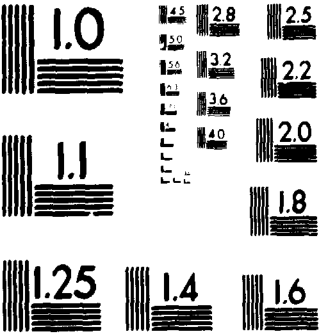

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COMITATO NAZIONALE PER LA RICERCA E PER LO SVILUPPO DELL'ENERGIA NUCLEARE E DELLE ENERGIE ALTERNATIVE

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G. DATTOLI, A. RENIERI
ENEA - Dipartimento Tecnologie Intersettoriali di Base, Centro ricerche energia Frascati

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SUMMARY

In this paper we present a quantum analysis of the Free Electron Laser. We develop the theory both in single and longitudinal multimode regimes. We finally present a self-consistent procedure to study the growth of the laser signal from the vacuum to the macroscopic level.

RIASSUNTO

In questo lavoro viene presentata un'analisi quantistica del Free Electron Laser. La teoria viene sviluppata nell'ipotesi di singolo modo e di molti modi. In fine presentiamo un procedimento autoconsistente per studiare la nascita del segnale laser dal vuoto fino ad un livello macroscopico.

Submitted to FEL-HANDBOOK

TABLE I

Symbols and notations

c	speed of light
m_0	electron mass
$\lambda\!\!\!^{-}_c = \hbar/mc$	electron Compton wave-length
$r_0 = e^2/m_0 c^2$	electron classical radius
σ_ε	e-beam relative energy spread
$\varepsilon_{x,y}$	e-beam transverse emittances
$K = \frac{e\langle B^2\rangle^{1/2}\lambda_U}{2\pi m_0 c^2}$	undulator parameter
λ_U	undulator wave-length
B	undulator on axis magnetic field
N	number of undulator periods
λ^*	pseudo radiation field wave-length
$g = \frac{2\pi c^2 r_0}{\omega V}$	coupling constant
V	interaction volume
$\underline{k}$	laser field wave number
$-\underline{k}$	undulator field wave number
λ	laser field wave-length
$a_{L,U}, a^+_{L,U}$	annihilation, creation operators for the laser and undulator fields

1. INTRODUCTION

The analysis of the Free Electron Laser (FEL) in terms of quantum mechanics is a rather old subject.

During the pioneering days of FEL Physics, Madey deduced the gain formula using a quantum mechanical treatment [1].

It was later pointed out that the basic mechanism underlying the FEL interaction is not quantum but classical [2-4]. The gain, indeed, does not depend crucially on the Planck constant, which can be eliminated by suitably defining the coupling constant in terms of classical quantities, like the wave electric field amplitude.

Classical theories [5] work quite well in accounting for the main features of the experimental results [6] which do not support any interpretation in terms of genuine quantum effects.

In spite of the above rather crude statement against the quantum analysis of the FEL dynamics, they still deserve interest for at last three reasons:

(1) Quantum effects must be taken into account for a correct analysis of the start-up phenomena.

(2) An appropriate description of the statistical properties of the FEL radiation needs quantum mechanics.

(3) Under specific experimental situations, discussed below, the well-established gain curve could be modified by higher order quantum corrections.

There is a natural distinction between the above three points.

The first two are "intrinsic" effects in the sense that they are common to all the FEL experiments. The third, on the other side, is peculiar to certain types of experiments and defines a "strong quantum regime".

In this paper we will be mainly concerned with the "intrinsic" effects, because they are the most significant.

The "strong quantum regime" will be discussed in this introduction and we will show that it holds under too specific conditions and that, for the moment, it is far from having any sizeable effect.

Before going into technical details let us recall that there are, in principle, four ways to describe the interaction between the electromagnetic (e.m.) fields and the matter. In fact, we can quantize the fields, the matter or both as schematically shown in Table II.

TABLE II

Classical and Quantum Mechanics Possible Descriptions of Field-Matter Interaction

MATTER \ FIELD	QUANTUM MECHANICS	CLASSICAL
QUANTUM MECHANICS	I	II
CLASSICAL	IV	III

In region I both field and matter are described quantum-mechanically. This is the most correct description to study the physics of the interaction process, no ambiguity in the interpretation of the results arises, as pointed out by Senitzky [7]. However a good description of the process, regardless of the vacuum field fluctuations, can be done in region II (semiclassical approximation) where the field is treated classically and the matter is quantized.

Region III is relevant to fully classical treatment. Conventional lasers cannot be treated within this framework. As already remarked the FEL on the other side seems for the moment a genuine classical "laser". The fourth region is also a sort of semiclassical approximation region where the fields are treated quantum-mechanically and the matter classically.

We can now try an answer to the following question: What is the range of validity of the classical approximation for the FEL?

We will discuss this point in a simple heuristic way.

Let us consider, for simplicity, the FEL process in the so-called "resonant reference frame" [4], namely the frame in which laser and "undulator wave" have the same frequency (see next sections). In this frame the electron can be viewed as a "classical current" if the width of its wave-packet remains small, compared to the fields (laser and undulator) reduced wave-length, for all the interaction time, namely

$$\Delta x \ll \lambda\!\!\!^{-}. \qquad (1.1)$$

We can assume, following Ref.[8], a classicity threshold when the electron wave-packet width equals 1% of the fields reduced wave length

$$\Delta x \le \frac{\lambda_{th}}{100} \tag{1.2}$$

This choice may sound somewhat arbitrary, but it is just an "order of magnitude" criterion to understand the role of the physical quantities involved.

If the electron motion is weakly perturbed by the FEL interaction the electron wave-packet spreads in time according to [9](*)

$$\Delta x^2 = \Delta x_0^2 + \left(\frac{\Delta P_0}{m_0}\right)^2 t^2 \tag{1.3}$$

where m_0 is the electron rest mass and $(\Delta x_0, \Delta P_0)$ are the initial electron wave-packet width and momentum spread respectively.

The condition (1.2) can be satisfied, requiring

$$\Delta x_0 \le \frac{10^{-2}}{\sqrt{2}} \lambda_{th} \quad , \quad \Delta P_0 \le \frac{10^{-2}}{\sqrt{2}} \lambda_{th} \cdot \frac{m_0 c}{L} \tag{1.4}$$

where c is the light velocity and L is the undulator length. Multiplying (1.4) term by term and using the indetermination principle, one gets the following inequalities

$$\frac{\lambda_{th}^2 m_0 c}{2L} \cdot 10^{-4} \ge \Delta x_0 \Delta P_0 \ge \hbar/2 \tag{1.5}$$

Introducing the electron reduced Compton wave length $(\lambda_c = \hbar/m_0 c)$, we obtain from (1.5) the following threshold condition

---

(*) The assumption of weakly perturbed motion may be understood as an ad hoc assumption. A more complete analysis, including the interaction, leads to the same result (see Ref.[9] for further comments).

$$\frac{\lambdabar_{th}^2}{\lambdabar_c L} \geq 10^4 \qquad (1.6)$$

Transforming back to the laboratory frame (1.6) reads ($\gamma = E/mc^2$)(*)

$$\lambda_{th} \geq 10^2 \pi \sqrt{\frac{\lambdabar_c L}{\gamma^3}} \qquad (1.7)$$

If this condition does not hold, higher order quantum mechanical corrections must be included.

In Figure 1 we have reported in a (λ,γ) plane some FEL sources. The continuous line describes the "classicity threshold" (1.7) (for L=1 m), while the broken line refers to the "Quantum mechanical regime"; namely when the wave--packet dimension is equal to the reduced laser wave length(**)

$$\lambda = \pi \sqrt{\frac{\lambdabar_c L}{\gamma^3}} \qquad (1.8)$$

The figure shows that the existing IR single passage devices and the SR-FELs are largely within the limits of a classical treatment. In the case of the two-stage Santa Barbara FEL, in which a real e.m. millimeter wave is utilized as undulator, the quantum effects cannot be neglected. An idea of what happens in the short wave-length regions (VUV,X range) is given by the operating points of a λ =100 nm low γ FEL working with a 80 μm wave length

(*) Note that λ_{th} and L are now laboratory frame variables.

(**) Eventually the condition (1.8) can be easily transformed in an equality condition between the undulator wave-length and and the energy, namely $\lambda_U = 4\pi^2 \gamma N \lambdabar_c$ (N≡ number of undulator periods).

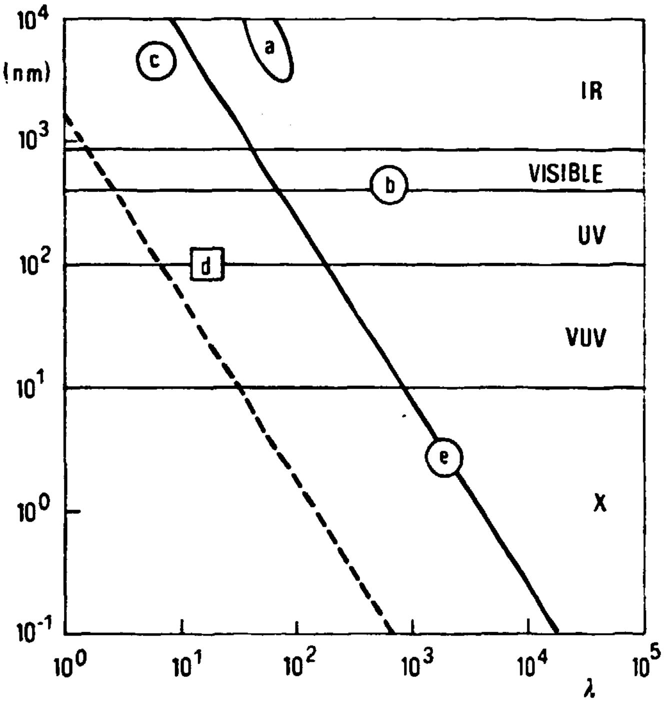

Fig. 1

Wavelength (λ) and energy (γ) working points for some FEL sources (see Ref. [1]). Continuous line ≡ classicity threshold for L = 1 m (Eq. (8)) Broken line ≡ strong quantum mechanical regime for L = 1 m (Eq. (9)); (a) Single pass IR FELs (Stanford, TRW, Los Alamos, UK Project, MSNW, Frascati-ENEA); (b) SR-FEL (Orsay, Frascati-INFN, Brookhaven); (c) Two stages S. Barbara FEL; (d) typical low energy VUV FEL source (λ = 100 nm, E = 10 MeV, e.m. wave undulator with λ_U = 80 μm); (e) Typical X SR-FEL source (λ = 2.5 nm, E = 1 GeV, undulator Period λ_U = = 1 cm.

undulator and of a $\lambda = 2.5$ nm high energy Storage-Ring (S.R.) FEL working with a 1 cm undulator wave length. The figure shows that the low energy device may exhibit quantum effects while the S.R. one lies just on the board line.

To summarize, at the moment the Santa Barbara second stage FEL only, could be affected by macroscopic quantum effects.

Since the macroscopic quantum effects amount to a correction of the gain curve, we must take into account all the other (less exotic) effects which can modify the gain curve, namely the inhomogeneous broadening due to the e.b. energy spread and emittances.

As elsewhere pointed out (see Ref.[10] and also next sections) the quantity which controls the quantum corrections to the gain formula is a quantity linked to the electron recoil and, in the laboratory frame, reads

$$\mu_q = 2N \cdot \frac{8\pi^2 \lambda\!\!\!^{-}_c}{\gamma\lambda} \tag{1.9}$$

On the other side, the presence of modifications in the gain due to the energy spread and the emittances is controlled by the following parameters(*) [10]

$$\mu_\varepsilon = 4N\sigma_\varepsilon \qquad \sigma_\varepsilon \equiv \text{e-beam relative energy spread}$$

$$\mu_{x,y} = 2N\sqrt{2}\ \frac{K}{1+K^2}\ \frac{\gamma\varepsilon_{x,y}}{\lambda_U}\ , \qquad \varepsilon_{x,y} \equiv \text{e-beam transverse emittances.} \tag{1.10}$$

(*) The $\mu_{x,y}$ coefficients are relevant to circularly polarized undulators.

To appreciate quantum corrections, the coefficient μ_q must be of the same order of magnitude as the coefficients (1.10). Imposing $\mu_q \cong (\mu_\varepsilon, \mu_{x,y})$ we find the following two conditions(*)

$$\sigma_\varepsilon \simeq 4\pi^2 \frac{\lambda\!\!\!^{-}_c}{\gamma\lambda}$$

$$\varepsilon_{x,y} \simeq 8\sqrt{2}\,\pi^2 \frac{(1+K^2)}{K}^{1/2} \lambda\!\!\!^{-}_c \qquad (1.11)$$

The first equality gives the same indication as before. Namely, macroscopic quantum effects could be observed in an FEL operating, with a low energy accelerating device, at short wave-lengths. To give an example for $\gamma \simeq 30$, $\lambda \simeq 10^{-3}\,\mu m$ we get $\sigma_\varepsilon \simeq 5.10^{-4}$ which is not far from the capabilities of a microtron. On the other side, the required emittances turn out to be so small that they are not conceivable for any realistic accelerating machine.

The above qualitative considerations show that at the moment macroscopic quantum effects are hard to be seen in any FEL experiment. Furthermore, if any, they cannot be easily distinguished by other classical effects such as the inhomogeneous broadening.

2. QUANTUM FEL HAMILTONIAN

We will develop the quantum theory of the FEL in a moving frame where the electron motion can be treated using non-relativistic mechanics.

(*) The second of conditions (1.11) may appear incorrect: this is due to the different role played by the relativistic factor in (1.9) and (1.10). In (1.9) indeed γ should be replaced by $\gamma/\sqrt{1+K^2}$ where the square root is due to the transverse electron motion.

The reason for this choice is twofold, namely it allows both a simple mathematical treatment of the problem and, what is more important, offers a transparent physical insight into the interaction process.

The details of the non-relativistic Hamiltonian picture of the FEL have been reviewed in Ref.[10]. Here we will not rediscuss this topic at length but rather we will summarize its basic elements.

According to the Weizsäcker-Williams approximation the undulator field is treated as a radiation field with a wave-length double its period

$$\lambda^{*} = 2\,\lambda_U \tag{2.1}$$

The meaning and the validity of this kind of approximation have been also discussed in Ref.[10]. Let us stress that it was just this point of view that indicated the concrete possibility of building FELs with high power operating in a broad range of wave-lengths. Previous suggestions (see Ref.[10] for a historical review) considered as pump field sources as microwaves or laser fields which had too low a photon intensity to provide enough gain(*).

One of the practical advantages of the "undulator-wave" approximation is that the emission process can be understood as a Compton Scattering or better as a stimulated Compton Scattering.

The choice of the reference frame where the process can be studied is quite arbitrary. Any frame where the electron motion can be treated classically is well-suited.

(*) To compare the orders of magnitude we emphasize that a CO_2 100J, 100 ns laser has a photon density of about $10^{18}\,cm^{-3}$, while a 3 kG λ_U=5 cm undulator magnet has an equivalent photon intensity of about $10^{22}\,cm^{-3}$. A simple formula to evaluate the undulator equivalent photon density is $\bar{n}=\frac{\alpha}{4}\,K^2/(\lambda \cdot r_0^2)$, $\alpha \equiv$ fine structure constant.

However, in the single mode hypothesis the natural choice is the so-called "resonant frame", already defined in the previous section.

The velocity v^* of this frame can be chosen according to its definition namely [4]

$$2\lambda_L \gamma^* = \frac{\lambda_U}{\gamma^*} \qquad v^* = c\left(1-2\,\frac{\lambda_L}{\lambda_U}\right)^{1/2} \tag{2.2}$$

where λ_L is the laser wave-length in the laboratory frame. In this frame the scattering of undulator photons into laser ones is resonant by definition and the problem of FEL interaction has been reduced to that of an electron moving in the field of two counterpropagating equal frequencies e.m. waves.

In this frame, in the transverse gauge, the quantized laser and undulator vector potentials read

$$\begin{aligned} \underline{A}_L &= \left(\frac{2\pi c\hbar}{\omega V}\right)^{1/2} [a_L e^{ikz}\underline{e} + a_L^+ e^{-ikz}\underline{e}^*] \\ \underline{A}_U &= \left(\frac{2\pi c\hbar}{\omega V}\right)^{1/2} [a_U e^{-ikz}\underline{e} + a_U^+ e^{ikz}\underline{e}^*] \end{aligned} \tag{2.3}$$

where a and a^+ are the annihilation and creation operators ($[a_\ell, a_m^+] = \delta_{\ell,m}$), both laser and undulator fields are assumed circularly polarized so that

$$\underline{e} = \frac{1}{\sqrt{2}}(\underline{x} + i\underline{y}) \tag{2.4}$$

finally V is the interaction volume.

