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COERCIVE PROPERTIES OF ELLIPTIC-PARABOLIC OPERATOR

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ABSTRACT

Using a generalized Poincaré inequality, we study the coercive properties of a class of elliptic-parabolic partial differential equations, which contains many degenerate elliptic equations considered by the other authors.

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INTRODUCTION .

The coercive properties of uniformly elliptic equations on bounded domains rely on the ellipticity of the equations and the Poincare inequality . For unbounded domains we can use weighted Poincare inequalities to establish them (cf.[4,5,6,10]) . In the present paper we shall study the coercive properties of a class of degenerate elliptic equations with general domains . This class contains many degenerate elliptic equations considered by others authors . We shall establish a generalized Poincare inequality for each denegerate elliptic equation and use it to study the coercive properties .

The paper consists of two sections. In the first section we shall define and study the flow-domains. The $(w_1,\dots,w_n)$-elliptic partial differential operators and their coercive properties will be considered in the second section.

Throughout of the paper we denote

G : an open subset of the euclidian space $\mathbb{R}^{n-1}$.

$x \longmapsto b_x$: an application from G into $(0,\infty]$.

$D = \{ (t,x) \in \mathbb{R}\times G \; : \; 0 \leq t < b_x \}$

$\overset{\circ}{D}$: the interior of D .

∂D : the boundary of D .

h : a one-to-one continuous mapping from D into $\mathbb{R}^n$.

w_j : a nonnegative measurable function on $\overline{V} = \overline{h(D)}$.

$D_t = \frac{\partial}{\partial t}$

$D_i u = D_{x_i} u$ for any integer i in $\{1,\dots,n\}$.

$$((u,v)) = \int_V [\, uvw_o + \sum_{j=1}^{n} (D_j u D_j v) w_j \,]\, dx$$

$C_c^{\infty}(B)$: the family of all function u in $C^{\infty}(\mathbb{R}^n)$ having compact support contained in B .

And we assume that

(i) $|u|$ is a norm on $C_c^{\infty}(V)$, where $|u| = ((u,u))^{1/2}$.

(ii) $\mathring{D} = \{ (t,x) \in D \;:\; 0 < t < b_x \}$.

(iii) h is of class $C^1(\mathring{D})$ and its Jacobian determinant $J_h(t,x) \neq 0$ at every (t,x) in D .

(iv) For every (t,x) in D , the following integeral is finite

$$s(h,x,t) = \int_o^t |D_y (y,x)|^2 \, dy$$

Finally we write

$$d(h,w_o,\dots,w_n) = \sup_{\substack{(y,x)\in D \\ 1 \leq j \leq n}} \int_y^{b_x} \frac{s(h,x,t)\, w_o(h(t,x))\, |J_h(t,x)|}{w_i(h(y,x))\, |J_h(y,x)|} \, dt$$

if these integral are defined .

1 . FLOW-DOMAINS .

Let h , D and V be as in the introduction, we say V is a flow-domain defined by (h,D) and denote by W_o the closure in the norm $|.|$ of the following set

$$\{ u \in C^1(h(D)) \;:\; u(h(0,x)) = 0 \quad \text{for every x in G} \} .$$

We have the generalized Poincare inequality as follows :

Lemma 1 .

(i) If $d(h,w_o,w_1,\dots,w_n)$ is finite , then we have for every u in W_o

$$\int_V |u(x)|^2 w_o(x)\, dx \leq d(h,w_o,w_1,\dots,w_n) \sum_{j=1}^{n} \int_V |D_j u(x)|^2 w_i(x) dx$$

(ii) If there exists an i in $\{1,\dots,n\}$ such that $D_t h(t,h) = (0,\dots,0,D_t h_i(t,x),0,\dots,0)$ for every (t,x) in D , and if $d(h,w,w_{o\,i},\dots,w_i)$ is finite , then we have for every u in W_o

$$\int_V |u(x)|^2 w_o(x)\, dx \leq d(h,w_o,w_i,\dots,w_i) \int_V |D_i u(x)|^2 w_i(x)\, dx$$

Proof .

