and Svetitsky[19] in connection with a second-order fermion formulation. They considered forms for $eF_{\mu\nu}$ other than the second-nearest neighbor one that we use. Based on their work, we would expect alternate forms for $eF_{\mu\nu}$ to change the coefficient of the mass correction, but not the general conclusion that there is dynamical mass generation. If one *could* find a form for $eF_{\mu\nu}$ that produced no mass generation in $O(e^2)$, then, in the absence of a protective chiral symmetry, it would almost certainly generate mass corrections in higher orders.

We conclude that, in approaching the continuum limit in the Rebbi formalism, one would need to adjust the bare scalar mass m_0 in order to maintain a zero physical mass. Furthermore, since the quantity $eF_{\mu\nu}$ is a gauge invariant, the couplings to the gauge field contained within it could, in general, be renormalized by the quantum corrections in such a way that they do not 'track' with the gauge couplings contained in Δ. Thus, the strength of the couplings contained in $eF_{\mu\nu}$ would also need to be adjusted as the continuum limit is taken. This complicated approach to the continuum would make the Rebbi formulation a rather awkward basis for lattice simulations.

4. Conclusions

We present here a table which summarizes our results for the three fermion derivative proposals that we have considered in this paper. It is clear from these results that one can learn much about the behavior of various lattice fermion schemes by subjecting them to some simple perturbative analyses. Such perturbative tests might be useful minimal criteria for judging the utility of future lattice fermion proposals.

Table 2. Summary of Results

Derivative	Calculation	Result
DWY (SLAC) (Lagrangian)	Schwinger model (all orders)	spectrum doubling anomaly$=0$ $\langle\overline{\psi}\psi\rangle=0$
Quinn-Weinstein	Π_{ij} in 4-d	$\sim\delta_{ij}a^{-2}$ gauge field mass term
Rebbi	fermion self-energy	dynamically generated fermion mass

After this material was presented at the DPF meeting, we obtained some new results with regard to the Rebbi proposal.[16] A detailed examination of the vacuum polarization in two dimensions reveals that fermion doubling actually arises in a subtle way. This phenonmenon occurs because the expressions for fermion loops in terms of the complete fermion propagator with external legs contain inverse powers of the free fermion propagator. As a result, the second-nearest neighbor numerator in Rebbi's propagator can appear in the denominator of the loop expressions, and thereby cause doubling.

We wish to acknowledge helpful discussions with G. P. Lepage and D. K. Sinclair. One of us (E. K.) wishes to thank the Argonne high energy theory group for its hospitality.

5. REFERENCES

1. Drell, S. D., Weinstein, M. and Yankielowicz, S., Phys. Rev. D 14, 487 (1976); 14, 1627 (1976).
2. Quinn, H. R. and Weinstein, M., Phys. Rev. D 34, 2440 (1986).
3. Rebbi, C., *Chiral-invariant Regularization of Fermions on the Lattice*, Brookhaven preprint BUHEP-86-6.
4. Wilson, K. G., Phys. Rev. D 10, 2445 (1974); in *New Phenomena in Subnuclear Physics*, edited by A. Zichichi (Plenum, New York, 1977).
5. Kogut, J. and Susskind, L., Phys. Rev. D 11, 395 (1975).
6. Nielsen, H. B. and Ninomiya, M., Nucl. Phys. B185, 20 (1981); Nucl. Phys. B193, 173 (1981).
7. Nielsen, H. B. and Ninomiya, M., Phys. Lett. 105B, 219 (1981).
8. Karsten, L. H. and Smit, J., Nucl. Phys. B183, 103 (1981).
9. Ninomiya, M. and Tan, C.-I, Phys. Rev. Lett. 53, 1611 (1984); in *Q.C.D. and Beyond*, edited by J. Tran Thanh Van (Kim Hup Lee, Singapore, 1985).
10. Rabin, J. M., Phys. Rev. D24, 3218 (1981).
11. Karsten, L. H. and Smit, J., Nucl. Phys. B144, 536 (1978).
12. Karsten, L. H. and Smit, J., Phys. Lett. 85B,100 (1979).
13. Bodwin, G. T. and Kovacs, E. V., *Lattice Fermions in the Schwinger Model*, Argonne preprint ANL-HEP-PR-86-96, October, 1986. To be published in Phys. Rev. D.
14. Schwinger, J., Phys. Rev. 128, 2425 (1962).
15. Longhitano, A. C. and Svetitsky, B., Phys. Lett. 126B,259 (1983).
16. Bodwin, G. T. and Kovacs, E. V., in preparation.