The Hamiltonian of the coupled electron-field system can be immediately written as

$$H = \frac{1}{2m_0} (\underline{P} - \frac{e}{c} \underline{A})^2 + \hbar\omega(a_L^+ a_L + 1/2) + \hbar\omega(a_U^+ a_U + 1/2) \quad (2.5)$$

$\underline{P} \equiv$ canonical electron momentum.

The last two terms in the Hamiltonian account for the free field energy, while the first contains the electron kinetic energy and the electron-fields interaction part. Since $\underline{A}$ is transverse, the $\underline{P}\cdot\underline{A}$ terms vanish and the $\underline{A}\cdot\underline{A}$ terms only survive, so that (2.5) finally writes

$$\begin{aligned} H = {} & \frac{p^2}{2m_0} + \hbar g\{a_L^+ a_U e^{-2ikz} + a_U^+ a_L e^{2ikz}\} \\ & + \hbar(\omega+g)(a_L^+ a_L + 1/2) + \hbar(\omega+g)(a_U^+ a_U + 1/2) \end{aligned} \quad (2.6)$$

where p is the longitudinal momentum, and

$$g = \frac{2\pi c^2 r_0}{\omega V} \ (<< \omega) \qquad (r_0 = e^2/m_0 c^2) \quad (2.7)$$

is a quantity with the dimension of a frequency, which acts as a coupling constant and as renormalization of the photon frequency.

The physical interpretation of the coupling term in (2.6) is straightforward. The first term describes the creation of a laser photon, the destruction of an undulator one and a loss of $2\hbar k$ of electron momentum, the second term describes the complementary process (for further comments see Ref.[10]).

We can now discuss a few physical implications relevant to the laws of conservation.

According to Ref.[11] we can embed the electron and field variables a la Schwinger [12] and get the invariance group of the FEL. We define, indeed

$$R_1 = \frac{1}{2}(a_L^+ a_U e^{-2ikz} + a_U^+ a_L e^{2ikz})$$

$$R_2 = \frac{1}{2i}(a_L^+ a_U e^{-2ikz} - a_U^+ a_L e^{2ikz}) \tag{2.8}$$

$$R_3 = \frac{1}{2}(a_L^+ a_L - a_U^+ a_U)$$

It is easy to verify that the above operators obey the following commutation relations

$$[R_\ell , R_m] = i\,\varepsilon_{\ell,m,k} R_k \qquad \varepsilon_{\ell,m,k} \equiv \text{Ricci tensor}$$

$$\ell \neq m \neq k = 1,2,3$$

$$[\frac{p}{2\hbar k} , R_\ell] = (-1)^\ell i R_m (\ell \neq m = 1,2)$$

$$[\frac{p}{2\hbar k} , R_3] = 0 \tag{2.9}$$

The commutation relations (2.9) reflect the "dynamical" symmetry of the single mode FEL which is rotational and translational.

It is easy to verify that the vector $\underline{R} \equiv (R_1, R_2, R_3)$ obeys the following equations of motion

$$\dot{\underline{R}} = \underline{R} \times \underline{\Omega} + i\tilde{\omega}\hat{D}\underline{R} \qquad (\tilde{\omega} = \frac{2\hbar k^2}{m_0})$$

$$\underline{\Omega} \equiv (-2g, 0, \frac{\tilde{\omega} p}{\hbar k}) \quad , \quad \hat{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Two well-known laws of conservation follow quite straightforwardly, namely

$$|\underline{R}| = \hat{n}_L + \hat{n}_U = \text{constant}$$

$$\frac{p}{2\hbar k} + R_3 = \text{constant} \tag{2.11}$$

where $\hat{n}$ is the photon number operator.

The previous laws of conservation are relevant to the total number of photons (laser + undulator) and to the total linear momentum (electron + fields). In this simple picture the FEL process is understood as a photon exchange between the undulator and the laser and the electron provides the necessary momentum.

The model we have developed is rather simplified but displays some nice features as a direct analogy with other topics in quantum optics like the two levels system [13]. In the next sections we will reconsider this problem from a dynamical point of view. We will now spend a few more words on the multimode FEL problem treated within the group-theoretical framework.

Recently the theory of n-levels quantum systems has been formulated in terms of unitary symmetries [14, 15] In particular it has been shown that the dynamics of the system can be described by means of a generalized Bloch equation, namely by the rotation of an (n^2-1)-dimensional coherence vector of a SU_n space.

Furthermore a number of previously unforeseen non linear constants of motion, linked to the Casimir invariants of the algebra, have been discovered.

The tools of unitary symmetries can be exploited for the FEL too [16], in the hypothesis of a longitudinal multimode analysis of the laser field(*).

The quantum version of the classical multimode FEL Hamiltonian [17] can be written as [18]

(*) The necessity of a multimode analysis will be discussed in Sections (5,6).

$$H = \frac{p^2}{2m_0} + \hbar\omega_U(a_U^+ a_U + 1/2) + \sum_{s=1}^{M} \hbar\omega_s(a_s^+ a_s + 1/2) +$$

$$+ \hbar \sum_{s=1}^{M} g_{s,U}\{a_s^+ a_U e^{-i(k_s+k_U)z} + h.c.\} + \tag{2.12}$$

$$+ \hbar \sum_{s<j}^{M} g_{s,j}\{a_s^+ a_j e^{-i(k_s-k_j)z} + h.c.\}, \quad (g_{s,j} = \frac{2\pi c^2 r_c}{(\omega_i \omega_j)^{1/2} V})$$

For the meaning of the symbols see the introductory Table. The index s labels the s-th laser mode and the summation is extended to all the longitudinal modes. The Hamiltonian (2.12) is written in a non-relativistic frame(*).

The physics described by (2.12) is transparent. The third term describes the interaction between the undulator and the laser modes. The last term accounts for the laser modes interaction.

The SU_n(**) invariance of (2.12) can be easily understood by generalizing the Schwinger procedure previously described for the two modes case.

To give an idea we will explicitly discuss the SU_3 FEL interaction [16] because it is quite simple and can be immediately compared to the three level system discussed by Elgin [14], Eberly and Hioe [15].

We consider two copropagating modes 1,2 together with the electron and the undulator field moving in the opposite direction. In this connection the Hamiltonian (2.12) becomes

---

(*) Note that the resonant frame makes no sense in the multimode case. The new frame has been chosen in such a way that the central frequency coincides with that of the undulator.

(**) $n=M+1$, where M is the number of the laser modes and 1 refers to the undulator field.

$$H = \frac{p^2}{2m_0} + \hbar \sum_{j=1}^{3} \omega_j a_j^+ a_j +$$

$$+ \hbar g_{1,2} [a_1^+ a_2 e^{-i(k_1 - k_2)z} + h.c.] +$$

$$+ \hbar g_{1,3} [a_1^+ a_3 e^{-i(k_1 + k_3)z} + h.c.] +$$

$$+ \hbar g_{2,3} [a_2^+ a_3 e^{-i(k_2 + k_3)z} + h.c.] \tag{2.13}$$

The undulator "mode" has been indicated with the subscript 3 for later convenience.

Just following the prescription (2.8) we define the following pseudospin vector operators $\underline{R}_{\ell,j}$ $(\ell<j=1,2,3)$

$$R^{(1)}_{\ell j} = \frac{1}{2} (a^+_\ell a_j S_{\ell j} + h.c.)$$

$$R^{(2)}_{\ell j} = \frac{1}{2i} (a^+_\ell a_j S_{\ell j} - h.c.) \tag{2.14}$$

$$R^{(3)}_{\ell j} = \frac{1}{2} (a^+_\ell a_\ell - a^+_j a_j)$$

where

$$S_{\ell j} = e^{-ik_{\ell j} z}$$

$$k_{\ell j} = \begin{cases} k_\ell - k_j & , \quad j=2 \\ k_\ell + k_j & , \quad j=3 \end{cases} \tag{2.15}$$

Following the eightfold way notation [19] we make the following identification

$$\underline{R}_{1,2} = \underline{T}$$

$$\underline{R}_{1,3} = \underline{V} \tag{2.16}$$

$$\underline{R}_{2,3} = \underline{U}$$

in analogy with the usual definitions of isospin, V-spin and U-spin of the elementary particles SU_3 scheme [19]. The Hamiltonian (2.12) therefore writes

$$H = \frac{p^2}{2m_0} + \hbar \sum_j \omega_j a_j^+ a_j + 2\hbar[g_{1,2}T_1 + g_{1,3}V_1 + g_{2,3}U_1] \qquad (2.17)$$

The dynamics of the system can be specified following the evolution of the pseudospin vectors ($\underline{T},\underline{V},\underline{U}$). Such a description is redundant; the three modes FEL dynamics can be determined by the equation of motion of a single eight component vector $\underline{F}$ with components [16]

$$\underline{F} \equiv (T_1, T_2, T_3, V_1, V_2, U_1, U_2, M) \qquad (2.18)$$

where M is linked to the standard "hypercharge" by

$$M = \frac{\sqrt{3}}{2} Y = (a_1^+ a_1 + a_2^+ a_2 - 2a_3^+ a_3)/2\sqrt{3} \qquad (2.19)$$

It can be verified that the commutation rules of the components of $\underline{F}$ are the well-known ones of the SU_3 algebra, namely

$$[F_\alpha, F_\beta] = i\, f_{\alpha,\beta,\gamma} F_\gamma \qquad (\alpha \neq \beta \neq \gamma = 1 \ldots 8) \qquad (2.20)$$

where $f_{\alpha,\beta,\gamma}$ are the structure constants.

Further details on this topic can be found in Ref. [16]. Let us however stress that the evolution of the vector $\underline{F}$ is governed by the following equation

$$\dot{F}_\alpha = f_{\alpha,\beta,\gamma} \Omega_\beta F_\gamma + \text{dephasing term} \qquad (2.21$$

For the explicit expression of the dephasing term (unessential in this framework) see Ref.[16], while the vector $\underline{\Omega}$ is given by

$$\underline{\Omega} \equiv (2g_{1,2}, 0, -w_{1,2}, 2g_{1,3}, 0, 2g_{2,3}, 0, w_{1,2}/2 - w_{1,3})$$

$$(w_{i,j} = \omega_{i,j} p/m c + \Delta_{i,j}, \quad \omega_{i,j} = k_{i,j} c, \quad \Delta_{i,j} = \omega_i - \omega_j) \qquad (2.22)$$

What is remarkable is that apart from the dephasing term (due essentially to the non-commutivity between the electron variables) the equation (2.21) is identical to those derived in Ref.[16] for the molecular case. This fact is a further indication of the possibility of treating the FEL and other problems in quantum optics from a unified point of view.

We must finally stress that even if the SU_n algebra possesses n Casimir invariants, they do not bring any new constant of motion except those already discussed relevant to the total number of photons and to the total linear momentum, namely(*)

$$\sum_j a_j^+ a_j = \text{constant}$$

$$p + \hbar \sum_j k_j a_j^+ a_j = \text{constant} \qquad (2.23)$$

For more technical aspects the reader is addressed to Ref.[16], where a comparison with the molecular case too has been accomplished.

3. THE FEL SCHRÖDINGER EQUATION, AN INTRODUCTION

In the previous section we have discussed the problem of the FEL interaction, within the framework of a single electron multimode theory. We have analyzed the problem from the Heisenberg point of view, deducing the motion equations for electron and fields operators. Furthermore the group theory has been the essential tool to derive the motion invariants.

(*) Strictly speaking the second law of conservation is not linked to the Casimir invariants of SU_n but to translational invariance.

Even if the above analysis could be further exploited to understand the conditions under which coherence is preserved in FEL, we prefer to continue the discussion of the problem using a less abstract dynamical treatment.

In this section we will introduce the Schrödinger equation governing the FEL dynamics and study the solution in a particular but important case.

We have quantized both the laser and undulator fields, although the latter can be treated classically and its variations can be neglected during the interaction.

In this section we will continue to treat the undulator as a quantized field to discuss the problem of the FEL quantum coherence within the most general and rigorous framework.

According to the previous model the FEL process can be understood, in the resonant frame, as a stimulated Thomson Scattering. Furthermore, the laws of conservation relevant to the total momentum and total number of photons allow, as elsewhere remarked (see Ref.[20]), the characterization of the quantum state, describing the electron-fields coupled system, in terms of a single integer.

If we assume that initially both laser and undulator are two Glauber coherent fields and that the electron has a definite energy, the following states only can be coupled(*)

(*) Since a coherent Glauber state is a superposition of n states, we denote by $n_\pm$ the initial number of photons for each individual n-state of the laser and undulator fields respectively.

$$|\psi\rangle = e^{-i(p_0^2/(2m_0\hbar)+\omega)t} \sum_{n_-=0}^{\infty} \frac{e^{-\frac{1}{2}|\beta_0|^2}}{\sqrt{n_-!}} \cdot \beta_0^{n_-} \cdot$$

$$\cdot \sum_{n_+=0}^{\infty} \frac{e^{-\frac{1}{2}|\alpha_0|^2}}{\sqrt{n_+!}} \alpha_0^{n_+} \sum_{\ell=-n_+}^{n_-} C_\ell(t)\,|p_0-2\ell\hbar k; n_- -\ell, n_+ +\ell\rangle$$

$$(\alpha_0 = |\alpha_0|\, e^{i\omega t}, \quad \beta_0 = |\beta_0|\, e^{i\omega t}) \tag{3.1}$$

Where $|\alpha_0|$ and $|\beta_0|$ are linked to the laser and undulator average number of photons, ℓ is the number of exchanged photons during the interaction and $C_\ell(t)$ are the probability amplitudes of emitting ℓ photons.

It is easy to verify that the Schrödinger equation yields for the time-dependent coefficients C_ℓ the following equation

$$iC'_\ell = (-W_0+\varepsilon\ell)\ell C_\ell + \bar{g}[\sqrt{(n_+ +\ell+1)(n_- -\ell)}\; C_{\ell+1} +$$

$$+ \sqrt{(n_+ +\ell)(n_- -\ell+1)}\; C_{\ell-1}] \quad , \qquad C_\ell(0)=\delta_{\ell,0} \tag{3.2}$$

The two parameters W_0 and ε are given by

$$W_0 = 2\omega\Delta t p_0/m_0 c$$
$$\varepsilon = 2\,\frac{\hbar k^2}{m_0}\,\Delta t \quad . \tag{3.3}$$

It is easily realized that ε is linked to the electron recoil. Furthermore $\bar{g}=g\cdot\Delta t$ and the prime means derivative with respect to $\tau=t/\Delta t$, where Δt is the total interaction time.

The Equation (3.2) is the so-called spherical Raman-Nath (R.N.) equation [21]. It belongs to a class of differential difference equations which appear in many problems in quantum optics and have been widely utilized in the quantum description of the FEL (see the Appendix).

The exact solution of (3.2) can be written, in principle, in terms of special functions which are a generalization of the Mathieu ones [22]. However, since we need solutions of practical interest we will present a useful non-trivial perturbative solution in terms of the electron recoil.

In this section we will discuss the unperturbed case, which, for our purposes, is particularly interesting.

We discuss therefore the following equation(*)

$$i\,\overset{0}{C}{}_\ell' = -W_0\ell\overset{0}{C}_\ell + \bar{g}\Big[\sqrt{(n_+ + \ell+1)(n_- - \ell)}\;\overset{0}{C}_{\ell+1} + \sqrt{(n_+ + \ell)(n_- - \ell+1)}\;\overset{0}{C}_{\ell-1}\Big] \quad , \qquad \overset{0}{C}_\ell(0)=\delta_{\ell,0} \tag{3.4}$$

The above equation can be exactly solved. The solution technique we follow is that suggested in the above quoted reference [21].