(i) Let $u \in C^1(h(D))$ be such that $u(h(o,x)) = 0$ for every x in G , we have

$$u(h(t,x)) = \int_o^t D_y (u(h(y,u))\, dy = \int_o^t \nabla u(h(y,x)).\, D_y h(y,x)\, dy$$

By Cauchy-Schwartz inequality we have

$$|u(h(t,x))|^2 \leq \int_o^t |D_y h(y,x)|^2 dy \int_o^t |\nabla u(h(y,x))|^2 dy$$

$$= s(h,x,t) \int_o^t \sum_{j=1}^{n} |D_j u(h(y,x))|^2 dy$$

Thus by Fubini's theorem and the theorem of change of variables we have

$$\int_V |u(y)|^2 w_o(y)\,dy = \int_D |u(h(y))|^2 w_o(h(y))|J_h(y)|\,dy$$

$$= \int_G \int_o^{b_x} |u(h(t,x))|^2 w_o(h(t,x))|J_h(t,x)|\,dt\,dx .$$

$$\leq \int_G \int_o^{b_x} [s(h,x,t) \sum_{j=1}^n \int_o^t |D_j u(h(y,x))|^2 dy] w_o(h(t,x))dtdx$$

$$= \int_G \sum_{j=1}^n \int_o^{b_x} \int_y^{b_y} s(h,t,x)|D_j u(h(y,x))|^2 w_o(h(t,x))|J_h(t,x)|\,dtdydx$$

$$\leq \int_G \sum_{j=1}^n \int_o^{b_x} d(h,w_o,w_1,\dots,w_n)|D_j u(h(y,x))|^2 w_j(h(y,x))|J_h(y,x)|dydx$$

$$= d(h,w_o,w_1,\dots,w_n) \sum_{j=1}^n \int_D |D_j u(h(y))|^2 |J_h(y)| w_j(h(y))dydx$$

$$= d(h,w_o,w_1,\dots,w_n) \sum_{j=1}^n \int_V |D_j u(x)|^2 w_j(x)\,dx .$$

Thus (i) is proved . In the case of (ii) we have

$$u(h(t,x)).D_t h(t,x) = D_i u(h(t,x))D_t h_i(t,x) .$$

Therefore we have (ii) .

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2 . THE $(w_1,\dots,w_n)$-ELLIPTIC PATIAL DIFFERENTIAL EQUATIONS .

Let a_{ij} be an enough smooth function on $\overline{V}$ for every i and j in {1,...n} . Let's consider the following partial differential operator

$$Lu = - \sum_{i,j=1}^n D_i(a_{ij} D_j u) + \sum_{j=1}^n b_j D_j u + cu .$$

We say L is $(w_1,\dots,w_n)$ - elliptic if for every $(y_1,\dots,y_n)$ in $\mathbb{R}^n$ we have

$$\sum_{i,j=1}^n a_{ij}(x)\, y_i y_j \geq \sum_{i=1}^n w_i(x)\, y_i^2 \quad \text{for every } x \text{ in } \overline{V} .$$

If $w_i(x) = F(d(x))$ for every i , where d(x) is the distance from x to a part of the boundary of V, the $(w_1,\dots,w_n)$-elliptic operator have been studied in [3,11,12,13,14] . For different w_i , we can find them in [1,2,16] . As in [7] and its reference we see that the boundary conditions of $(w_1,\dots,w_n)$-elliptic equations may be different to those of strongly elliptic equations .

We put for every $x = (x_1,\dots,x_n)$ in $\overline{V}$ and i in {1,...,n}

$$y_i(x) = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$$

$$J(i,y_i(x)) = \{ t \in \mathbb{R} : (x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_n) \in V \}$$

$$\partial_i V = \{ x \in \partial V : 0 \text{ belongs to the boundary of } J(i,y_i(x)) \}$$

We say $\partial_i V$ is the boundary of V in the i-direction . In this section we assume that $J(i,y_i(x))$ is a finite union of open intervals and $a_{ij} = a_{ji}$ for every x in V and i and j in {1,...,n} .

