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TWO– AND THREE–LOOP AMPLITUDES IN THE BOSONIC STRING THEORY

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A b s t r a c t

Explicit formulae are obtained for two-and three-loop vacuum amplitudes in the theory of closed oriented bosonic strings at $d = 26$ in terms of the theta-constants, with the module space being parametrized by period matrices.

The problem of multiloop calculations in the string theory have recently become a subject of special interest. The main practical aim of these investigations is validation of finiteness of superstring models. Moreover, they may have other important implications as soon as the general structure of many-loop corrections is established. In any case, the model of closed oriented bosonic strings (ESVM) is a good laboratory in such studies.

In this letter we present explicit formulas for the two- and three-loop vacuum amplitudes in ESVM at critical dimension d=26 (generalisation to scattering amplitudes is straightforward). In papers /1/ this problem was reduced to derivation of a certain measure on the space M_p of complex structures of Riemann surfaces of genus p (p=2,3 in this paper), which is known to have real dimension 6p-6 for $p \geqslant 2$ and be a complex manifold. In ref./2/ the analytical properties of this measure as a function of complex coordinates $y_1, \ldots, y_{3p-3}$ on M_p were clarified. These properties have been proved to unambigously determine the measure up to a constant multiplyer. The main difficulty in derivation of explicit formulas for general p is the absense of good parametrization of the space M_p. However, for p=2,3 (and, in fact, p=1) the complex structures may be parametrized by period matrices. In conjunction with the analytical properties of the measure established in /2/ this allows to express it through θ -constants.

1. Analytical properties of the measure /2/.

Complex coordinates y_i on the module space M_p can be introduced as in /3/. Consider a Riemann surface S_p of genus $p \geqslant 2$ with coordinates ξ^1, ξ^2 and metric $ds^2 = g_{ab} d\xi^a d\xi^b$. Consistent with this metric is complex structure $J^{(o)}{}_a{}^b = \epsilon_{ac} g^{cb} \sqrt{g}$. In harmonic coordinates $z, \bar{z}$, satisfying $\frac{\partial z}{\partial \xi^a} = i J^{(o)}{}_a{}^b \frac{\partial z}{\partial \xi^b}$ the metric looks like $ds^2 = \rho\, dz\, d\bar{z}$. Making use of some basis $f_i(z)(dz)^2$, $i = 1,\dots,3p-3$ in the space of holomorphic quadratic differentials on S_p and the dual basis $\eta^k(z,\bar{z}) \frac{d\bar{z}}{dz}$, $k = 1,\dots,3p-3$ in the space of Beltrami differentials, $\int \eta^k f_i d^2\xi = \delta_i^k$, find out that all the complex structures in the vicinity of $J^{(o)}$ may be parametrized by coordinates $y_i, \bar{y}_i$ so that the complex structure with coordinates $y_i, \bar{y}_i$ is consistent with metric $ds^2 = \rho \left| dz + y_i \eta^i d\bar{z} \right|^2$. It is known, that $y_i, \bar{y}_i$ so defined are complex analytical coordinates on M_p. In ref./2/ the measure in ESVM was proved to be

$$Z_p = \int d\mu_p\,; \quad d\mu_p = F(y)\, dV \wedge \overline{F(y)}\, d\bar{V} (\det \operatorname{Im} T)^{-13}\,;$$
$$dV = dy_1 \wedge \dots \wedge dy_{3p-3}\,; \quad p \geq 2 \qquad (1)$$

Here T is the period matrix (see below), and $F(y)dV$ is the holomorphic on M_p $(3p-3,0)$-form with no zeros and with double poles at infinities D_q $q = 0,1,\dots,[p/2]$ of the space M_p. At D_q surface S_p breaks down into two surfaces of genera q and $p-q$; D_o consists of surfaces with one degenerate handle.