We first perform the following transformation

$$\overset{0}{C}_\ell(x) = \overset{0}{M}_\ell(x)\cdot e^{-i\ell\beta x} \quad , \quad \beta=-W_0/\bar{g} \quad , \quad x=\bar{g}\tau \tag{3.5}$$

Inserting (3.5) in (3.4) gives

(*) The superscript "o" means zero-th order solution in ε.

$$i\,\frac{dM^0_\ell(x)}{dx} = \sqrt{(n_+ + \ell + 1)(n_- - \ell)}\; M^0_{\ell+1}(x)\, e^{-i\beta x} +$$

$$+ \sqrt{(n_+ + \ell)(n_- - \ell + 1)}\; M^0_{\ell-1}(x)\, e^{i\beta x} \quad , \qquad M^0_\ell(0) = \delta_{\ell,0}$$

(3.6)

It can be easily understood that the above expression can be derived from the following fictitious Hamiltonian

$$\bar{H} = a^+_+ a_- e^{i\beta x} + a^+_- a_+ e^{-i\beta x} \tag{3.7}$$

where $a^+_\pm$, $a_\pm$ are creation, annihilation operators ($[a^+_\pm, a_\mp] = 0$, $[a^+_\pm, a_\pm] = -1$) acting on the Fock space $|n_+, n_->$. According to the Schwinger procedure [12], outlined in the previous section, we can combine $a^+_\pm$ and $a_\pm$ in order to get the following angular momentum operators

$$J_+ = a^+_+ a_-$$
$$J_- = a^+_- a_+ \tag{3.7}$$
$$J_3 = \frac{1}{2}(a^+_+ a_+ - a^+_- a_-)$$

The Hamiltonian (3.7) can be therefore rewritten as

$$\bar{H}(J_+, J_-, x) = J_+ e^{i\beta x} + J_- e^{-i\beta x} \tag{3.8}$$

The above Hamiltonian is "time" dependent. The coefficient $M^0_\ell(x)$ can be evaluated according to(*)

(*) We have defined

$$|\bar{\psi}(x)> = \sum_{\ell=-n_+}^{n_-} M^0_\ell(x)|n_+ + \ell, n_- - \ell>$$

furthermore

$$|\bar{\psi}(x)> = \mathcal{U}(x)|\bar{\psi}(0)> \; (|\bar{\psi}(0)> = |n_+, n_->).$$

$$M_\ell^0(x) = \langle n_+ + \ell,\ n_- - \ell | \mathcal{U}(x) | n_+, n_- \rangle \tag{3.9}$$

Where $\mathcal{U}(x)$ is the evolution operator obeying the equation

$$i \frac{d\mathcal{U}}{dx} = \overline{H}(J_+, J_-, x)\mathcal{U} \quad , \quad \mathcal{U}(0) = 1 \tag{3.10}$$

The explicit time dependence of the evolution operator can be found using the Wei-Norman algebraic procedure for time ordering [23].

A rather detailed discussion of this technique can be found in Ref.[21]. Here we quote only the result of relevance to the FEL problem, namely the explicit expression of C_ℓ^0

$$C_\ell^0(\tau) = (-1)^\ell \exp\{-i \frac{n_+ n_-}{2} W_0 \tau\} \exp\{2i[\frac{n_+ - n_-}{2} + \ell] tg^{-1}(\frac{W_0}{\delta} tg \frac{\delta\tau}{2})\} \frac{1}{(1-p(\tau))^{\frac{n_+ - n_-}{2} + \ell}} (g(\tau))^\ell [\binom{n_-}{\ell}\binom{n_+ + \ell}{\ell}]^{1/2}$$

$$\cdot {}_2F_1(-n_+, n_- + 1; \ell + 1; g(\tau) f(\tau)) \tag{3.11}$$

where ${}_2F_1$ (....) is the hypergeometric function [24] and

$$p(\tau) = (\bar{g} \cdot \frac{\sin(\delta\tau/2)}{\delta/2})^2 \quad , \quad \delta = \sqrt{W_0^2 + 4\bar{g}^2}$$

$$g(\tau) = [p(\tau) \cdot (1-p(\tau))]^{\frac{1}{2}} \cdot e^{-itg^{-1}(\frac{W_0}{\delta} tg \frac{\delta\tau}{2})}$$

$$f(\tau) = [\frac{p(\tau)}{1-p(\tau)}]^{\frac{1}{2}} e^{itg^{-1}(\frac{W_0}{\delta} tg \frac{\delta\tau}{2})} \tag{3.12}$$

The above equation is rather complex; however, using both (3.11), (3.1) and the properties of the contiguous hypergeometric functions(*), one can show that

$$a_+|\psi>_{\varepsilon=0} = (\alpha_0+\alpha_+(\tau))|\psi>_{\varepsilon=0}$$

$$a_-|\psi>_{\varepsilon=0} = (\beta_0+\alpha_-(\tau))|\psi>_{\varepsilon=0} \quad (3.13)$$

where $\alpha_\pm(\tau)$ are combinations of the already defined functions $p(\tau)$, $g(\tau)$, $f(\tau)$ and have been omitted for the sake of conciseness. Since at the zero-th order in ε (3.1), is a simultaneous eigenstate of $a_\pm$, the results (3.13) could be interpreted as proof that quantum coherence, in the sense of Ref.[25], is preserved in the recoilless approximation.

This last statement deserves a few words of comment. We have already remarked that when a laser photon is created an undulator photon is annihilated and the electron loses $2\hbar k$ of momentum and vice versa. Any operator acting on the photon fields acts therefore on the electron variables. The operators $a_\pm$ are not pure field operators but implicitly contain also the operator shifting the electron momentum ($S^\pm = e^{\pm 2ikz}$) so that the above conclusion about coherence is not rigorously true. However since the electron motion in the recoilless approximation is unperturbed, the operators $S^\pm$ may be substituted by c-numbers ($S^\pm = e^{\pm iW_0\tau}$), and in this connection the state $|\psi>_{\varepsilon=0}$ could be considered coherent at any time. This answer

(*) Namely [24]

$$(c-a-1)\,{}_2F_1(a,b;c;z) = (c-1)\,{}_2F_1(a,b-1;c-1;z) - a(1-z)\,{}_2F_1(a+1,b;c;z)$$

and

$${}_2F_1(a,b;c;z) = {}_2F_1(a,b-1;c;z) + \frac{a}{c}\, z\,{}_2F_1(a+1;b;c+1;z)$$

is still ambiguous and a rigorous treatment of the problem needs the introduction of a new coherent state which accounts for the electron translational part.

Let us now learn something more from the above solution (3.11). If we assume that the laser field starts from the vacuum, we must calculate $C^0_\ell(t)$ at $n_+=0$; it can be easily shown that

$$C^0_\ell(\tau) = \binom{n_-}{\ell}^{\frac{1}{2}} e^{-\{\frac{i}{2} n_-(W_0\tau - 2tg^{-1}(\frac{W_0}{\delta} tg \frac{\delta\tau}{2}))\}} \cdot (\bar{\alpha}(\tau))^\ell$$

$$\cdot [1-|\bar{\alpha}(\tau)|^2]^{(n_- - \ell)/2}$$

$$(\bar{\alpha}(\tau) = -i(\bar{g} \frac{\sin(\delta\tau/2)}{\delta/2}) e^{itg^{-1}[\frac{W_0}{\delta}tg(\frac{\delta\tau}{2})]}) \qquad (3.14)$$

In other words, the probability of emitting ℓ-photons is a binomial distribution. Furthermore, if we assume that the undulator is initially a state with a very large number of photons, the corresponding $C^0_\ell(\tau)$ can be calculated from (3.14) taking the very large n_- limit, so obtaining(*)

$$C^0_\ell(\tau) = e^{-i\frac{W_0}{2}\int_0^\tau |\alpha(\tau')|^2 d\tau'} \cdot \frac{(\alpha(\tau))^\ell}{\sqrt{\ell!}} \cdot$$

$$\cdot e^{-(|\alpha(\tau)|^2)/2}, \quad \alpha(\tau) = -i\, e^{i\frac{W_0\tau}{2}} \bar{g}_R(\frac{\sin(W_0\tau)/2}{W_0/2})$$

$$\bar{g}_R = \bar{g}\sqrt{n_-} \qquad (3.15)$$

In this case the probability of emitting ℓ photons is a Poisson distribution. Two other interesting solutions

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(*) The correct procedure to deduce the limits can be found in Ref. [21] where the group contraction technique has been exploited.

can be derived from (3.11). The first is relevant to large undulator photon numbers and arıbitrary laser intensity. In this case we must keep the limit $n_- \to \infty$ in (3.11) thus obaining(*)

$$C^0_\ell(\tau) = e^{-i\frac{W_0}{2}\int_0^\tau |\alpha(\tau')|^2 d\tau'} \cdot \sqrt{\frac{n_+!}{(n_+ + \ell)!}} (\alpha(\tau))^\ell e^{-|\alpha(\tau)|^2/2} L^\ell_{n_+}(|\alpha(\tau)|^2) \tag{3.16}$$

Where $L^\ell_{n_+}(\cdot)$ is the generalized Laguerre polynomial. Finally if both laser and undulator have an arbitrarily large number of photons, taking the $n_+ \to \infty$ limit in (3.16) one gets(**)

$$C^0_\ell(\tau) = (-i)^\ell e^{i\ell\frac{W_0\tau}{2}} J_\ell\left(2\Omega_R \frac{\sin(W_0\tau/2)}{W_0/2}\right) \qquad (\Omega_R = g_R\sqrt{n_+}) \tag{3.17}$$

All the above discussed solutions have been the starting point of different quantum analyses of the FEL.

In particular (3.11) has been usefully exploited to study the FEL coherence properties and has also allowed a useful analogy with the two-level system physics [13,26].

(*) In deriving (3.16) we exploited the following asymptotic relation [24]

$$\lim_{n_-\to\infty} \left[\binom{n_-}{\ell}\binom{n_+ + \ell}{\ell}\right]^{\frac{1}{2}} x^\ell (1-x^2)^{(n_- - n_+ - \ell)/2}$$

$${}_2F_1(-n_+;\ n_- + 1;\ \ell+1;\ x^2) = \sqrt{\frac{n_+!}{(n_+ + \ell)!}}\, e^{-n_- x^2/2} (\sqrt{n_-}\, x)^\ell L^\ell_{n_+}(n_- x^2) \ .$$

(**) The following asymptotic properties of the Laguerre polynomials have been used [24]

$$\lim_{n\to\infty} L^\ell_n(x) \simeq x^{-\ell/2} e^{x/2} J_\ell(2\sqrt{nx}) \ .$$

The solution (3.15) was particularlv useful to state that, in the recoilless approximation, starting from the vacuum the FEL radiates into coherent Glauber states [27].

This point can be easily understood as follows: assuming that the laser field is initially the vacuum and that the undulator is an "n-state" with a large photon number, the state $|\psi>_{\varepsilon=0}$, describing the evolution of the FEL quantum system can be written as [27]

$$|\psi>_{\varepsilon=0} = e^{-i(\frac{p_0^2}{2m\hbar}\Delta t+\omega\Delta t)\tau - i\frac{W_0}{2}\int_0^\tau |\alpha(\tau')|^2 d\tau'} \cdot$$

$$\cdot\ e^{-|\alpha(\tau)|^2/2} \sum_{\ell=0}^{\infty} \frac{(\alpha(\tau))^\ell}{\sqrt{\ell!}} |\ell> \tag{3.18}$$

The proof that (3.18) is a coherent Glauber state is straightforward; indeed it is an eigenstate of the laser annihilation operator, namely

$$a|\psi>_{\varepsilon=0} = \alpha(\tau)|\psi>_{\varepsilon=0} \tag{3.19}$$

Furthermore, in this approximation one cannot find non standard effects like antibunching and squeezing. It can be immediately shown that the variance of the emitted photon number is equal to the average number of the emitted photons [27] ($<\ell>_{\varepsilon=0} = \sum_{\ell=0}^{\infty} \ell|C_\ell^0|^2$) ,

$$<\Delta\ell^2>_{\varepsilon=0} = <\ell>_{\varepsilon=0} = |\alpha(\tau)|^2 \tag{3.20}$$

In the next section we will see how the inclusion of the electron recoil is responsible for the "coherence breaking" and the non-standard effects.

Let us now come to (3.16) which has been exploited to understand the quantum aspects of an FEL amplifier [28,29]. This solution is applied to the physical situation in which a laser beam undergoes FEL amplification and one is interested in the modification of the input laser coherence properties due to the interaction.

The $|\psi\rangle_{\varepsilon=0}$ state in this case writes

$$|\psi\rangle_{\varepsilon=0} = e^{-i\left(\frac{p_0^2}{2m_0\hbar}\Delta t+\omega\Delta t\right)\tau - i\frac{W_0}{2}\left|\int_0^\tau |\alpha(\tau')|^2 d\tau\right|} \cdot \sum_{n_+=0}^{\infty} \frac{e^{-|\alpha_0|^2}}{\sqrt{n_+!}} \alpha_0^{n_+} \sum_{\ell=-n_+}^{\infty} \sqrt{\frac{n_+!}{(n_++\ell)!}} \cdot (\alpha(\tau))^{\ell} e^{-|\alpha(\tau)|^2/2} L_{n_+}^{\ell}[|\alpha(\tau)|^2]|\ell\rangle \tag{3.21}$$

In this case, assuming that the electron motion is weakly perturbed by the interaction, it can be shown that (3.21) is coherent at any time, namely(*)

$$a|\psi\rangle_{\varepsilon=0} = (\alpha(\tau)+\alpha_0 e^{iW_0\tau})|\psi\rangle_{\varepsilon=0} \tag{3.22}$$

Even in this case non standard effects do not arise. The variance is still equal to the average photon number as can be easily seen, indeed

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(*) In deriving (3.22) the following properties of the Laguerre polynomials have been used $L_n^{\ell}(\cdot)=L_{n-1}^{\ell}(\cdot)+L_n^{\ell-1}(\cdot)$. Furthermore, strictly speaking, since a is an electron-field operator the relation (3.22) is not correct and should be substituted by

$$a|\psi\rangle_{\varepsilon=0} = (\alpha(\tau) + \alpha_0 S^+)|\psi\rangle_{\varepsilon=0}$$

and this result leaves undetermined the answer whether $|\psi\rangle_{\varepsilon=0}$ is coherent or not [29].

$$\langle\Delta(n+\ell)^2\rangle_{\varepsilon=0} = \langle n+\ell\rangle_{\varepsilon=0} = |\alpha_0|^2+|\alpha(\tau)|^2 \qquad (3.23$$

To derive (3.23), it has been necessary to evaluate sums of the type $\sum_{\ell=-n_+}^{\infty} \ell^k |C_\ell|^2$ (where k is an arbitrary integer). These sums have been evaluated with the help of the following relation [30,31]

$$\sum_{\ell=-n}^{\infty} \lambda^\ell \frac{n!}{(n+\ell)!} x^{2\ell} e^{-x^2} (L_n^\ell(x^2))^2 =$$
$$= e^{(\lambda-1)x^2} L_n\left(-x^2\frac{(\lambda-1)^2}{\lambda}\right) \quad , \qquad 0<|\lambda|<\infty \qquad (3.24)$$

which will be also widely utilized in the next sections. The cases relevant to (3.17) are discussed in the Appendix.

To conclude, we stress that this section has been devoted to the essential mathematical background necessary to understand the technical details of the quantum FEL theory as developed in the current scientific literature. In the next sections we will also see how the perturbed solutions can be expressed as combinations of the different cases discussed here.

4. FEL SCHRÖDINGER EQUATION: PERTURBED SOLUTIONS

This section is devoted to the rather cumbersome problem of finding a solution for the FEL Schrödinger equation leading to physically meaningful results.

We must stress that, even if the previous unperturbed solutions gave us interesting information, it has not been possible to recover the correct gain formula. The average number of emitted photons, calculated e.g. from (3.15) gives indeed

$$\langle \ell \rangle = |\alpha(\tau)|^2 = g_R^2 \left(\frac{\sin(W_0\tau/2)}{W_0/2}\right)^2 \tag{4.1}$$

which is the well-known expression of the spontaneous emission.

The reason why we have been unable to recover the gain is straightforward. The simple physical picture we discussed in Sect.2, clearly shows that the gain mechanism is strictly linked to the $\pm 2\hbar k$ variations of the electron linear momentum. It is easily understood that the ε parameter defined in the previous section is nothing but $\frac{(\Delta p)^2}{2m_0\hbar}\Delta t$. The proper inclusion of this parameter in the solution of the R.N. equation discussed before will give us the correct gain expression. However, before discussing the solution technique including ε, let us choose the most convenient form of R.N. equation to deal with.

The spherical (or SU_2) R.N. equation introduced in the previous section is too general for our purposes. In any FEL experiment the number of undulator photons is so large that its variations can be indeed neglected. Furthermore the search of a solution, even of perturbative nature, is particularly complicated [21,32,33].

For these two reasons it is more convenient to assume, from the very beginning, that the state describing the undulator is an n state with a large number of photons ($n_U \gg \ell$). Moreover if we assume that initially the laser is the vacuum state, we easily find the following equation for the C_ℓ coefficients

$$iC'_\ell = (-W_0 + \varepsilon\ell)\ell C_\ell + \bar{g}_R[\sqrt{\ell+1}\; C_{\ell+1} + \sqrt{\ell}\; C_{\ell-1}] \qquad C_\ell(0) = \delta_{\ell,0} \tag{4.2}$$

which is known as the Harmonic R.N. equation.