6. FIGURE CAPTIONS

1. The momentum space representations of the operators (a) $-i\nabla_\mu^\pm$ and (b) $(-\nabla_\mu^+\nabla_\mu^-)^{1/2}$.
2. The DWY derivative function in momentum space.
3. The 'smeared' DWY derivative function in momentum space.

4. The Feynman graphs that give the vacuum polarization and the anomaly in the lattice Schwinger model. A solid line represents a fermion propagator and a wavy line represents a photon propagator. (a) The non-seagull graph. (b) The seagull graph. The numeral 5 indicates that in the anomaly graphs there is a factor γ_5 to the right of one of the vertices.

5. The geometric series of vacuum polarization bubbles that gives the complete photon propagator.

6. The expansion of $\langle\overline{\psi}(x)\psi(x)\overline{\psi}(0)\psi(0)\rangle$ according to the number of complete photon propagators. A wavy line with a blob on it represents a complete photon propagator.

7. (a) The $O(e^2)$ fermion self-energy graph for the continuum theory. (b) The leading behavior of that graph in the limit of vanishing external momentum p.

8. (a) The graphs in the Rebbi formulation that give the leading contributions to the $O(e^2)$ fermion self-energy in the limit of vanishing external momentum p. A dashed line represents the spinor-like numerator of the Rebbi propagator and a double line represents the scalar-like denominator. The photon-scalar vertices with an asterisk come from the Pauli term $e\sigma_{\mu\nu}F_{\mu\nu}$, while those without an asterisk come from the ordinary scalar term Δ. (b) The leading behavior of these graphs in the limit of vanishing p.