2. Spaces M_2 and M_3

The period matrix T is defined as follows. Consider a symplectic basis of $2p$ cycles (i.e. closed orientable uncontractable paths) a_i, b_i, $i = 1,\ldots,p$:

$$a_i \circ a_j = b_i \circ b_j = 0 \quad \text{for } i \neq j; \qquad (2)$$
$$a_i \circ b_j = \delta_{ij},$$

where o denotes algebraic number of cycle intersections. Basis a_i, b_i defines a basis $\omega_i = \phi_i(z)dz$, $i = 1,\ldots,p$ of holomorphic 1-differentials, which satisfy the conditions

$$\oint_{a_i} \omega_j = \delta_{ij} . \qquad (3)$$

Matrix

$$T_{ij} = \oint_{b_i} \omega_j \qquad (4)$$

is referred to as the period matrix of surface S_p. It is known that

$$T_{ij} = T_{ji} \text{ and } \operatorname{Im} T_{ij} \text{ is positive definite.} \qquad (5)$$

Space H_p of all the matrices satisfying (5) is called Siegel's upper half-space. Thus period matrices T belong to Siegel's half-space. One can easily show, that symplectic basis $\{a_i, b_i\}$ is but ambigously defined by conditions (2), and the sole complex structure corresponds to a set of period matrices, related through transformations from modular group $\Gamma_p = Sp(p, \mathbb{Z})$ of integer-valued $2p \times 2p$ matrices $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, which satisfy $AB^T - BA^T = CD^T - DC^T = 0$, $AD^T - BC^T = I$. Γ_p acts on H_p according to the following rule :

$$M(T) = (AT + B)(CT + D)^{-1} . \qquad (6)$$

The complex dimension of H_p, $\dim_{\mathbb{C}} H_p = p(p+1)/2$, for low $p = 1,2,3$ coincides with dimension of module space M_p. In such cases M_p may be represented as a fundamental domain $\mathfrak{S}_p = H_p/\Gamma_p$ of modular group Γ_p in the Siegel's upper half-space H_p, i.e.

$$M_p = \mathfrak{S}_p \qquad \text{for } p = 1,2,3.$$

3. The measure for $p = 2,3$

The results, cited in pp.1,2, suggest, that the measure in question has the form of

$$d\mu_p = d\nu_p \ |\chi_{12-p}(T)|^{-2} \cdot (\det \operatorname{Im}T)^{p-12}, \quad p = 2,3 \qquad (7)$$

Here

$$d\nu_p = \left[\prod_{k\leq j} \frac{i}{2}\, dT_{kj} \wedge d\bar{T}_{kj}\right] \cdot (\det \operatorname{Im}T)^{-(p+1)} \qquad (8)$$

is the modular invariant measure on H_p. The requirement for (7) to be invariant under modular transformations (6) implies that

$$\chi_{12-p}(M(T)) = (\det(CT+D))^{12-p}\ \chi_{12-p}(T). \qquad (9)$$

Thus for $p=2$ χ_{10} is the modular form of weight 10 with no zeroes inside $\mathfrak{S}_2$. Besides, at the infinity D_0 (where T_{11} of T_{22} tends to $i\infty$) the measure $\left[\prod_{i\leq j} dT_{ij}\right]$ has first-order pole. Hence the analytical properties of the entire measure $\left\{\prod_{i\leq j} dT_{ij} \cdot \left(\chi_{12-p}(T)\right)^{-1}\right\}$, cited in p.1, imply that the form χ_{10} has first-order zero at D_0 and double zero at D_1 (where $T_{12}\to 0$), i.e. χ_{10} is parabolic. The form χ_{10} is unambiguously defined by the weight and location of zeroes (up to a constant multiplyer) and is equal to a product of all the even θ-constants /4/ :

$$\chi_{10} = \prod_m \theta_m^2(T) \tag{10}$$

Theta-constants are defined as

$$\theta_m(T) = \sum_{n\in Z^p} \exp\left\{\pi i\left(n+\frac{m'}{2}\right)^T T\left(n+\frac{m'}{2}\right) + 2\pi i\left(n+\frac{m'}{2}\right)^T \frac{m''}{2}\right\}, \tag{11}$$
$$m \equiv (m', m'')$$

(p=2), and components of vectors m',m'', which form the "theta-characteristic" m=(m',m''), take values 0,1. Number $e(m) = (m')^T m'' (\mathrm{mod}\,2)$ is referred to as parity of characteristic m, and the product in (10) is over all even characteristics. For genus p there are as many as $2^{p-1}(2^p+1)$ even and $2^{p-1}(2^p-1)$ odd characteristics. For odd $e(m)=1$ $\theta_m(T)\equiv 0$. Using (11), one can easily check that χ_{10} has the first- and second-order zeroes at D_0 and D_1 respectively. Also, non-vanishing of χ_{10} inside $\mathfrak{S}_2$ may be easily demonstrated. For this purpose, we make use of the following formula for fermionic determinant /2/ :

$$\det{}_m \bar\partial_{1/2} \left(\det \bar\partial_0\right)^{1/2} = \theta_m(T). \tag{12}$$