An exact solution of (4.2) in terms of known functions does not exist. The best one can do is a non-trivial perturbative analysis.

By means of the transformation (3.5) we can rewrite (4.2) as

$$i \frac{dM_\ell}{dx} = 2\ell^2 M_\ell + \sqrt{\ell+1}\, M_{\ell+1} e^{-i\beta x} + \sqrt{\ell}\, M_{\ell-1} e^{+i\beta x} \qquad (4.3)$$

In analogy to what has been done in the previous section we note that the above equation can be derived from a "Hamiltonian" of the type(*)

$$\bar{H} = \rho(a^+a)^2 + f(x)a + f^*(x)a^+, \qquad f(x) = e^{-i\beta x}$$
$$\rho = \varepsilon/\bar{g}_R \qquad (4.4)$$

where $a(a^+)$ are annihilation (creation) operators.

Hamiltonians of this type are well-known in quantum optics [34], and it is also well-known that they do not preserve coherent states [35], owing to the presence of the quadratic term(**).

The technique we follow to evaluate the coefficients $M_\ell(x)$ is similar to that discussed in Sect.3. The presence

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(*) It has been indeed defined $|\bar{\psi}(x)\rangle = \sum_{\ell=0}^{\infty} M_\ell(x)|\ell\rangle$.

(**) Since the term responsible for the "coherence breaking" is linked to the electron recoil, we have the first proof that non standard effects are due to the gain. This point will be better clarified in the following.

of the quadratic term in $a^{+}a$ forces us to write an interaction Hamiltonian of the type(*)

$$\bar{H}_I(x) = e^{i\rho(a^{+}a)^2 x}\{f(x)a+f^*(x)a^{+}\}e^{-i\rho(a^{+}a)^2 x} = \tag{4.5}$$
$$= e^{-i\rho(2a^{+}a+1)x}f(x)a + e^{+i\rho(2a^{+}a-1)x}f^*(x)a^{+}$$

The interaction Hamiltonian is a rather complicated time dependent function and there is no hope of finding a simple closed expression for the evolution operator. At this moment to work out practical solutions, it is necessary to introduce some approximations. The most convenient thing to do is an expansion in terms of ρ which is linked to the electron recoil. Strictly speaking such an expansion is not in terms of ε but in terms of $\varepsilon\ell$, therefore the following condition must be satisfied $\varepsilon\ell \ll 1$. We note however that the magnitude of ℓ is also a function of the coupling constant $\bar{g}_R$ and increases rapidly with this quantity. We must therefore require the auxiliary condition

$$\varepsilon\bar{g}_R^2\,\ell < 1 \tag{4.6}$$

which is nothing but the small signal regime condition(**).

(*) In deriving (4.5) we make use of the operatorial identity

$$e^{-\xi B}Ae^{\xi B} = A + \xi[A,B] + \frac{\xi^2}{2}[[A,B],B] + \frac{\xi^3}{3!}[[[A,B],B],B] + \ldots..$$

(**) ε can be written in terms of laboratory frame variables as

$$\varepsilon = \frac{8\pi^2\lambda\!\!\!^{-}_c L}{\gamma^3\lambda^2}$$

to give an order of magnitude for the Stanford experiment we have $\varepsilon \sim 10^{-7}$

A perturbed solution up to the first order can be found with a lengthy and tedious algebra, discussed e.g. in Ref.[21]. We omit the details and write the final result

$$C_\ell = e^{-i\frac{W_0}{2}\int_0^\tau |\alpha(\tau')|^2 d\tau'} \frac{e^{-|\alpha(\tau)|^2/2}}{\sqrt{\ell!}} (\alpha(\tau))^\ell \cdot \tag{4.7}$$

$$\cdot \{A_\ell(\tau) + i\, D_\ell(\tau)\}$$

where A_ℓ and D_ℓ are given by (*)

$$A_\ell(\tau) = 1 - \frac{\varepsilon}{|\alpha(\tau)|}\frac{\partial}{\partial W_0}|\alpha(\tau)|\, L_1^{\ell-1}(\cdot) - \frac{\varepsilon}{|\alpha(\tau)|}[R(\tau) + \frac{2}{9}\frac{\partial}{\partial W_0}|\alpha(\tau)|^3]$$

$$L_1^{\ell-1}(\cdot) - \frac{2\varepsilon}{|\alpha(\tau)|}\frac{\partial}{\partial W_0}|\alpha(\tau)|\, L_2^{\ell-2}(\cdot)$$

$$D_\ell(\tau) = -\varepsilon\ell^2\tau + \varepsilon[2C(\tau)(1+|\alpha(\tau)|^2) + G(\tau) - \tau|\alpha(\tau)|^2 \cdot$$

$$\cdot (\frac{7}{6}|\alpha(\tau)|^2+1)] + \frac{\varepsilon\tau}{2} L_1^{\ell-1}(\cdot) + \varepsilon[2\tau|\alpha(\tau)|^2-3C(\tau)] \cdot$$

$$\cdot L_1^{\ell-1}(\cdot) - \frac{2\varepsilon}{|\alpha(\tau)|^2}[C(\tau) + \frac{\tau}{2}|\alpha(\tau)|^2]\, L_2^{\ell-2}(\cdot)] + \tag{4.8}$$

$$+\varepsilon[2\tau|\alpha(\tau)|^2-3C(\tau)]\, L_1^{\ell-1}(\cdot)$$

where the argument of the Laguerre polynomial is $|\alpha(\tau)|^2$ and

$$C(\tau) = (\frac{\bar{g}_R}{W_0})^2 (\tau - \frac{\sin W_0\tau}{W_0})$$

$$G(\tau) = -\frac{2}{3}(\frac{\bar{g}_R}{W_0})^4 \cdot [-\frac{\tau}{4}\cos 2W_0\tau - \frac{13}{4}\tau -$$

$$-2\tau\cos W_0\tau + \frac{1}{6}W_0^2\tau^3 + \frac{3}{4W_0}\sin 2W_0\tau + \frac{4}{W_0}\sin W_0\tau]$$

$$R(\tau) = +\frac{2}{3}(\frac{\bar{g}_R}{W_0})^3 [7\tau\cos\frac{W_0\tau}{2} + 2\tau\cos W_0\tau\cos\frac{W_0\tau}{2} -$$

(*) Note that within this framework $|\alpha(\tau)|$ means $|\alpha(\tau)| = g_R(\frac{\sin(W_0\tau/2)}{W_0/2})$ and not absolute value.

$$- \frac{7}{2W_0} \sin(\frac{3}{2}W_0 \tau) - \frac{15}{2W_0} \sin \frac{W_0\tau}{2}] \tag{4.9}$$

The first order perturbed solution of C_ℓ is considerably more complicated than the unperturbed case. The most immediate conclusion we can draw using (4.7) is that the state $|\psi\rangle = \sum_{\ell=0}^{\infty} C_\ell(t)|\ell\rangle$ is not an eigenstate of a, so that we have definitively proved that the Glauber coherence is destroyed in the limit of non-zero electron recoil(*).

We can now evaluate the average number of emitted photons, after some algebra one finds(**)

$$\langle \ell \rangle = |\alpha(\tau)|^2 - \varepsilon \frac{\partial |\alpha(\tau)|^2}{\partial W_0} + \Gamma(\bar{g}_R^4) \tag{4.10}$$

The most remarkable feature in (4.10) is the presence of the gain term, proportional, as expected, to the electron recoil. To clarify its physical meaning we stress that, within this framework, it is a genuine quantum effect due to the vacuum field fluctuations (see Ref.[27] for further comments).

The probability $|C_\ell|^2$ is not a Poisson distribution one can expect therefore sub or super Poissonian effects. This is clearly seen evaluating the second normalized moment of the distribution, which gives [27,36]

$$\Delta\ell^2 - \langle \ell \rangle = - \varepsilon \frac{\partial |\alpha(\tau)|^4}{\partial W_0} \tag{4.11}$$

(*) Recall that the vacuum $|0\rangle$ is by definition a Glauber state.

(**) where

$$\Gamma(\bar{g}_R^4) = - \frac{4}{3}\varepsilon\{ \bar{g}_R^3[\frac{7}{W_0^3} \tau\cos\frac{W_0\tau}{2} + \frac{2}{W_0^2} \tau\cos W_0\tau \cos\frac{W_0\tau}{2} - \frac{7}{2W_0^4} \sin(\frac{3}{2}W_0\tau) - \frac{15}{2W_0^4} \sin\frac{W_0\tau}{2}] + |\alpha(\tau)|^2 \frac{\partial}{\partial W_0} |\alpha(\tau)|\} |\alpha(\tau)| ,$$

and can be neglected in a small signal analysis.

and indicates that when $W_0 > 0$, $\Delta\ell^2 - \langle\ell\rangle > 0$ so that we have super-Poissonian statistics and vice versa for $W_0 < 0$.

We can also expect squeezed states [37], defining indeed(*)

$$A_1 = \frac{a+a^+}{2} \quad , \quad A_2 = \frac{a-a^+}{2i} \tag{4.12}$$

one can find [27,36]

$$\Delta A_1^2 = \frac{1}{4} + \frac{1}{2} \varepsilon \frac{\partial}{\partial W_0} |\alpha(\tau)|^2 \cos(W_0\tau)$$

$$\Delta A_2^2 = \frac{1}{4} - \frac{1}{2} \varepsilon \frac{\partial}{\partial W_0} |\alpha(\tau)|^2 \cos(W_0\tau) \tag{4.13}$$

The "squeezing" term is proportional to the electron recoil. In conclusion we have proved that, as predicted, gain, bunching and squeezing are all due to the electron recoil. We remark that the non standard effects are due to the presence of the gain: it must however be stressed that coherence is not preserved even in a gainless process ($W_0=0$) as can easily understood inspecting (4.7).

We have up to now discussed the case in which the laser field develops from the vacuum. We will analyze the problem of the evolution of the FEL optical field in the hypothesis that the input signal is a coherent Glauber state.

This study has a twofold motivation

(1) It allows the understanding of the statistical properties of an FEL amplifier

(*) The operators A_1 and A_2 correspond to the laser field p,q variables, but since a and a^+ are not pure field operators their physical meaning is ambiguous.

(2) It turns out to be useful, in a self-consistent procedure, to follow the evolution of the optical field from the vacuum to the macroscopic level [38]. (This point will be discussed in some detail in the next sections).

The problem reduces in this case to that of following the time evolution of the state

$$|\psi(\tau)\rangle = e^{-i\left(\frac{p_0^2}{2m\hbar}\Delta t+\omega\Delta t\right)\tau} \cdot \sum_{n_+=0}^{\infty} \frac{e^{-\frac{1}{2}|\alpha_0|^2}}{\sqrt{n_+!}}\,\alpha_0^{n_+} \cdot$$

$$\cdot \sum_{\ell=-n_+}^{\infty} C_\ell(\tau) \cdot |p_0 - 2\ell\hbar k;\ n_++\ell\rangle \qquad (4.14)$$

The equation for the coefficients C_ℓ now reads

$$iC'_\ell = (-W_0+\varepsilon\ell)\ell C_\ell + \overline{g}_R[\sqrt{n_++\ell+1}\ C_{\ell+1} + \sqrt{n_++\ell}\ C_{\ell-1}]$$

$$C_\ell(0)=\delta_{\ell,0} \qquad (4.15)$$

The only difference between the above equation and (4.2) is the presence of the integer n_+ in the square root. The search for a perturbed solution requires in this case few minor changes with respect to the previously discussed technique. We will omit the details and write directly the first order perturbed solution

$$C_\ell(\tau) = e^{-i\frac{W_0}{2}\int_0^\tau |\alpha(\tau')|^2 d\tau'}\ e^{-\frac{1}{2}|\alpha(\tau)|^2} \sqrt{\frac{n_+!}{(n_++\ell)!}}\ (\alpha(\tau))^\ell \cdot$$

$$\cdot \{A_\ell^{n_+}(\tau) + i\ D_\ell^{n_+}(\tau)\} \qquad (4.16)$$

where

$$A_\ell^{n_+}(\tau) = L_{n_+}^{\ell}(\cdot) - \frac{\varepsilon}{|\alpha(\tau)|}\frac{\partial}{\alpha W_0}|\alpha(\tau)|\ [|\alpha(\tau)|^2 L_{n_+-1}^{\ell+1}(\cdot) \quad +$$

37F

$$+ (n_+ +1)L^{\ell-1}_{n_+ +1}(\cdot)] + \frac{\varepsilon}{|\alpha(\tau)|}[R(\tau) + \frac{2}{9}\frac{\partial}{\partial W_0}|\alpha(\tau)|^3] \cdot$$

$$\cdot [|\alpha(\tau)|^2 L^{\ell+1}_{n_+ -1}(\cdot) - (n_+ +1)L^{\ell-1}_{n_+ +1}(\cdot)] + \frac{\varepsilon}{|\alpha(\tau)|}\frac{\partial}{\partial W_0}|\alpha(\tau)| \cdot$$

$$\cdot [|\alpha(\tau)|^4 L^{\ell-2}_{n_+ -2}(\cdot) - (n_+ +1)(n_+ +2)L^{\ell+2}_{n_+ +2}(\cdot)]$$

$$D^{n_+}_{\ell}(\tau) = -\varepsilon\ell^2\tau + \varepsilon\{[2C(\tau)|\alpha(\tau)|^2 + G(\tau) - \frac{7}{6}\tau|\alpha(\tau)|^4 +$$

$$+(2n_+ +1)(2C(\tau) - \tau|\alpha(\tau)|^2 L^{\ell}_{n_+}(\cdot) -$$

$$-\frac{\varepsilon\tau}{2}[|\alpha(\tau)|^2 L^{\ell+1}_{n_+ -1}(\cdot) - (n_+ +1) L^{\ell-1}_{n_+ +1}(\cdot)]$$

$$\varepsilon[2\tau|\alpha(\tau)|^2 - 3C(\tau)] [|\alpha(\tau)|^2 L^{\ell+1}_{n_+ -1}(\cdot) + (n_+ +1) L^{\ell-1}_{n_+ +1}(\cdot)] -$$

$$-\frac{\varepsilon}{|\alpha(\tau)|^2}[C(\tau) + \frac{\tau}{2}|\alpha(\tau)|^2][|\alpha(\tau)|^4 L^{\ell+2}_{n_+ -2}(\cdot) +$$

$$+ (n_+ +1)(n_+ +2) L^{\ell-2}_{n_+ +2}(\cdot)] + \varepsilon[2\tau|\alpha(\tau)|^2 - 3C(\tau)]|\alpha(\tau)|^2 L^{\ell+1}_{n_+ -1} + (n_+ +1)L^{\ell-1}_{n_+ +1}] \tag{4.17}$$

As an obvious comment, we note that the structure of the above solution is essentially that of the $n_+ = 0$ case, with the only difference that the Poisson functions have been substituted by the Laguerre orthonormal functions

$$\phi^{\ell}_{n_+}(|\alpha(\tau)|^2) = \sqrt{\frac{n_+!}{(n_+ +\ell)!}}\, e^{-|\alpha(\tau)|^2/2}(\alpha(\tau))^{\ell} \cdot L^{\ell}_{n_+}(|\alpha(\tau)|^2) \tag{4.18}$$

Needless to say coherence is destroyed in this case too. The number of total photons present after the interaction can be evaluated, with the aid of (3.24) and reads [27]

$$<(n_+ +\ell)> = |\alpha_0|^2 + |\alpha(\tau)|^2 - (2|\alpha_0|^2 + 1)\,\varepsilon\,\frac{\partial}{\partial W_0}|\alpha(\tau)|^2 + O(\bar{g}_R^4) \tag{4.19}$$

The first term in (4.19) is the average number of photons of the input wave, the second is the FEL spontaneous emission, the third can be split into two parts, one is the small quantum correction already discussed, the other is the classical gain expression, for the fourth term see Eq.(4.10).

We could now evaluate the amount of non standard effects in this case too, but we will not deal with this point since, within this framework, it is a straightforward excercise. The interested reader is referred to the papers quoted in [27,36]. We must however stress that, as to this last point, a clear operative definition is necessary which is able to indicate a test, possibly e.g. in an amplification experiment.