Let x in V and i in {1,...,n} , suppose $w_i(x) = 0$, then by the $(w_1,\dots,w_n)$-ellipticity of L , we see that

$$\sum_{j=1}^n a_{ij}(x)y_j = 0 \quad \text{for every } (y_1,\dots,y_n) \text{ in } \mathbb{R}^n .$$

This implies that $a_{ij}(x) = a_{ji}(x) = 0$ for every j . We have

<u>Lemma 2</u> . <u>Let</u> i $\in$ {1,...,n} and u, v $\in C_c^1(V)$. Suppose $w_i(x)v(x) = 0$ for every x in $\partial_i V$. Then

$$\int_V D_i(a_{ij} D_j u)v dx = - \int_V a_{ij} D_j u D_i v\, dx \quad \text{for every } j \text{ in } \{1,\dots,n\}$$

<u>Proof</u>.

We can suppose i = 1 and write $x = (t,y) \in \mathbb{R}\times\mathbb{R}^{n-1}$ and $V_2 = pr_2 V$. By Fubini's theorem and the integration by part we have

$$\int_V D_i(a_{ij} D_j u)vdx = \int_{V_2} \int_{J(1,y)} D_i(a_{ij} D_j u)vds =$$

$$= -\int_{V_2}\int_{J(1,y)} a_{ij} D_j u D_i v \, dx = -\int_V a_{ij} D_j u D_i v \, dx .$$

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Let $\overset{\circ}{W}$ be the closure in the norm $|.|$ of the following set

$$\left\{ u = v|_{\bar V} : \begin{array}{l} v \in C_c^{\infty}(R^n) ,\ v(h(o,x)) = 0 \ \text{ for every } x \text{ in } G \text{ and} \\ w_i(x)u(x) = 0 \ \text{ for every } x \text{ in } \partial_i V \ \text{ and } i \text{ in } \{1,\dots,n\} \end{array} \right\}$$

We assume that $a_{ij} D_i u D_j v$ is integrable on V for every u, v in $\overset{\circ}{W}$ and i, j in $\{1,\dots,n\}$. Then lemma 2 still holds for u , v in $\overset{\circ}{W}$. We have the coercive properties of L as follows .

Theorem 1 . Let L be a $(w_1,\dots,w_n)$-elliptic partial differential operator with $d(h,w_o,w_1,\dots,w_n) < \infty$. Suppose

(C_1) $\quad [n \sup_{1\le j\le n} d(h,|b_j|^2 w_j^{-1},w_1,\dots,w_n)]^{1/2} + d(h,|c|,w_1,\dots,w_n) < 1$

Then there exists a constant A such that for every u in $\overset{\circ}{W}$ we have

$$|u|^2 \leq A \int_V [\sum_{i,j=1}^{n} -D_i(a_{ij} D_j u) + \sum_{j=1}^{n} b_j D_j u + cu]u \, du$$

Proof .

Let u be an element of $\overset{\circ}{W}$, we have by the $(w_1,\dots,w_n)$-elliptic property of L and Lemma 2

$$(1) \quad \int_V \sum_{i,j=1}^{n} - D_i(a_{ij} D_j u)u \, dx = \int_V \sum_{i,j=1}^{n} a_{ij} D_j u D_i u \, dx \geq$$

$$\geq \sum_{j=1}^{n} \int_V |D_j v|^2 w_j \, dx$$

By lemma 1 we have

$$(2) \quad \int_V \sum_{i,j=1}^{n} -D_i(a_{ij} D_j u)u \, dx \geq d(h,w_o,w_1,\dots,w_n)^{-1} \int_V |u|^2 w_o \, dx$$

By Cauchy-Schwartz inequality and lemma 1 we get

$$(3) \quad \left| \int_V b_j u D_j u \, dx \right| \leq [\int_V |b_j|^2 w_j^{-1} |u|^2 dx]^{1/2} [\int_V |D_j u|^2 w_j \, dx]^{1/2} \leq$$

$$\leq d(h,|b_j|^2 w_j^{-1},w_1,\dots,w_n)^{1/2} [\int_V \sum_{i=1}^{n} |D_i u|^2 w_i \, dx]^{1/2} [\int_V |D_j u|^2 w_j \, dx]^{1/2}$$