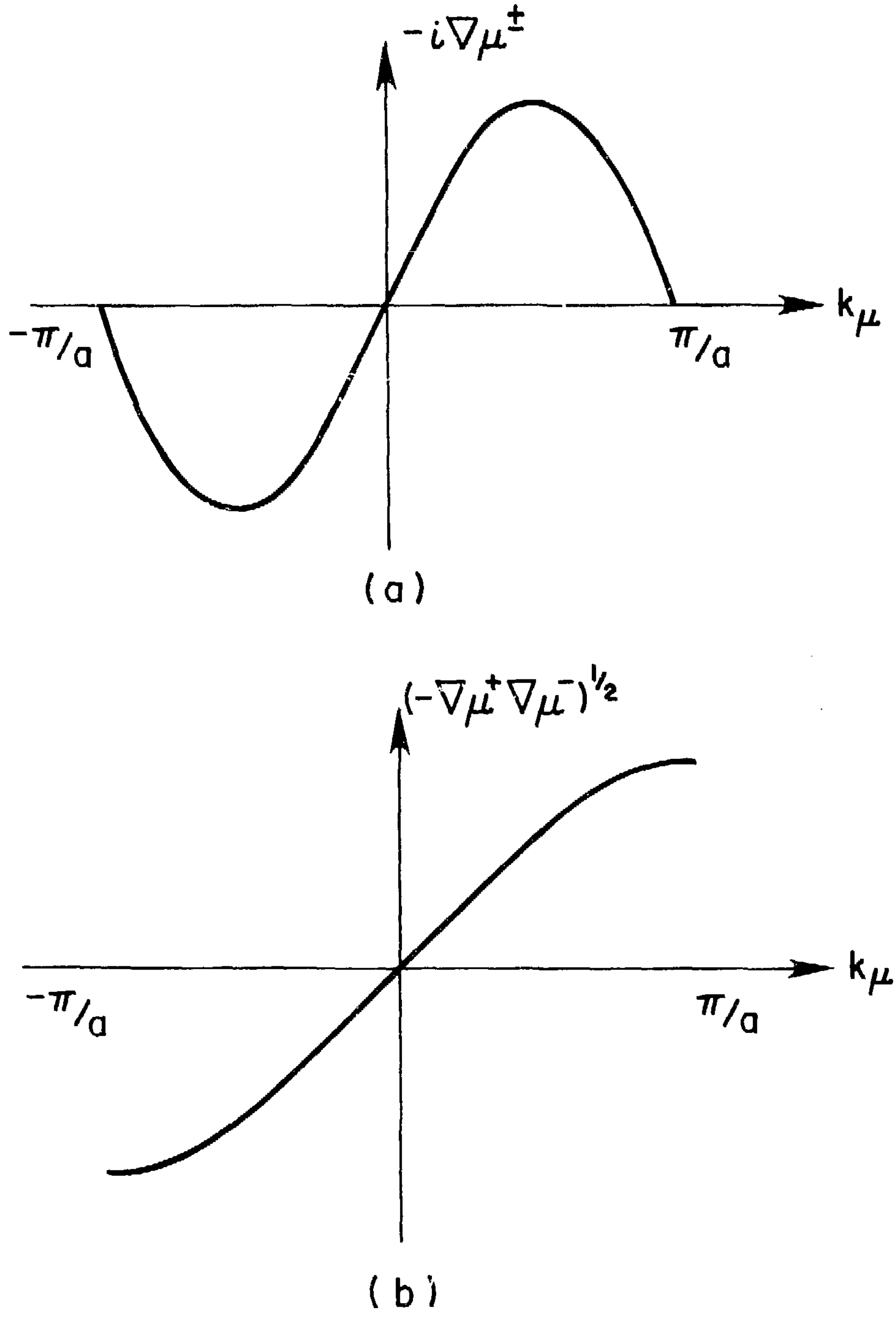

Fig. 1

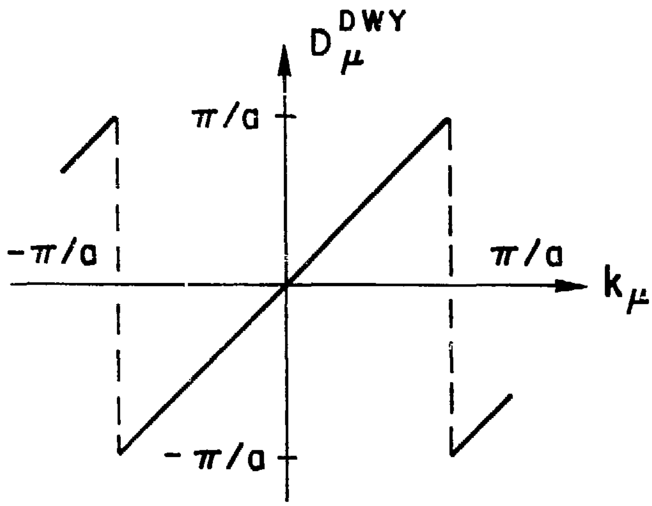

Fig. 2

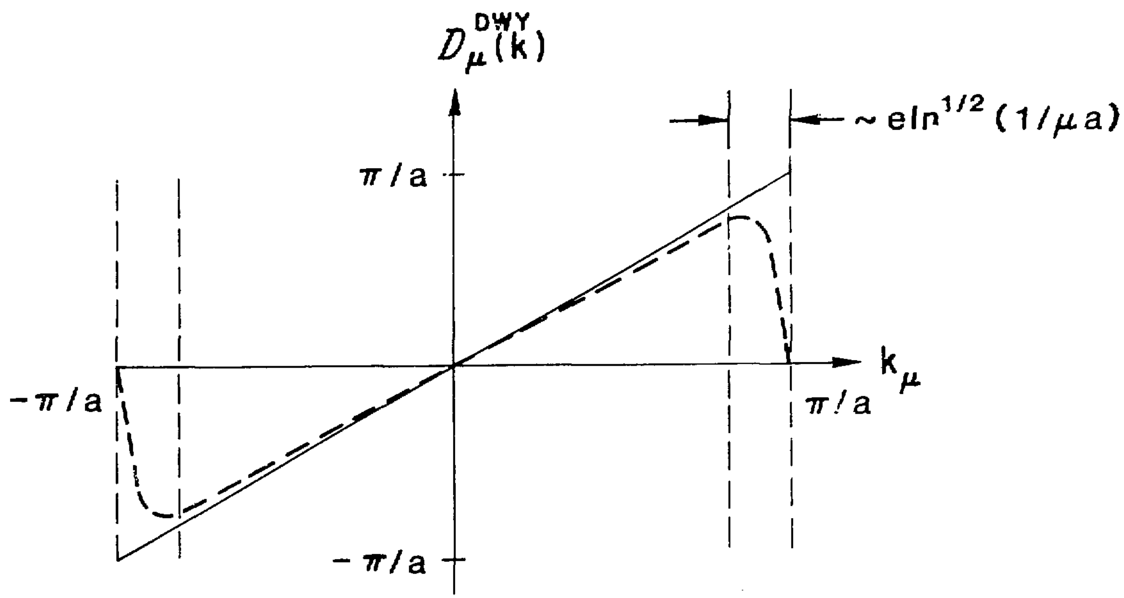