The analysis we have so far developed is basically a one electron theory. If FELs were operating with one electron there would be noisy devices only, in the sense that the stimulated part could never become larger than the spontaneous one. The inclusion of many electrons may seem, at first sight, trivial, i.e. just multiply by N_e (i.e. the total number of electrons) the above obtained expressions. However, this is not the case, as in Refs [39,40], where it has been pointed out that an extra term proportional to $N_e(N_e-1)$ should be included in the gain formula. This term, which is of classical nature, can be understood as an amplified spontaneous emission, occurring because the initial spontaneous emission proportional to N_e is afterwards amplified by one of the other electrons if gain is positive or attenuated otherwise, so that the total effect is proportional to $N_e(N_e-1)$.

According to Refs [39,40] the average number of emitted photons in the many electron case is given by

$$<\ell> = N_e[|\alpha(\tau)|^2 - \varepsilon(2|\alpha_0|^2+1)\frac{\partial}{\partial W_0}|\alpha(\tau)|^2 + \Gamma(\bar{g}_R^4)]$$
$$- N_e(N_e-1)\frac{\varepsilon \bar{g}_R^2}{W_0^2}\left(\frac{\partial}{\partial W_0}|\alpha(\tau)|^2 - W_0|\alpha(\tau)|^2\frac{\partial^2|\alpha(\tau)|^2}{\partial W_0^2}\right) \quad (4.20)$$

The last term is not necessarily small. To fix the orders of magnitude, requiring that the last term is of the same order of magnitude as the conventional spontaneous emission term, we find that the peak e-beam current must fulfill the following identity (*)

$$\hat{I} = \frac{I_0\alpha}{8\pi^2}\frac{\lambda^2 L}{n_u \sigma_z r_0^2} \quad (4.21)$$

where α is the fine structure constant, I_0 the Alfvén current and σ_z is the microbunch r.m.s. longitudinal length (all the quantities are relevant to the laboratory frame). For typical operating FEL parameters (4.21) gives for $\hat{I}$ few Amperes, largely within the limits of an R.F. conventional accelerator.

The "collective" term causes also a shift of the peak of the spontaneous emission curve towards the positive gain side by an amount given by

$$\Delta\lambda = \frac{\pi}{10\gamma}\frac{r_0 \lambda\!\!\!^{-}_c}{L^2}\sigma_z\frac{\hat{I}}{I_0} \quad (4.22)$$

(*) The relation (4.21) has been obtained in the hypothesis that the transverse e-beam dimensions are contained in the laser mode waist ($\sim \lambda L$). This condition may not be fulfilled at short wavelengths. In this case (4.21) should be substituted by the identity

$$\hat{\hat{I}} = \frac{1}{4\pi^2 k^2 N^3}\frac{I_0}{\sigma_z \lambda_u^2}$$

where $\hat{\hat{I}}$ is the peak current per volume.

Such a shift, if detectable, would be the signal marking the presence of a multielectron effect. In any practical case it is, however, negligible and, if of some relevance, would be undistinguishable from the shift caused by the beam emittance.

In any case, according to the authors of Refs [39, 40], the light statistical properties, previously discussed, are modified in a many electron theory. To give an idea, the second normalized moment of the distribution reads

$$\langle \Delta \ell^2 \rangle - \langle \ell \rangle = N_e (N_e - 1) |\alpha|^4 - \varepsilon N_e^2 \frac{\partial |\alpha|^4}{\partial W_0} \qquad (4.23)$$

It can be easily checked that the first contribution is always dominant, indicating that spontaneous emission from several electrons is incoherent (for further comments see Refs [39,40]). Also the expression giving the amount of the squeezing is modified. However we should again point out that any conclusion drawn on the physical meaning of this effect is, within this framework, highly doubtful and may be regarded as an artifact of the formalism, since the A_1 and A_2 are not pure field operators.

5. MULTIMODE THEORY AND SELF-CONSISTENT OSCILLATORY SOLUTION

In the previous sections we have developed a one electron, one mode FEL Theory. We have later stressed the necessity of including the "collective" effects, which modifies in a substantial way the results of the one body calculations.

In this section we make a further step towards a more realistic quantum FEL Theory, by including the multimode operation too.

The necessity of a multimode anlaysis was stressed by the present authors in Ref.[17], where it was pointed out that, in a bunched e-beam FEL operation, a natural mode locking of the laser-beam arises due to the discontinuous structure of the electron pulses. As has been later stressed [38], a correct description of the FEL signal starting from the vacuum requires a multimode analysis, the vacuum being, by definition, an infinite superposition of longitudinal and transverse modes(*).

The quantum multimode Hamiltonian has been written in Sect.2 (Eq.(2.12)) and the relevance of the SU_n algebra to this case has been stressed. It has been furthermore proved that the laws of conservation, relevant to the total number of photons and to the total linear momentum, are a simple generalization of the single mode case (see Eq.(2.23)).

Using the above constants of motion and assuming that no longitidunal mode is occupied initially, the state describing the evolution of the FEL can be characterized and explicitly specified by a set of integers ($\{\ell\} = \ell_1, \ell_2, \ldots, \ell_M$) that are the number of photons emitted in each mode.

In this case the state describing the evolution of the system will be more complicated than those discussed in Sects.3,4, but its generalization is quite straightforward

(*) We will discuss longitudinal modes only, a more complete analysis requires the inclusion of the transverse modes too.

$$|\psi(\tau)\rangle = \int dk_e g(k_e) e^{-i[\frac{\hbar k_e^2}{2m_0}\Delta t + (\omega_U \Delta t)(n_U^0 + 1/2) + \frac{1}{2}\sum_{j=1}^{M} \omega_j \Delta t]\tau} \sum_{\{\ell\}} C_{\{\ell\}}(k_e,\tau)\,|k_e - \sum_{j=1}^{M}(k_j + k_U)\ell_j, \{\ell\}, n_U^0 - \sum_{j=1}^{M} \ell_j\rangle \tag{5.1}$$

To treat a more general case we have assumed that the electron is not initially in an energy eigenstate and that $g(k_e)$ is initially the electron wave function in momentum representation (k_e being the electron wave number). The time dependent coefficients $C_{\{\ell\}}(k_e,\tau)$ are the probability amplitudes of exchanging ℓ_j at time t at given electron momentum $\hbar k_e$.

The Schrödinger equation for the state (5.1) yields for the coefficients $C_{\{\ell\}}(k_e,\tau)$ the following equation

$$\begin{aligned} i\frac{d}{d\tau} C_{\{\ell\}} = \sum_{j=1}^{M} \{ &-\eta_j + \sum_{s=1}^{M} \varepsilon_{s,j}\,\ell_s\}\ell_j\, C_{\{\ell\}} + \\ &+ \sum_{j=1}^{M} \bar{g}_{j,U}\{\sqrt{(\ell_j+1)(n_U^0 - \sum_{s=1}^{M}\ell_s)}\; C_{\{\ell,\ell_j+1\}} + \\ &+ \sqrt{\ell_j(n_U^0 - \sum_{s=1}^{M}\ell_s + 1)}\; C_{\{\ell;\ell_j-1\}} + \\ &+ \sum_{s=1}^{M}\sum_{\substack{j=1\\ j>s}}^{M} \bar{g}_{s,j}\cdot\{\sqrt{\ell_s(\ell_j+1)}\; C_{\{\ell;\ell_s-1;\ell_j+1\}} + \\ &+ \sqrt{\ell_j(\ell_s+1)}\; C_{\{\ell;\ell_s+1;\ell_j-1\}}\} \end{aligned} \tag{5.2}$$

where $\{\ell;\ \ell_s \pm 1;\ \ell_j \mp 1\} \equiv (\ell_1,\ldots,\ell_{s-1},\ \ell_s \pm 1,\ \ell_{s+1},\ldots,\ell_{j-1},\ell_j \mp 1,\ldots,\ell_N)$

with the initial condition

$$C_{\{\ell\}}(k_e,0) = \prod_{j=1}^{M} \delta_{\ell_j,0} \tag{5.3}$$

The quantities η_j and $\varepsilon_{s,j}$ are the generalization of the detuning and recoil parameters respectively and are defined by

$$\eta_j = (\omega_U - \omega_j + (\omega_U + \omega_j)\frac{\hbar k_e}{m_0 c})\,\Delta T$$

$$\varepsilon_{s,j} = \frac{\hbar}{2m_0 c^2}(\omega_s + \omega_U)(\omega_j + \omega_U)\,\Delta T \qquad (5.4)$$

The last parameter derives from the exchange of a j-photon in the presence of an s-photon and vice versa.

The Equation (5.2) reflects the algebraic structure of the Hamiltonian (2.12) and has been called SU_n R.N. equation in Ref. [38]. An analytical solution of (5.2) is not known even for the case of zero electron recoil. We can speculate that the zero-th order solution can be written in terms of the generalized hypergeometric functions (the so-called Lauricella functions [42]), but this requires a rather complicated analysis, irrelevant in the present context. We can indeed reduce (5.2) to a more manageable form by assuming that the number of undulator photons is extremely larger than the exchanged photons ($n_U^0 >> \sum_{s=1}^{M} \ell_s$) and consequently we can assume n_U^0 to be unchanged during the interaction. Moreover the laser--laser cou,ling term can be neglected in comparison with the undulator-laser term becuase

$$\bar{g}_{s,j} << \bar{g}_{Rj}$$

The SU_n R.N. equation (5.2) reduces therefore to(*)

(*) For convenience of notation we have omitted the explicit (k_e,τ) dependence of the coefficients $C_{\{\ell\}}$.

$$i \frac{d}{d\tau} C_{\{\ell\}} = \sum_{j=1}^{M} (- \eta_j + \sum_{s=1}^{M} \varepsilon_{s,j} \ell_s) \ell_j C_{\{\ell\}} + \\ + \sum_{j=1}^{M} \bar{g}_{Rj} [\sqrt{(\ell_j+1)}\, C_{\{\ell;\ell_j+1\}} + \sqrt{\ell_j}\, C_{\{\ell;\ell_j-1\}}] \tag{5.5}$$

The above equation can be treated perturbatively in the recoil paramters $\varepsilon_{s,j}$. The technique we adopt is a generalization of that discussed in the previous sections, but we omit the details for the sake of conciseness. In zeroth order the solution can be easily written as

$$C_{\{\ell\}}(k_e,\tau) = \prod_{j=1}^{M} \frac{1}{\sqrt{\ell_j!}} e^{-\frac{1}{2}\eta_j \int_0^\tau d\tau' |\alpha_j(\tau')|^2)} (\alpha_j(\tau))^{\ell_j} e^{-\frac{1}{2}|\alpha_j(\tau)|^2}. \tag{5.6}$$

where

$$\alpha_j(\tau) = - i \bar{g}_{Rj} e^{i\eta_j \frac{\tau}{2}} (\frac{\sin(\eta_j \tau/2)}{\eta_j/2}) \tag{5.7}$$

The result (5.6) states that the probability of emitting $\{\ell\} = (\ell_1, \ldots, \ell_N)$ photons in each longitudinal mode disperses as the product of M Poissonians.

Furthermore one can easily prove that the state given by Eq. (2.4) in zero-th order of the electron recoil parameter evolves similar to independent Glauber coherent states, i.e.

$$\prod_{j=1}^{M} \hat{A}_j |\psi\rangle_{\varepsilon=0} = \prod_{j=1}^{M} a_j e^{i(k_j+k_U)z} |\psi\rangle_{\varepsilon=0} = \prod_{j=1}^{M} \alpha_j(\tau)|\psi\rangle_{\varepsilon=0} \tag{5.8}$$

As already stressed the recoilless case only refers to spontaneous emission. The stimulated process requires the inclusion of at least the first order in the electron recoil parameter. The search for a first order perturbed solution is extremely complicated and lengthy, however the $C_{\{\ell\}}(k_e,\tau)$ can be finally written as

$$C_{\{\ell\}}(k_e,\tau) = \prod_{j=1}^{M} (-i)^{\ell_j} e^{-\frac{1}{2}\eta_j \int_0^\tau d\tau' |\alpha(\tau')|^2} \cdot$$

$$\cdot\ e^{i\ell_j \eta_j \tau/2 - \frac{1}{2}|\alpha_j(\tau)|^2} \qquad [A_{\{\ell\}}(k_e,\tau) + i\ D_{\{\ell\}}(k_e,\tau)]$$

(5.9)

Where $A_{\{\ell\}}$ and $D_{\{\ell\}}$ generalize the functions defined in (4.8), (4.17). Their explicit expressions are omitted for brevity and in any case they do not bring any particular physical insight.

The average number of emitted photons in the j-th mode can be evaluated and reads

$$\langle \ell_j \rangle = \int dk_e |g(k_e)|^2 [|\alpha_j(\tau)|^2 - \varepsilon_{j,j} \frac{\partial}{\partial \eta_j} |\alpha_j(\tau)|^2 +$$

$$+ \sum_{s=1}^{M} \varepsilon_{s,j} \ \Gamma(\alpha_j,\alpha_s;\eta_j,\eta_s)] \qquad (5.10)$$

The first two terms are not coupled, i.e. the spontaneous emission and "vacuum field fluctuations" gain in the j-th mode, the last terms describe the coupling between the different longitudinal modes. The explicit expression of the coupling function is given in Ref.[38], we emphasize that it is proportional to the fourth power of the Rabi-frequency ($\bar{g}_{Rj}$) and gives only significant contribution in the start up.

In the case of multimodes we have non standard effects too. To give an idea, the normalized second factorial moment of the distribution of the j-th mode is given by

$$(\Delta \ell_j^2) - \langle \ell_j \rangle = - \int dk_e | g(k_e)|^2 \ \varepsilon_{j,j} \frac{\partial}{\partial \eta_j} |\alpha_j(\tau)|^4$$

(5.11)

Let us note that this is exactly the single mode result and the non-diagonal longitidunal mode contributions cancel exactly.

We have so far discussed the first step of the FEL interaction. Namely when the e-beam is injected in the resonator and the FEL signal starts from the vacuum, i.e. no photon in the longitudinal modes at t=0. This is, according to Ref.[38], the initial step of a multipass process taking place in a resonator. It has been indeed envisaged [38] a self-consistent iterative approach which will use the photon distribution created in the m-th pass through the undulator as the initial photon state of the radiation in the (m+1)th pass. After the m-th step through the resonator, the state of the system, electron-undulator--laser can be represented, to first order in the $\varepsilon_{s,j}$ parameter by [38]

$$|\psi(\tau)\rangle_{m+1} = \sum_{\{n^0\}=0}^{\infty} \langle C^{m}_{\{n^0\}}(k_e,1)\rangle_{k_e} \int dk_e g(k_e) \cdot$$
$$\cdot e^{-i\Delta t\tau\left[\frac{\hbar k_e^2}{2m_0}+\omega_U(n_U^0+1/2)+\sum_{j=1}^{M}\omega_j(n_j^0+1/2)\right]} \cdot$$
$$\cdot \sum_{\{\ell\}=-\{n^0\}}^{\infty} C^{m+1}_{\{\ell\}}(k_e,\tau)\left|k_e - \sum_{j=1}^{M}\ell_j(k_j+k_U),\{n^0+\ell\},\ n_U^0 - \sum_{j=1}^{M}\ell_j\right\rangle \tag{5.12}$$

where we denote the average over the electron momentum of the probability amplitude of producing $\{n^0\}$ photons in the m-th pass as

$$\langle C^{m}_{\{n^0\}}(k_e,\tau=1)\rangle_{k_e} = \int dk_e |g(k_e)|^2 \, C^{m}_{\{n^0\}}(k_e,1) \tag{5.13}$$

The equation governing the evolution of the coefficients $C^{m+1}_{\{\ell\}}(k_e,\tau)$ can be straightforwardly written as(*)

$$i\frac{d}{d\tau}C^{m+1}_{\{\ell\}} = \sum_{j=1}^{M}\left(-\eta_j + \sum_{s=1}^{M}\varepsilon_{s,j}\ell_s\right)\ell_j\, C^{m+1}_{\{\ell\}} + + \sum_{j=1}^{M}\bar{g}_{Rj}\left[\sqrt{n^0_j+\ell_j+1}\; C^{m+1}_{\{\ell;\ell_j+1\}} + \sqrt{n^0_j+\ell_j}\; C^{m+1}_{\{\ell;\ell_j-1\}}\right] \quad (5.14)$$

with the same initial condition (5.3).

We can now evaluate quantities of physical interest as the average number of photons present after the (m+1)-th passage

$$\langle L_j\rangle^{m+1} = \int dk_e |g(k_e)|^2 [\langle L_j\rangle^m + |\alpha_j(\tau)|^2 - - \varepsilon_{j,j}(1+2\langle L_j\rangle^m)\frac{\partial}{\partial\eta_j}|\alpha_j(\tau)|^2 + \sum_{s=1}^{M}\varepsilon_{s,j}\Gamma(\alpha_j,\alpha_s;\eta_j,\eta_s)] \quad (5.15)$$

where, to simplify the notation we have introduced $\langle L_j\rangle^m$ which is the total number of photons in the j-th mode at the end of the m-th pass.