Applying Cauchy-Schwartz inequality we have from (3)

$$(4) \quad \left| \sum_{j=1}^{n} \int_V b_j u D_j u \, dx \right| \leq$$

$$\leq [\sup_{1\le j\le n} d(h,|b_j|^2 w_j^{-1},w_1,\dots,w_n)]^{1/2} \sum_{j=1}^{n} (\int_V |D_j u|^2 w_j \, dx)^{1/2} [\int_V \sum_{i=1}^{n} |D_i u|^2 w_i \, dx]^{1/2}$$

$$\leq [n \sup_{1\le j\le n} d(h,|b_j|^2 w_j^{-1},w_1,\dots,w_n)]^{1/2} \int_V \sum_{i=1}^{n} |D_i u|^2 w_i \, dx$$

By the proof of Lemma 1 we have

$$(5) \quad \left| \int_V c|u|^2 dx \right| \leq d(h,|c|,w_1,\dots,w_n) \int_V \sum_{j=1}^{n} |D_j u|^2 w_j \, dx$$

By (1),(4),(5) (C_1) and (2) we have the theorem .

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Theorem 2 . Let L be a $(w_1,\dots,w_n)$-elliptic operator with $d(h,w_o,w_1,\dots,w_o) < \infty$. Suppose

(C_2) For every j , b_j is continuously differentiable on V and $u(x)b_j(x) = 0$ for every u in W and x in $\partial_j V$, and there is a constant $k < 1$ such that for almost x in V we have

$$\left| c(x) - \frac{1}{2} \sum_{j=1}^{n} D_j b_j(x) \right| \leq k \, d(h,w_o,w_1,\dots,w_n)^{-1} w_o(x).$$

Then there exist a constant A such that for every u in $\overset{\circ}{W}$ we have

$$A \int_V [- \sum_{i,j=1}^{n} D_i(a_{ij} D_j u) + \sum_{j=1}^{n} b_j D_j u + cu]u \, dx \geq |u|^2$$

Proof.

For each u in $\overset{\circ}{W}$ we have by lemma 2 and (C_2)

$$(6) \quad \left|\int_V \Big(\sum_{j=1}^n b_j D_j u + cu\Big) u\,dx\right| = \left|\int_V \Big(c|u|^2 + \frac{1}{2}\sum_{j=1}^n b_j D_j |u|^2\Big)dx\right|$$

$$= \left|\int_V \Big(c - \frac{1}{2}\sum_{j=1}^n D_j b_j\Big)|u|^2 dx\right| \leq k\,d(h,w_o,w_1,\dots,w_n)^{-1}\int_V |u|^2 w_o\,dx$$

From (1), (2) and (6) we obtain the theorem.

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Applying the part (ii) of lemma 1, we obtain

Theorem 3. Let L be a $(w_1,\dots,w_n)$- elliptic operator. Suppose there exists an i in $\{1,\dots,n\}$ such that $d(h,w_o,w_i,\dots,w_i) < \infty$ and $D_t h(t,x) = (0,\dots,0,D_t h_i(t,x),0,\dots,0)$ for every (t,x) in D, and the following condition is satisfied

(C_3) For every j in $\{1,\dots,n\}$, b_j is continuously differentiable on V and $u(x)b_j(x) = 0$ for every u in W and x in $\partial_j V$, and there exists a constant k in the interval $(0,1)$ such that for almost x in V

$$\left| c(x) - \frac{1}{2}\sum_{j=1}^n D_j b_j(x) \right| \leq k\, d(h,w_o,w_i,\dots,w_i)^{-1} w_o(x) .$$

Then there exists a constant A such that for every u in $\overset{\circ}{W}$

$$|u|^2 \leq A \int_V Lu.u\,dx$$

Remark. The boundary condition of u in $\overset{\circ}{W}$ can be written as follows $u|_{\partial_o V} = 0$, where $\partial_o V$ is the following set

$$h(\{0\}\times G) \cup \bigcup_{i=1}^n [\, \{x : w_i(x) \neq 0\} \cap \partial_i V \,]$$

Now we consider some examples of $(w_1,\dots,w_o)$-elliptic equations.