Fig. 3

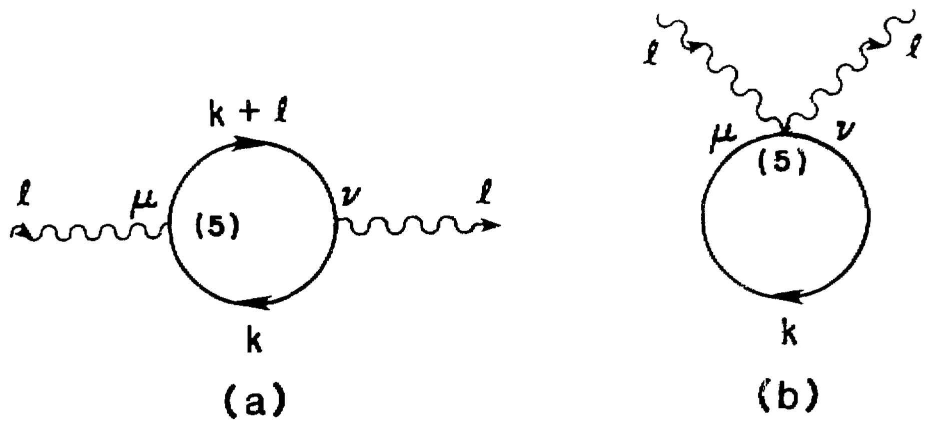

Fig. 4

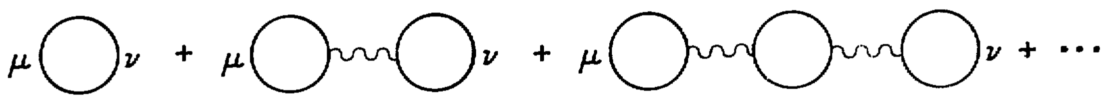

Fig. 5