The Equation (5.15) provides a simple and useful formalism to study numerically the start up of the laser signal from the vacuum. We have not included for the moment multielectron terms which however can be taken into account generalizing the last term in (4.20). In this case (5.15) should be rewritten as

$$\langle L_j\rangle^{m+1} = \int dk_e |g(k_e)|^2 [\langle L_j\rangle^m + N_e|\alpha_j(\tau)|^2 -$$

(*) The Equation (5.14) has been written in the hypothesis of large and constant number of undulator photons and neglecting the laser--laser contributions.

$$- N_e \varepsilon_{j,j} (1+2\langle L_j\rangle^m) \frac{\partial}{\partial \eta_j} |\alpha_j(\tau)|^2 + N_e \sum_{s=1}^{M} \varepsilon_{s,j} \Gamma(\alpha_j,\alpha_s;\eta_j,\eta_s) -$$

$$- N_e(N_e-1) \frac{\varepsilon_{j,j} \bar{g}_{Rj}^2}{\eta_j^2} \left(\frac{\partial}{\partial \eta_j} |\alpha_j(\tau)|^2 - \eta_j |\alpha_j(\tau)| \frac{\partial^2 |\alpha_j(\tau)|^2}{\partial \eta_j^2}\right)] \qquad (5.16)$$

In Figures 2-4 we have collected a few numerical results derived using the above self-consistent picture. In Fig.2 we show the laser spectrum for the first 50 passages; in the first passage the spectrum is dominated by the spontaneous emission, passage after passage the stimulated terms give more and more significant contributions. The stimulated part of the spectrum becomes larger than the spontaneous part when the threshold value(*) [17]

$$\langle L\rangle_{th} = \frac{\gamma\lambda}{32\pi^2 N \bar{\lambda}_c} \qquad (5.17)$$

is reached. In the next passages the spectrum becomes narrower and narrower and the peak moves towards the maximum gain region. This effect is clearly shown in Fig.3 where we have plotted the position of the peak and the width of the spectrum, vs. the number of passages. It is evident that at the first passage the peak is positioned at zero and moves after 40 passages to 2.6 where the maximum of the gain curve is located. As to the width of the spectrum, at the first passage it is the spontaneous emission curve width and it is significantly reduced and remains practically constant after the 40th passage.

In Figure 4 we have plotted the stimulated part namely

(*) The relation refers to laboratory frame quantities.

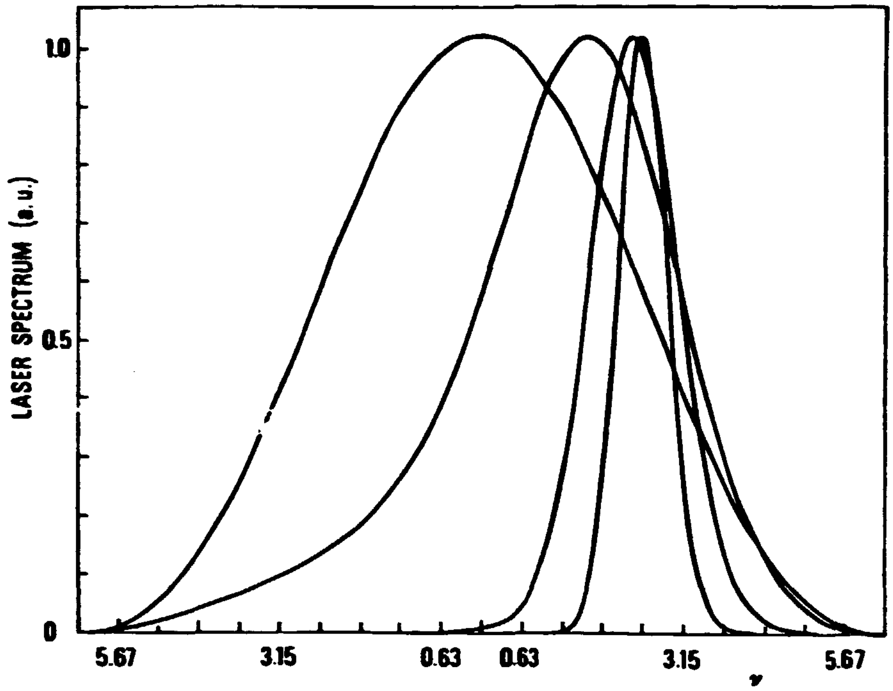


Fig. 2

Laser spectrum vs. $\nu = 2\,\pi N\,(\omega_U - \omega)\omega_U$ at different passages (1 - 60), $\dot{g}_R \cong 0.01$, $N_e \sim 10^9$, $\varepsilon \simeq 10^{-7}$.

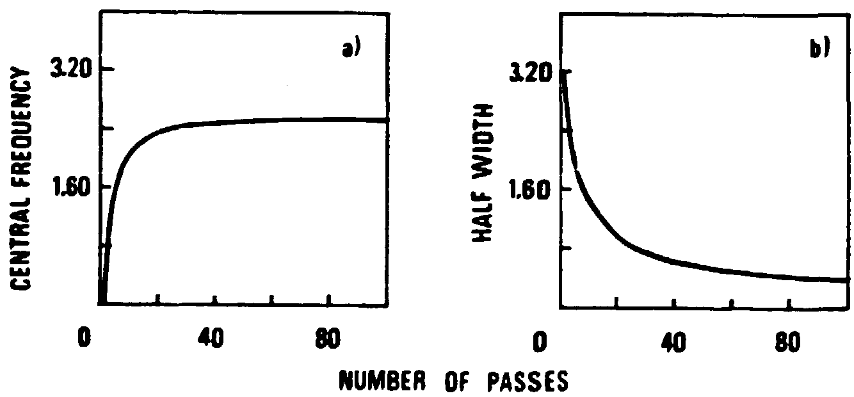


Fig. 3

(a) Central peak spectrum frequency (in units of ν) vs. number of passages (same values as the parameters in 2); (b) Spectrum half width vs. number of passages (same values as the parameters in 2).

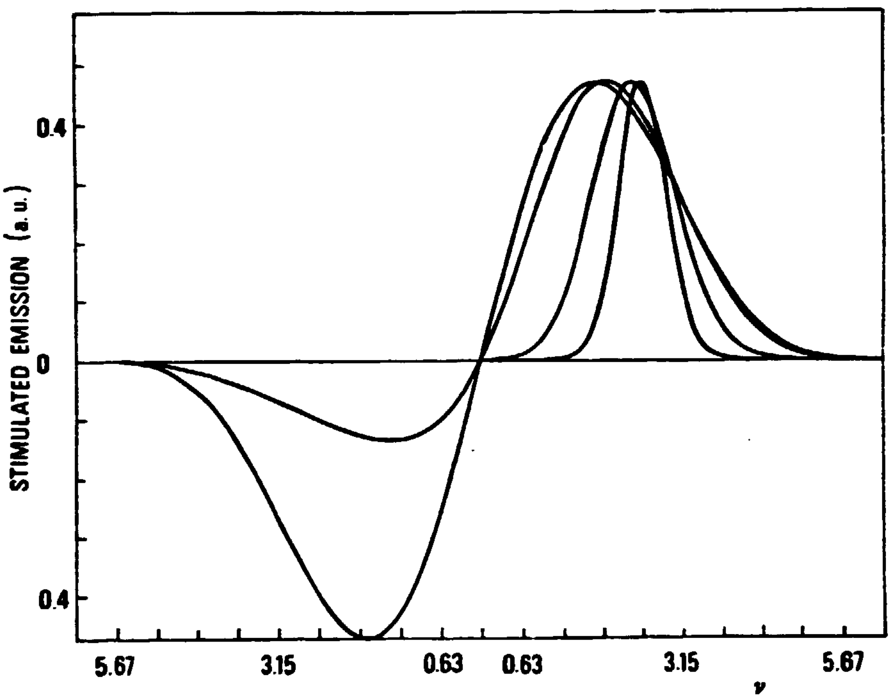


Fig. 4

Stimulated part vs. ν (same values as the parameters in 2).

$$\text{Stim.} = -\int dk_e |g(k_e)|^2 [N_e \varepsilon_{j,j} (1+2\langle L_j \rangle^m) \frac{\partial}{\partial \eta_j} |\alpha_j(\tau)|^2$$

$$\cdot - N_e \sum_{s=1}^{M} \varepsilon_{s,j} \Gamma(\alpha_j, \alpha_s; \eta_j, \eta_s)] \qquad (5.18)$$

The plot shows that at the first passage the curve is the well-known antisymmetrical FEL gain curve. When the number of passages increases the negative gain part reduces more and more, while the positive gain part becomes narrower about the maximum peak gain value and after the 20th passage reproduces essentially the spectrum of Fig.2.

We should finally point out that the above numerical results have been obtained without including the multielectron term, which, for some values of the detuning parameter gives a negative value of the spectrum. This effect is evident at the first passages (up to the 20th with the parameters of Fig. 2) where the collective term may give a signficant contribution. After it disappears and the results coincide with those obtained without the many bodies correction. The presence of a negative value of the spectrum is however a clear indication that the calculation of the many electrons effect should be reconsidered, as also stressed by the authors of Refs [39,40].

6. CONCLUSIONS

In this paper we have given a description of the state of the art of the quantum analysis of the FEL. We have developed the thoery following a rather pedagogical procedure, including both multielectrons and multimodes analyses.

In the introduction we made a distinction between two types of quantum effects, i.e. the intrinsic effects and those peculiar to a particular experiment. We concluded that the latter are practically undetectable since they can be masked by the inhomogeneous broadenings due to the e-beam energy spread and in particular to the emittances. As to the former, we must stress that non standard effects can be derived if one neglects the many electrons contributions. Including these corrections, those effects do not seem to hold anymore, however we suggest that before drawing any conclusion one must properly evaluate the quadratic term in the electron number appearing in the various physical quantities.

As far as the problem of the start up is concerned, we must note that using a simple argument based on the rate of the spontaneous emission we can conclude that at the first passage the density number of laser photons is roughly given by

$$\bar{n}_L \simeq \frac{\alpha}{4}\,\frac{\sigma_z}{r_0}\,\frac{K^2}{L^2\lambda_U}\,\frac{\hat{I}}{I_0} \tag{6.1}$$

which for typical devices is of the order of 10^3, much larger than the unity. Therefore, according to a widespread definition of the classicity threshold [43], this should be reached in the first few passages.

However things are not so simple. An appropriate analysis of the problem requires both the inclusion of the e-beam qualities (energy spread and emittances) and the lethargic effect for FEL operating with short pulses, in the gain formula.

The effect of the inhomogeneous broadening in the self consistent procedure, described in the previous section, can be accounted for straightforwardly, making a simple convolution on the e-beam energy and emittances distribution for both the spontaneous and stimulated part [44].

The inclusion of the lethargy requires a more careful analysis. We have already stressed that one of the reasons for the multimode analysis is the fact that, in FEL operating with short electron pulses, the discontinuous structure of the e-beam induces a "phase-locking" in the laser field [17]. The multimode gain formula we presented in the previous section does not contain any information about the phase coupling. This fact has a twofold motivation

(1) The phases of the vacuum field are randomly distributed and the coupling due to the phases vanishes identically.

(2) That gain formula is appropriate to a continuous e-beam operation.

The phase locking can be inserted in our self-consistent procedure using the relations of number of photons and phase variations derived in Ref.[17], namely

$$\langle L_j\rangle^{m+1} - \langle L_j\rangle^{m} = - \sum_r A_{j,r}\sqrt{\langle L_j\rangle^{m}\langle L_r\rangle^{m}} \;\cdot$$

$$\cdot\,[B^{c}_{j,r}\cos(\phi^{m}_{j} - \phi^{m}_{r}) - B^{s}_{j,r}\sin(\phi^{m}_{j} - \phi^{m}_{r})]$$

$$\phi^{m+1}_{j} - \phi^{m}_{j} = \omega_j(T_c - \delta T) + \frac{1}{2}\sum_r A_{j,r}\sqrt{\frac{\langle L_j\rangle^{m}}{\langle L_r\rangle^{m}}} \;\cdot$$

$$\cdot\ [B^{s}_{j,r}\cos(\phi^{m}_{j}-\phi^{m}_{r}) + B^{c}_{j,r}\sin(\phi^{m}_{j}-\phi^{m}_{r})] \qquad (6.2)$$

where

$$A_{j,r} = 4\bar{g}_{Rj}\bar{g}_{R\tau}\varepsilon_{r,j}N_{e} \qquad (6.3)$$

is the gain coefficient and

$$B^{c}_{j,r} = C(\eta_j,\eta_r)\tilde{f}^{c}[2(k_j-k_r)]+S(\eta_j,\eta_r)\tilde{f}^{s}[2(k_j-k_r)]$$

$$B^{s}_{j,r} = S(\eta_j,\eta_r)\tilde{f}^{c}[2(k_j-k_r)] - C(\eta_j,\eta_r)\tilde{f}^{s}[2(k_j-k_r)]$$

$$C(\eta_j,\eta_r) = \mathrm{Re}\, E(\eta_j,\eta_r)$$

$$S(\eta_j,\eta_r) = -\mathrm{Im}\, E(\eta_j,\eta_r)$$

$$E(\eta_j,\eta_r) = e^{i\eta_j}\left\{ \frac{1}{\eta_j-\eta_r}\left[\left(\frac{1-e^{-i\eta_r}}{\eta_r^2}\right) - \frac{1-e^{-i\eta_j}}{\eta_j^2}\right] - \frac{i}{\eta_j\eta_r} \right\} \qquad (6.4)$$

Finally $\tilde{f}^{c,s}$ are the cosine and sine Fourier-transform of the electron beam longitudinal distribution $f(z_0)$ namely

$$\tilde{f}^{c,s}(2(k_j-k_r)) = \int_{-\infty}^{+\infty} f(z_0)\begin{pmatrix}\cos\\ \sin\end{pmatrix}(2(k_j-k_r)z_0)dz_0 \qquad (6.5)$$

T_c is the cavity round trip period and $\delta T(<<T_c)$ is the deviation from the perfect cavity tuning [45]. The insertion of (6.2) in the self-consistent picture previously described, should be able to account for the effects due to the e-beam bunched structure. We must stress that the iteration procedure is now significantly more complicated, we must indeed account for both number of photons and

phase variations. In the first passage owing to the randomness of the initial phases the gain formula will be essentially that given in Sect.5, at the same time the modes acquire a phase value $\phi^{1}_{j}=\omega_j(T_c-\delta T)$ which can be used as input for the second step and so on. Using this procedure one can follow step-by-step the evolution of the relevant FEL physical quantities, the effect of the lethargy etc. This analysis is the natural complement of the Super Mode (S.M.) picture developed in Ref.[45]. In the sense that we can follow the FEL evolution before the stationary S.M. solution takes place. Within this framework the definition of S.M. is straightforward [45]

$$\frac{\langle L_j\rangle^{m+1}-\langle L_j\rangle^{m}}{\langle L_j\rangle^{m}}=\alpha$$

$$\phi_j^{m+1}-\phi_j^{m}=\psi \qquad (6.6)$$

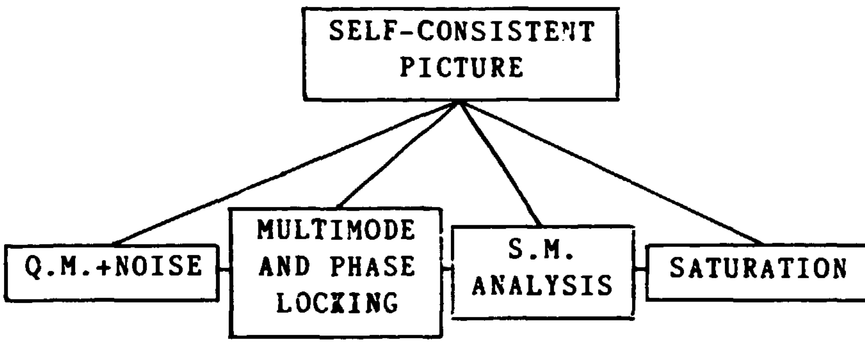
where α and ψ are identical for all the modes. The solution effects can be included calculating higher order terms in the Rabi frequency.