Let $V = \{ (x,t) \in \mathbb{R}^{n-1}\times\mathbb{R} : (t^{3/2},x)$ belongs to the interior of D $\}$

Let a_i be a nonnegative continuously differentiable function on V,

for every i in $\{1,\dots,n\}$. Assume a_n is positive and independent of the variable x_n. Let c be a measurable function on V such that there exists a constant k in the interval (0,1) such that

$|c(x_1,\dots,x_n)| \leq k\,(2x_n)^{-2} a_n(x_1,\dots,x_n)$ for every $(x_1,\dots,x_n)$ in V.

Let's consider the following Dirichlet problem

$$(P_1) \quad \begin{cases} \sum_{i=1}^n D_i(a_i D_i u) + cu = f \\ u|_{\partial_o V} = 0 \end{cases}$$

We put

$$h : D \longrightarrow V$$

$$h(t,x) = (x, t^{2/3})$$

$$w_o(x_1,\dots,x_n) = x_n^{-2} a_n(x_1,\dots,x_n)$$

$w_i = a_i$ for every i in $\{1,\dots,n\}$.

Then we have for every x in G and t,y in the interval $(0,b_x)$:

$$D_t h(t,x) = (0,\dots,0,\frac{2}{3}t^{-1/3})$$

$$|D_t h(t,x)| = |J_h(t,x)| = \frac{2}{3}t^{-1/3}$$

$$s(h,x,t) = \int_0^t |D_t h(s,x)|^2 ds = \frac{4}{9}\int_0^t s^{-2/3} ds = \frac{4}{3}t^{1/3}$$

$$F(w_o,w_n,x,y) = \int_y^{b_x} \frac{s(h,x,t)\,w_o(h(t,x))\,|J_h(t,x)|}{w_n(h(y,x))\,|J_h(y,x)|}\,dt$$

$$= \frac{4}{3}\int_y^{b_x} \frac{t^{1/3}\,t^{-4/3}\,a_n(x,t^{2/3})\,t^{-1/3}}{a_n(x,y^{2/3})\,y^{-1/3}}\,dt$$

Thus

$$F(w_o, w_n, x, y) = 4\, y^{1/3}\left(y^{-1/3} - b_x^{-1/3}\right) \leq 4 .$$

Therefore $d(h, w_o, w_n, \ldots, w_n) \leq 4$ and $|c| \leq k\, d(h, w_o, w_n, \ldots, w_n)^{-1}\, w_o$.

We have the following result.

<u>Theorem 4</u>. <u>Suppose there exists a contant</u> B <u>such that for every</u> v <u>in</u> $\mathring{W}$ <u>we have</u>

$$\left| \int_V f v\, dx \right| \leq B\, |v|$$

<u>Then the problem</u> (P_1) <u>has an unique solution</u> u <u>in</u> $\mathring{W}$.

<u>Proof</u>.

Put

$$Lu = \sum_{i=1}^{n} -D_i(a_i D_i u) - cu$$

By Theorem 3 and the proof of theorem 2, we see that the inner product $((.,.))$ is equivalent to the following

$$[u,v] = \int_V Lu.v\, dx$$

By the Riesz's theorem and a standard procedure, we can find an unique solution u in $\mathring{W}$ of the problem (P_1).

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If $n = 2$, the problem (P_1) has been studied in [1,16] in the form

$$y^p D_x^2 u + x^q D_y^2 u = 0 \qquad p, q > 0.$$

$$u\big|_{\partial V} = g$$

If $a_i = 1$ for every i, the problem (P_1) becomes the Schrödinger equation in bounded or unbounded domains, and is studied in [4,10].