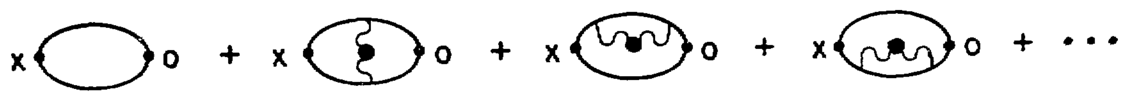

Fig. 6

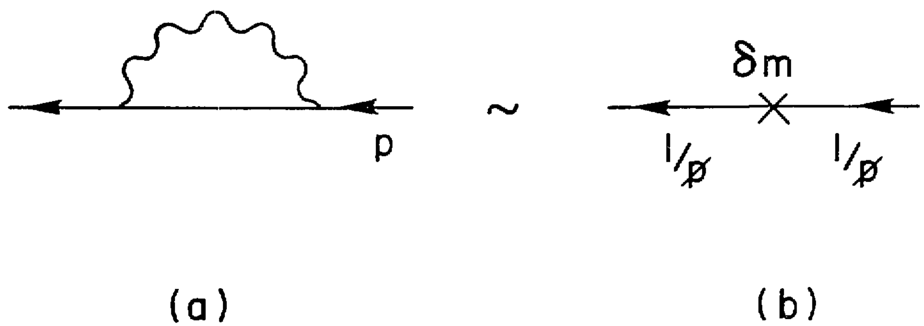

Fig. 7

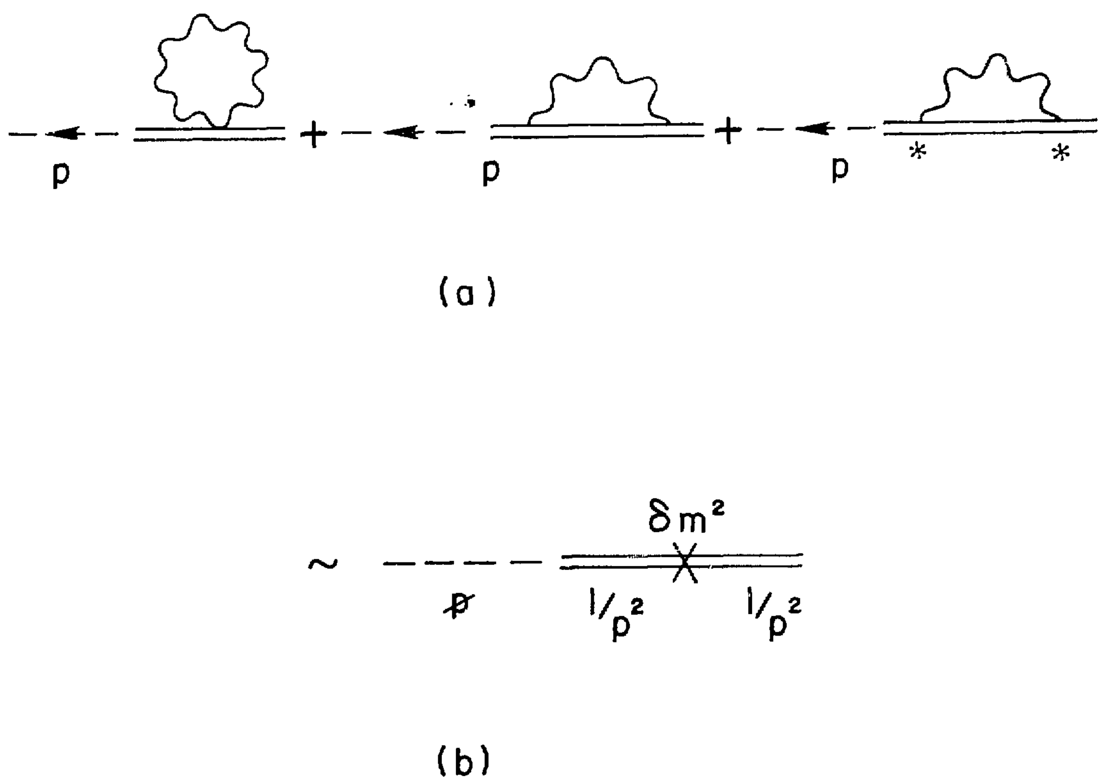

Fig. 8