To evaluate the buildup of the radiation we have assumed that initially only the vacuum field fluctuation noise is present. In any case if we assume that a noise with a discrete spectrum f_ℓ is present we should change the amplitude probability of emitting ℓ-photons according to the relation [21]

$$C_\ell(\tau)\Big|_{\varepsilon=0}=\Big[\sum_k f_k(-i)^{\ell-k}\sqrt{\frac{k!}{\ell!}}\,|\alpha(\tau)|^{\ell-k}L_k^{\ell-k}(|\alpha(\tau)|^2)\Big]\cdot$$

$$\cdot\, e^{-\frac{|\alpha(\tau)|^2}{2}-i\frac{W_0}{2}\int_0^\tau|\alpha(\tau')|^2d\tau'}$$

All these effects once inserted in the self-consistent picture should give a clear insight into the FEL rise time. To summarize, we can present the following block diagram



To conclude this section, we want to underline that analogous results to those obtained in Sect.4, have already been obtained, within the framework of a single mode anlaysis by the authors of Ref.[46]. In that paper a rate equation for the laser photon evolution has been obtained, which in our notations reads(*)

$$\frac{d\langle L(W_0,n)\rangle}{dn} = N_e\{|\alpha(W_0)|^2 - \varepsilon\frac{\partial}{\partial W_0}|\alpha(W_0)|^2\langle L(W_0,n)\rangle\} \qquad (6.8)$$

where n refers to the number of passages.

Since at the first passage the number of photons is zero (6.8) can be immediately solved as

$$\langle L(W_0,n)\rangle = \langle L\rangle_{th}\frac{|\alpha(W_0)|^2}{\frac{\partial}{\partial W_0}|\alpha(W_0)|^2}\{1 - e^{-n N_e \varepsilon\frac{\partial}{\partial W_0}|\alpha(W_0)|^2}\} \qquad (6.9)$$

This equation gives the photon number evolution depicted in Fig.5 which qualitatively reproduces the results of Fig.2.

(*) We indicate $|\alpha(W_0)|^2 = \bar{g}_R^2\left(\frac{\sin(W_0/2)}{W_0/2}\right)^2$.

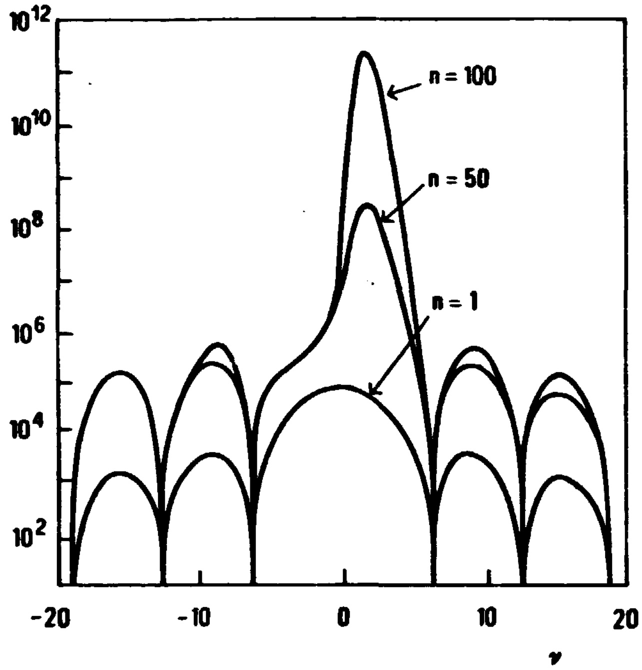


Fig. 5

Laser spectrum vs. ν (from Ref. [46]) λ_U = 3 cm, k = 0.3, N_e/V = = 10^{11} cm^{-3}, N = 50, γ = 50).

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APPENDIX

In this paper we have discussed the problem of the FEL quantum theory in a non-relativistic frame. However even if the most conspicuous part of the FEL quantum analyses has been developed in this frame, a large body of literature on this topic deals with a formulation in the laboratory one.

In particular, in the already quoted Ref.[46], the FEL oscillator evolution has been studied using relativistic quantum field theory to calculate the electron wave function, the angular distribution of the spontaneous emission and the transition rates for stimulated emission and absorption in each mode. In Ref.[47], because spin effects have been shown to be unimportant [48], the Klein--Gordon equation with classical undulator and laser fields has been used to describe the evolution of the electron motion,

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = \left(\frac{m'c}{\hbar}\right)^2 \left[1 + \frac{N^2}{\pi^2}\frac{\Omega^2}{2}\cos((k+k_U)z-\omega t)\right]\psi \qquad (A.1)$$

where

$$m' = m_0\sqrt{1+K^2} \qquad \Omega^2 = \frac{1}{c^2}\frac{e^2E_0B}{(m_0 c\gamma)^2}L^2 \qquad (A.2)$$

E_0 is the laser electric field and B the undulator magnetic field, furthermore $k = 2\pi/\lambda$ and (k,ω) are the laser wave-number and frequency respectively.

The equation (A.1) can be treated following two complementary methods.

The first has the advantage of being mathematically simpler and consists in the use of an appropriate coordinate transformation to turn the Klein-Gordon equation into a Mathieu equation which is then solved exactly using Green's functions in order to incorporate appropriate initial conditions [49]. Furthermore exploiting the asymptotic properties of the Mathieu functions [22] the expression for the small and strong signal gain can be obtained.

The second method [50] is certainly more complex from the mathematical point of view, but provides a clear physical insight. We will shortly discuss this method because it leads to equations formally identical to those discussed in Sect.2.

A wave equation satisfying the Eq.(A.1) can be written in the form

$$\psi = \sum_{n=-\infty}^{+\infty} a_n(t)\, e^{\frac{i}{\hbar}(p_n z - \varepsilon_n t)]} \tag{A.3}$$

where

$$p_n = p + \frac{n\hbar\omega}{c}(\omega+\omega_0), (\omega_0 = k_0 . c)$$
$$\varepsilon_n = c\sqrt{p_n^2 + m'^2c^2} \simeq \varepsilon + \frac{n\hbar(\omega+\omega_0)pc}{\varepsilon} + \frac{n^2\hbar^2\omega^2 m'^2 c^2}{\varepsilon^3} \tag{A.4}$$

and ε and p are the energy and momentum of the electron before the interaction with the laser-undulator potential. After a number of approximations followed by the assumption that

$$a_n(t) = C_n(t)\, e^{i(\varepsilon_n - n\omega - \varepsilon)t} \tag{A.5}$$

the Klein-Gordon equation reduces to the shift R.N. equation discussed in Sect.2, namely

$$i\hbar\dot{C}_n = \varepsilon_{anh.}(2n\frac{\Delta}{\hbar\omega}+n^2)C_n+\varepsilon_{int.}(C_{n+1}+C_{n-1}) \qquad C_n(0)=\delta_{n,0} \tag{A.6}$$

where

$$\varepsilon_{anh.} = \frac{(m'c^2\hbar)^2}{\varepsilon^3} \equiv \text{anharmonic energy}$$

$$\varepsilon_{int.} = \frac{N^2}{4\pi}\frac{m'c^2}{\varepsilon}\Omega \equiv \text{interaction energy}$$

$$\Delta \simeq -\frac{m'\gamma c^2}{2\omega_0}\Delta\omega \equiv \text{detuning from resonance} \tag{A.7}$$

It is easy to recover a one to one corrispondence between the above terms and those entering the equations discussed in Sect.2.

This short digression has been aimed at showing that both the moving and laboratory frame analysis can be accomplished with the same formalism, which, among the other things, has been widely exploited in the multiphoton dynamics [47].

We will not discuss the problems underlying the solution of the Eq.(A.6) within the above analysis, but we will refer to a paper by Kroll and Rosenbluth [51] which characterizes, in a very effective way, the FEL quantum interaction in the laboratory frame.

(*) The subscript anh. stands for anharmonic and is due to the analogy of (A.6) with the equation governing the multiphoton molecular dynamics where the anharmonic term appears too.

In that paper the authors started by quantizing the classical Hamiltonian [52] which describes the electron dynamics in a cosine-like potential provided by a static undulator and a classical laser field, namely(*)

$$H = \frac{k_U}{\gamma_R}(\delta\gamma)^2 - \frac{k\, a_U a_L}{\gamma_R}\cos\chi \tag{A.8}$$

where

$$\delta\gamma = \gamma - \gamma_R \tag{A.9}$$

is the shift of the electron energy with respect to resonance and

$$\chi = L(k+k_U)\bar{z} - \omega t \tag{A.10}$$

is the "phase" variable conjugate to $\delta\gamma$. (See Ref. [51] for further comments.) The quantization procedure of (A.8) simply requires the following identification

$$\delta\gamma = -i\hbar_1 \frac{\partial}{\partial\chi} \quad , \quad h_1 = \frac{\hbar k}{m_0 c} \tag{A.11}$$

so that

$$[\delta\gamma, \chi] = -i\hbar_1 \tag{A.12}$$

Within these variables assignment, one can write the following Schrödinger equation, describing the wave function evolution of the electron

---

(*) $a_{U,L}$ are the dimensionless vector potentials for the undulator and laser field ($a_{U,L} = e/(mc)\, A_{U,L}$).

$$- i \hbar_1 \frac{\partial \psi(\chi,\bar{z})}{\partial \bar{z}} = \frac{\hbar_1^2 k_U}{\gamma_R} \frac{\partial^2 \psi(\chi,\bar{z})}{\partial \chi^2} + \frac{k a_U a_L}{\gamma_R} (\cos \chi) \psi(\chi,\bar{z}) \qquad (A.13)$$

Because of the periodicity of the effective potential $\cos\chi$, one can write, according to the Bloch Theorem, the most general solution of (A.13) as

$$\psi(\chi,\bar{z}) = \sum_{n=-\infty}^{+\infty} C_n(\bar{z})\, e^{in\chi}\, e^{-i\frac{\varepsilon}{4}\mathcal{N}^2 \bar{z}} \qquad (A.14)$$

with the "free particle" initial condition

$$\psi(\chi,0) = e^{i\mathcal{N}\psi} \qquad (A.15)$$

where $\mathcal{N} = \delta\gamma/\hbar_1$ and ε is the recoil parameter defined in Sect.2. Substituting (A.14) in (A.13) we obtain the following equation for the C_n

$$i \frac{d}{d\bar{z}} C_n = \frac{\varepsilon}{4}(n^2 - \mathcal{N}^2) C_n - \Omega[C_{n+1} + C_{n-1}]$$

$$C_n(0) = \delta_{n,\mathcal{N}} \quad , \quad \Omega = Lka_U a_L/(2\hbar_1 \gamma_R) \qquad (A.16)$$

The Equation (A.16) is a slightly different version of the shift R.N. equation, we proceed to solve it making an expansion in Ω (small signal analysis) [53].

Following standard techniques we get

$$i \frac{d}{d\bar{z}} C_n^{(0)} = \frac{\varepsilon}{4} (n^2 - \mathcal{N}^2) C_n^{(0)} \quad , \qquad C_n^{(0)}(0) = \delta_{n,\mathcal{N}} \qquad (A.17)$$

(where the superscript "0" means unperturbed in Ω). The solution of (A.17) is

$$C_n^{(0)}(\bar{z}) = e^{-i\frac{\varepsilon}{4}(n^2-\mathcal{N}^2)\bar{z}} C_n^{(0)}(0) \tag{A.18}$$

Furthermore in first order in Ω, we obtain

$$i\frac{d}{d\bar{z}} C_n^{(1)} = \frac{\varepsilon}{4}(n^2-\mathcal{N}^2)C_n^{(1)} - C_{n+1}^{(0)} - C_{n-1}^{(0)} \qquad C_n^{(1)}(0)=0 \tag{A.19}$$

It is easy to see that the above equation splits in two uncoupled equations

$$\begin{aligned} i\frac{d}{d\bar{z}} C_{\mathcal{N}+1}^{(1)} &= \frac{\varepsilon}{4}(1+2\mathcal{N})C_{\mathcal{N}+1}^{(1)} - C_{\mathcal{N}}^{(0)} \\ i\frac{d}{d\bar{z}} C_{\mathcal{N}-1}^{(1)} &= \frac{\varepsilon}{4}(1-2\mathcal{N})C_{\mathcal{N}-1}^{(1)} - C_{\mathcal{N}}^{(0)} \end{aligned} \tag{A.20}$$

whose solutions read

$$\begin{aligned} C_{\mathcal{N}+1}^{(1)}(\bar{z}) &= i\, e^{-i\frac{\varepsilon}{8}(1+2\mathcal{N})\bar{z}} \frac{\sin[\frac{\varepsilon}{8}(1+2\mathcal{N})\bar{z}]}{[\frac{\varepsilon}{8}(1+2\mathcal{N})\bar{z}]} \\ C_{\mathcal{N}-1}^{(1)}(\bar{z}) &= i\, e^{-i\frac{\varepsilon}{8}(1-2\mathcal{N})\bar{z}} \frac{\sin[\frac{\varepsilon}{8}(1-2\mathcal{N})\bar{z}}{[\frac{\varepsilon}{8}(1-2\mathcal{N})\bar{z}]} \end{aligned} \tag{A.21}$$

The normalized wave function of the system can be obtained substituting (A.21) and (A.20) in (A.14), thus yielding

$$\begin{aligned} \Psi(\chi,\bar{z}) = \frac{1}{\sqrt{2\pi}} \{ 1 + i\Omega e^{i\chi - i\bar{z}\eta_+} \frac{\sin\eta_+\bar{z}}{\eta_+} + \\ + i\Omega e^{-i\chi - i\bar{z}\eta_-} \frac{\sin\eta_-\bar{z}}{\eta_-} \} e^{-i\frac{\varepsilon}{4}N^2\bar{z} + iN\chi} \end{aligned} \tag{A.22}$$

where it has been defined

$$\eta_\pm = \frac{\varepsilon}{4}(1\pm 2\mathcal{N}) = \frac{\varepsilon}{4} \pm \frac{\bar{W}_0}{2} \quad , \quad \bar{W}_0 = \varepsilon\mathcal{N} \tag{A.23}$$

To evaluate the change of energy of the electron at the position $\bar{z}$, one must evaluate the average value of the canonical "momentum" $\delta\gamma$, i.e.

$$\overline{\delta\gamma} = \langle\psi|\delta\gamma|\psi\rangle = -\hbar_1\Omega^2\Delta\left(\frac{\sin\eta\bar{z}}{\eta}\right)^2 \tag{A.24}$$

where

$$\Delta\left(\frac{\sin\eta\bar{z}}{\eta}\right) = \left(\frac{\sin\eta_+\bar{z}}{\eta_+}\right)^2 - \left(\frac{\sin\eta_-\bar{z}}{\eta_-}\right)^2 \tag{A.25}$$

Making a further expansion in ε we find

$$\overline{\delta\gamma}\Big|_{\bar{z}=1} \simeq -\frac{\varepsilon}{2}\hbar_1\frac{d}{d\bar{W}_0}\left(\frac{\sin(\bar{W}_0/2)}{\bar{W}_0/2}\right)^2 + O(\varepsilon^3) \tag{A.26}$$

i.e. the usual gain expression plus a negligibly small quantum correction in the electron recoil.