Let $V = (0,1)^3$, b and c be two positive continuously differentiable functions on $\bar{V}$. Suppose that for every (x,y,z) in $\bar{V}$ we have

$$b(x,0,z) = b(x,1,z) = 0$$

$$c(x,y,0) = 0$$

$$c(x,y,1) = 1$$

Let's consider the following problem

$$(P_2) \begin{cases} Lu = D_x^2 u + bD_y^2 u + cD_z^2 u + \lambda u = f \\ u\big|_{\partial_o V} = 0 \end{cases}$$

We put $\mathring{D} = V$, $h(t,(y,z)) = (t,y,z)$ for every (t,y,z) in V, $w_o = w_1 = 1$, $w_2 = b$ and $w_3 = c$. We have

$$d(h, w_o, w_1, \ldots, w_1) = \frac{1}{2}$$

$$\partial_o V = \left((0,1) \times [0,1]^2 \right) \cup \left([0,1]^2 \times \{1\} \right)$$

Arguing as in the proof of theorem 4 we have

<u>Theorem 5</u>. <u>For every</u> f <u>in</u> $L_2(V)$ <u>and</u> λ <u>with</u> $|\lambda| < 2$, <u>there exists an unique solution</u> u <u>in</u> W <u>of problem</u> (P_2).

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REFERENCES .

1. D. Amanov , Some boundary value problems for a degenerate eliptic equation in an unbounded domain . (Russian). Izv.Akd. Nauk UzSSR Ser. Fiz-Math.Nauk n.1 (1984), 8-13 .

2. M.S. Baoendi , Sur une classe d'operateurs elliptiques degeneres. Bull.Soc.Math.France 95 (1967), 45-87 .

3. M. Franciosi , Un teorema di esistenza ed unicita per la soluzione di un' equazione ellitico-parabolica , a coefficenti discontinui , in forma non divergenza . Bol.Un.Math.Ital. B(6) , n.1 (1985), 253-263 .

4. R.Janssen , Some variants of the Poincare lemma . Applicable Anal.14 (1983),303-315 .

5.__________ , The Dirichlet problem for second order elliptic operators on unbounded domains. Applicable Anal. 19,n.2-3 (1985) , 201-216 .

6.__________ , Elliptic problems on unbounded domains. SIAM J.Math.Anal. 17,n.6 (1986),1370-1389 .

7. J.J. Kohn and L. Nirenberg, Degenerate elliptic-parabolic equations of second order. Comm.Pure Appl.Math.20 (1967),797-872.

8. P.I. Lizorkin and S.M. Nikol'skki, Coercive properties of elliptic equations with degeneracies, variational methods. Proc.Steklov Inst.Math. 157,n.3 (1983),95-125 .

9. ________________________________, Coercive properties of an elliptic equation with degeneracy and generalized right-hand side. Proc.Steklov Inst.Math. 161,n.3 (1984),171-198 .

10. J. Mäulen , Lösung der Poissongleichung and harmonische Vectorfelder in unbeschränkten Gebieten. Math.Meth.in Appl.Sci.5 (1983) 303-315 .

11. N.V. Rogova , An equation with semi degeneration. (Russian) Differentzial'nye Urav. 20,n.11,2013-2015 (1984) .

12. Ya.A. Roitberg and Z.G. Shiftel , General boundary value problems for Douglis-Nirenberg elliptic systems with strong denegeracy Soviet Math.Dokl. 29,n.3 (1984),701-704 .

13. S. Rukauskas , The modified Dirichlet problem for a three dimensional elliptic equation strongly degenerate on the line . (Russian) Litovsk.Math.Sb. 24,n.2 (1984), 160-165 .

14. V.V. Shan'kov , Coercive properties of an elliptic equation with degeneracy. Soviet Math.Dokl.31,n.2 (1985), 440-442 .

15. A.K. Tyulina , Behavior of solutions of the first boundary value problem for a second-order elliptic quation in an unbounded domain. Moscow Univ.Math.Bull.39,n.4 (1984),48-54 .

16. A. Xasanov , On a mixed problem about the equation

$$\text{sign}\, y|y|^m u_{xx} + x^n u_{yy} = 0 .$$

Izv.Akd.Nauk UzSSR Ser.Phys.Math.Nauk,n.2 (1982),28-32 .