Here $\det_m \bar\partial_{1/2}$ is the determinant of the Dirac operator that acts on the space of the left-handed Weyl fermions "living" on S_p and satisfying the periodic (antiperiodic) boundary conditions under translations along the cycles a_i for $m_i' = 1(0)$ and b_i for $m_i'' = 1(0)$;

$$\left|\det \bar\partial_0\right|^2 \stackrel{\mathrm{def}}{=} \frac{\det'\left(-g^{-1/2}\partial_a g^{1/2} g^{ab}\partial_b\right)}{(\det \mathrm{Im}\, T)\left(\int\sqrt{g}\, d^2\xi\right)} . \tag{13}$$

From (12), (13) it follows that $\theta_m(T)$ vanishes at the surfaces S_2 where $\bar\partial_{1/2}$ acquires zero-modes with boundary conditions.

specified by m, i.e. when there are holomorphic 1/2-differentials with characteristic m on S_2. According to the Riemann singularity theorem the parity of the number of such 1/2-differentials coincides with the parity $e(m)$ of characteristic m. Moreover, their number is equal to $e(m)$ for generic surface. The vanishing of $\theta_m(T)$ for even $e(m)=0$ on the surface S_2 thus suggests that there are at least two holomorphic 1/2-differentials $\psi_1(z)(dz)^{1/2}$ and $\psi_2(z)(dz)^{1/2}$ with characteristic m on S_2. Besides, any holomorphic k-differential on a surface of genus p is known to have $2k(p-1)$ zeroes. There is therefore a meromorphic function $f(z)=\psi_1(z)/\psi_2(z)$ on S_2 with one pole and one zero, which is possible only for surfaces of genus zero. This discrepancy indicates that $\theta_m(T)$ has no zeroes inside $\mathfrak{S}_2$ for $e(m)=0$. The absense of holomorphic 1/2-differentials with even characteristic seems to also ensue from the fact that any Riemann surface of genus 2 may be represented as a hyperelliptic curve

$$y^2 = (z-a_1)\ \ldots(z-a_6) \tag{14}$$

in $\mathbb{C}^2 = (y,z)$. It can be easily ascertained that such surface has exactly 6 holomorphic 1/2-differentials:

$$\Psi_i = \sqrt{(z-a_i)/y}\,(dz)^{1/2}, \tag{15}$$

i.e. one for each of the six odd characteristics and none left for the even ones.

Let us now turn to $p=3$. The measure $\prod_{i\leq j}\cdot dT_{ij}$ has in this case the first-order pole at D_0 and the first-order zero at D_1. Besides, the χ_9 form of weight 9 may only be defined as taking values in the character of Γ_3, since in case of $p=3$ the right-hand side of eq.(9) appears to change the sign with substitution of $-m$ for m, while the left-hand side does not. The "real" modular form is

$$\chi_9^2 = \chi_{18} \tag{16}$$

It should have the second-order zero at D_0 and the sixth-order one at D_1. Such a form exists and equals to /5/ :

$$\chi_{18} = \prod_m \theta_m (T) \tag{17}$$

with the product over all the 36 even characteristics. However, χ_{18} vanishes not only at D_0 and D_1, but also at a manifold D_* of hyperelliptic curves inside $\mathfrak{S}_3$ /5/. Indeed, on a hyperelliptic curve of genus 3

$$y^2 = (z - a_1) \dots (z - a_8) \tag{18}$$

besides 28 fermionic zero-modes

$$\psi^{(ij)} = \sqrt{(z-a_i)(z-a_j)/y}\,(dz)^{1/2}, \quad i<j \tag{19}$$

(one for each of the 28 odd characteristics) there are two additional zero-modes

$$\psi^{(0)} = \frac{1}{\sqrt{y}}(dz)^{1/2}, \quad \psi^{(1)} = \frac{z}{\sqrt{y}}(dz)^{1/2} \tag{20}$$

with the common even characteristic, its specific value being dependent on the choice of the basis of cycles (2) on the curve (18). Thus, χ_{18} is actually vanishing at D_*, this zero in coordinates y_1 from p.1 is of the second order. Vice versa, if the form χ_{18} vanishes for some surface S_3^*, there are at least two holomorphic 1/2-differentials $\Psi_1(z)$ and $\Psi_2(z)$ with the same even characteristic on S_3^*, their ratio $f(z) = \Psi_1(z)/\Psi_2(z)$ being meromorphic function on S_3^* with two zeroes and two poles, i.e. S_3^* is in fact a two-sheet coverage of CP^1 and consequently a hyperelliptic surface.