We can now treat the problem making an expansion in ε rather than in Ω, to reduce the Eq.(A.16) to an identical form to those already studied, we perform the following shift in the integer or to $n'=n-\mathcal{N}$, thus obtaining

$$i\frac{d}{d\bar{z}}C_{n'} = \left(\frac{\varepsilon}{4}n'^2-n'\bar{W}_0\right)C_{n'} - \Omega(C_{n'+1}+C_{n'-1}) \qquad C_{n'}(0)=\delta_{n',0} \tag{A.27}$$

A first order perturbed solution in ε can be found along the lines discussed in Ref.[21] and reads [53]

$$C_n(\bar{z}) \simeq (-i)^{n-\mathcal{N}} e^{-i(n-\mathcal{N})\frac{\bar{W}_0}{2}\bar{z}} \cdot \{A_{n-\mathcal{N}}(\bar{z}) + i\,D_{n-\mathcal{N}}(\bar{z})\} \tag{A.28}$$

where

$$A_{n-\mathcal{N}}(\bar{z}) \simeq J_{n-\mathcal{N}}(\cdot) - \frac{\varepsilon}{4}\frac{\partial}{\partial\bar{W}_0}\left(\frac{\sin(\bar{W}_0/2)\bar{z}}{\bar{W}_0/2}\right)[(2n-1)\,J_{n-\mathcal{N}-1}(\cdot) -$$

$$-(2n+1)\,J_{n-\mathcal{N}+1}(\cdot)] - \frac{\varepsilon}{8}\Omega^2\frac{\partial}{\partial\bar{W}_0}\left(\frac{\sin(\bar{W}_0/2)\bar{z}}{\bar{W}_0/2}\right)^2 \cdot$$

$$\cdot\ [J_{n-\mathcal{N}+2}(\cdot) - J_{n-\mathcal{N}-2}(\cdot)] + \ldots$$

$$D_{n-\mathcal{N}}(\bar{z}) \simeq O(\varepsilon) \qquad (J_n(\cdot) = J_n(-2\Omega\,(\frac{\sin(\bar{W}_0/2)\bar{z}}{\bar{W}_0/2})) \cdot \tag{A.28}$$

To obtain the average energy value of the electron we need to calculate

$$\overline{\delta\gamma} = \sum_{n=-\infty}^{+\infty} n|C_n|^2 = \sum_{n=-\infty}^{+\infty} n\{J^2_{n-\mathcal{N}}(\cdot) - \frac{\varepsilon}{2}\,\Omega\,\frac{\partial}{\partial\bar{W}_0}(\frac{\sin(\bar{W}_0/2)\bar{z}}{\bar{W}_0/2})\cdot J_{n-\mathcal{N}}(\cdot)\ \cdot$$

$$\cdot\ [(2n-1)J_{n-\mathcal{N}-1}(\cdot)-(2n+1)J_{n-\mathcal{N}+1}(\cdot)] - \frac{\varepsilon}{4}\Omega^2\frac{\partial}{\partial\bar{W}_0}(\frac{\sin(\bar{W}/2)\bar{z}}{\bar{W}_0/2})^2\ \cdot$$

$$\cdot\ J_{n-N}(\cdot)[J_{n-\mathcal{N}+2}(\cdot) - J_{n-\mathcal{N}-2}(\cdot)] \tag{A.29}$$

Using the sum rule [31]

$$\sum_{n=-\infty}^{+\infty} t^n J_n^2(x) = I_0(x\,\frac{t-1}{\sqrt{t}}) \tag{A.30}$$

where $I_0(x)$ is the modified Bessel function of zero order, one can perform the sums indicated in (A.29) thus finding again the result (A.26).

The above results are relevant to a classical laser. The hypothesis of a quantized one can also be discussed along these lines; the details have been discussed in Ref.[53] and turn out to be identical to those worked out in the previous sections.

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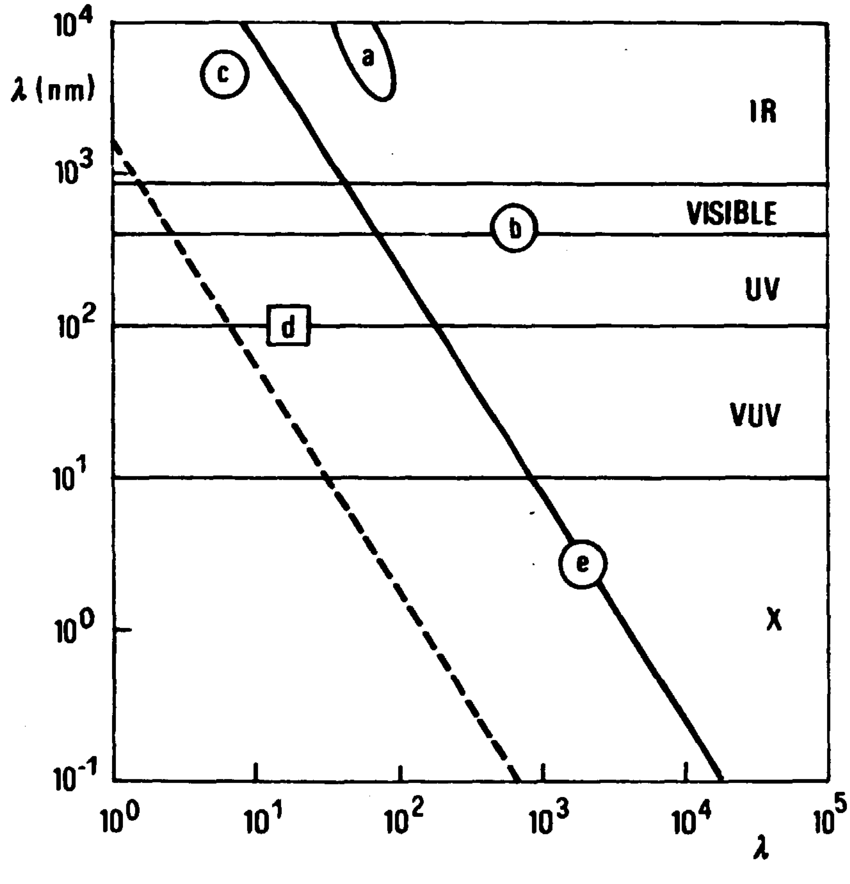


10^4
λ (nm)
10^3
10^2
10^1
10^0
10^{-1}
a
b
c
d
e
IR
VISIBLE
UV
VUV
X
10^0
10^1
10^2
10^3
10^4
10^5
λ

Fig. 1

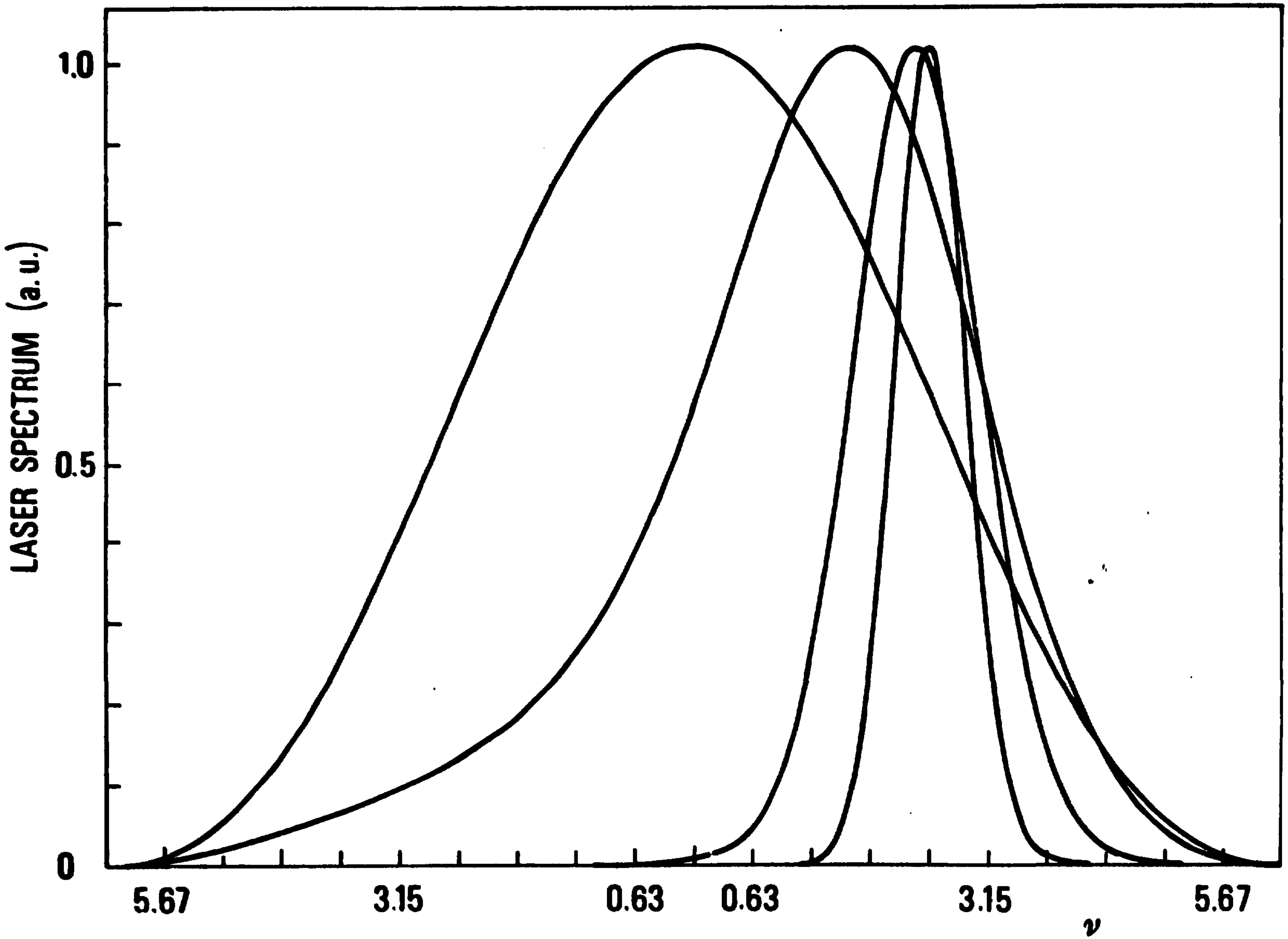
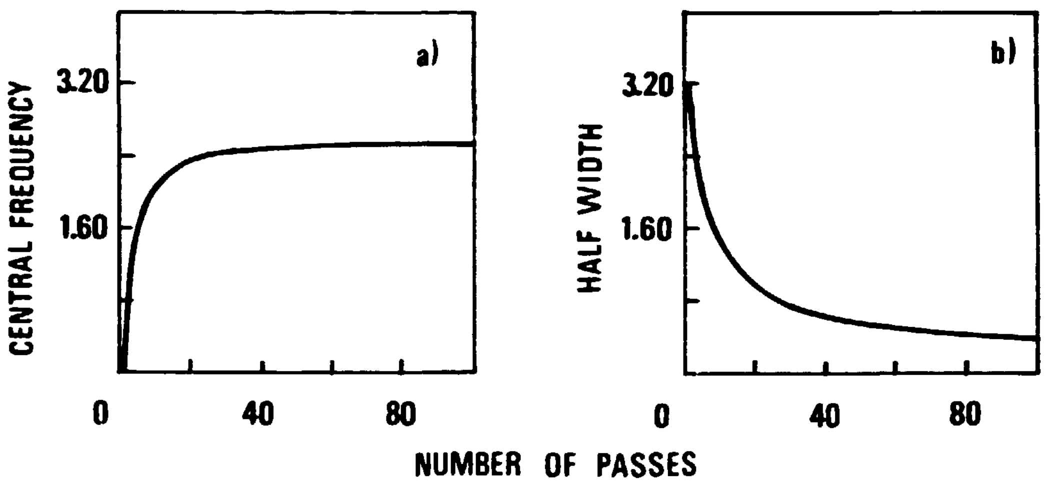


Fig. 2



a)
3.20
1.60
CENTRAL FREQUENCY
b)
3.20
1.60
HALF WIDTH
0
40
80
0
40
80
NUMBER OF PASSES

Fig. 3

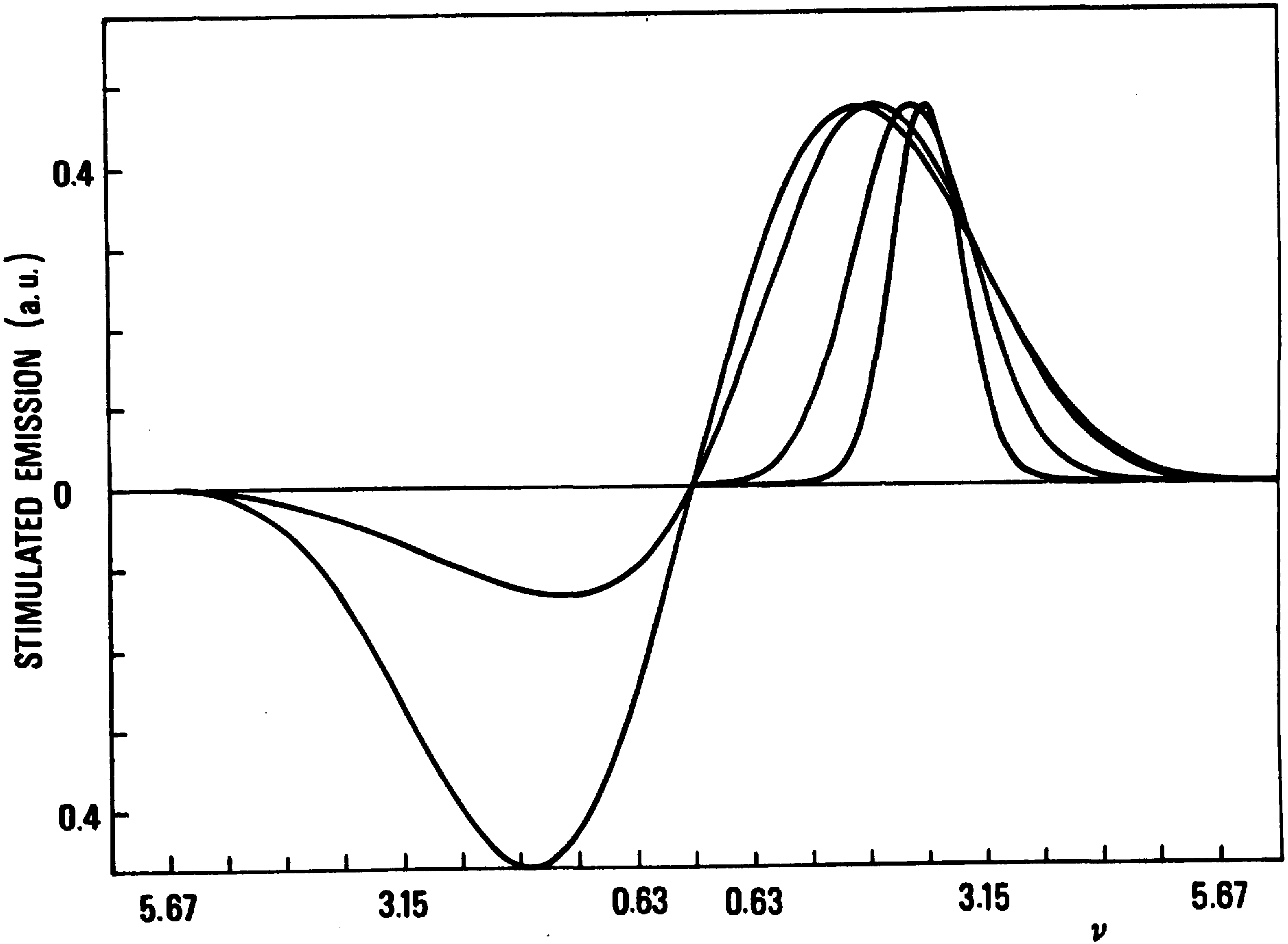


Fig. 4

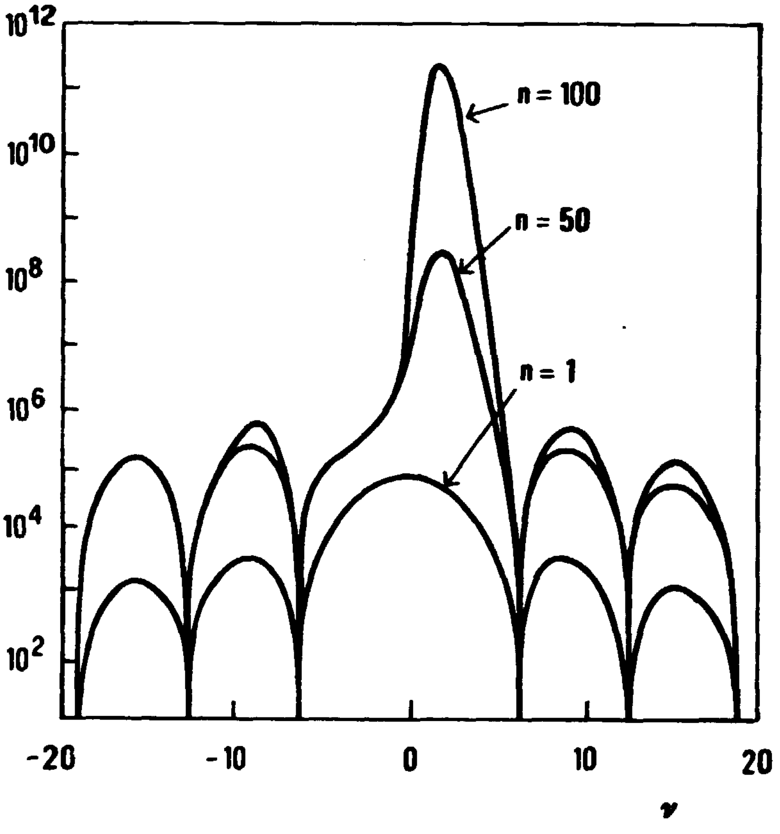


Fig. 5

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