To sum up, χ_{18} vanishes inside $\mathfrak{S}_3 = M_3$ exactly on hyperelliptic surfaces. This zero is of the second order in coordinates y_i from p.1, and the square root $\chi_9 = \sqrt{\chi_{18}}$ may be taken. It remains to be shown, that measure $\prod_{i \le j} dT_{ij}$ has the first-order zero on hyperelliptic curves. For this purpose, we choose the basis of holomorphic quadratic differentials on surface (18) as follows:

$$f_K = \frac{z^{K-1}}{y^2}(dz)^2, \quad K = 1, \dots, 5; \quad f_6 = \frac{(dz)^2}{y} \tag{21}$$

Take also the metric $\rho\, dz d\bar{z}$ symmetric under transform P: $(y, z) \to (-y, z)$. Then η^6 may be chosen odd under P :

$$\eta^6 = \text{const}\, \bar{f}_6 / \rho\, d\bar{z} dz \tag{22}$$

Furthermore, holomorphic Abelian differentials are linear combinations of

$$\frac{dz}{y}, \qquad \frac{z\,dz}{y}, \qquad \frac{z^2 dz}{y}.$$

Hence eq.(22) and the formula for variation of T/2/ *)

$$\frac{\partial T_{ab}}{\partial y_i} = -\int_{S_p} \eta^i \omega_a \wedge \omega_b, \tag{23}$$

which is valid for any genus, suggest that

$$\frac{\partial T_{ab}}{\partial y_6} = 0 \tag{24}$$

Taking $y_6\,(D_*) = 0$, we find :

$$\prod_{i \le j} dT_{ij} \underset{y_6 \to 0}{\sim} y_6\, dy_1 \dots dy_6 \tag{25}$$

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*) Formula (23) has been independently derived by Al.Zamolodchikov (unpublished).

Consequently, measure

$$\prod_{i\le j} dT_{ij}\,\chi_{18}^{-1/2} \underset{y\to 0}{\sim} \mathrm{Const}\cdot dy_1 \dots dy_6 \tag{26}$$

is holomorphic in coordinates y_i in the vicinity of D_* and does not vanish, in accordance with ref./2/. The integrable root-like singularity intrinsic in $d\mu_3$ wher expressed in terms of period matrix T (cf.(7)), does not contradict the holomorphic properties of the measure, because in the vicinity of hyperelliptic curves (those possessing the symmetry P) M_3 embeds into $\mathfrak{S}_3$ in a nonshooth way. To summarize, we have thus proved the validity of formulae (7),(16),(17) for the measure for genus p=3. It is worth noting, that analytical properties of forms χ_{10}, χ_{18} have been studied by Igusa /4,5/ with the use of other methods.

As for as genus 2 is concerned, the module space M_2 may be also parametrized by the coordinates of ramification points $\lambda_1, \lambda_2, \lambda_3$ of the curve

$$y^2 = z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3) \tag{27}$$

in $\mathbb{C}^2 = (y, z)$. In these coordinates the measure looks like

$$d\Omega = \prod_{i\le j} dT_{ij}\,[\chi_{10}(T)]^{-1} = [P(\lambda)]^{-2/5}[\chi_{10}(T)]^{-13/10}\cdot d\lambda_1 d\lambda_2 d\lambda_3, \tag{28}$$

$$P(\lambda) = \lambda_1\lambda_2\lambda_3(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)$$

and period matrix T may be expressed through $\lambda_1, \lambda_2, \lambda_3$ in terms of hyperelliptic integrals. To define the statistical sum

$$Z_3 \sim \int d\Omega \wedge d\overline{\Omega} \, (\det \mathrm{Im}\, T)^{-13} \tag{29}$$

one may integrate over each $d^2\lambda_i$ throughout the entire complex plane, since this accounts for the contribution of almost each Riemann surface as many as 720 times, thus giving rise to a simple numerical factor of 720.

To summarize we would like to note that in the recent paper by Manin /6/ was obtained the expression for the measure $d\mu_p$ for an arbitrary genus p, but this expression was of a more complicated structure (containing Abelian differentials).

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