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IS SMALL VIOLATION OF THE PAULI PRINCIPLE POSSIBLE? *

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ABSTRACT

The Pauli exclusion principle is one of the most fundamental laws of nature. Yet the experiment shows that many fundamental laws are in fact not absolute, but only approximate, i.e. are valid only to a certain accuracy. At present, however, there are no answers to the question: "To what accuracy is the Pauli principle valid?" This is so because there are no models capable of describing small deviations from the exclusion principle. In the present paper we consider the problem of constructing such models. We have constructed the simplest algebra of the creation and annihilation operators with a parameter β which incorporates the small violations of the Pauli principle (for $\beta = 0$ the Pauli principle holds absolutely true). The commutation relations in this model prove to be trilinear. We then present a model Hamiltonian based on the constructed algebra which describes the Pauli principle violating transitions i.e. transitions of two identical particles into the same state) with the probability suppressed by a factor of β^2 (notwithstanding the fact that the Hamiltonian itself does not contain any small parameters).

1. INTRODUCTION

Beginning with the pioneering works on the nonconservation of parity in weak interactions the problem of possible small violations of the fundamental physical principles has been constantly drawing much attention of both theorists and experimental physicists.

Recently much attention was paid to the problem of nonconservation of such quantum numbers as lepton and baryon charges. The nonconservation of these charges can be described in the framework of models, satisfying all the fundamental principles of quantum field theory (i.e. Lorentz invariance, relativistic causality and so on). The well-known examples of such models are grand unified theories.

Fewer papers were devoted to possible nonconservation of the electric charge [1,2], CPT-violation [3] and Lorentz invariance. The construction of models incorporating such effects apparently requires rather drastic modification of the existing theory.

The Pauli exclusion principle is certainly one of the most fundamental principles of the quantum theory. A possibility of small violations of this principle was discussed in Ref. [5]. However, this principle is very much different in its nature from those principles which are connected to the existence of some kind of symmetry (discrete or continuous). The symmetry principles are valid because of the special form (i.e. invariance) of the Lagrangian (and the vacuum) whereas the Pauli principle holds true due to the special form of the commutation relations. Therefore, the exact meaning of the phrase "small violation of the Pauli principle" is far from being trivial and does not seem to be unique. Here it should be noted that we here do not refer to the well-known facts that beside the Fermi-statistics there can exist other types of statistics (for instance, parastatistics [6] or interpolating statistics [7]; from our point of view parastatistics can be treated as "100% violation of Pauli principle").

Rather, we are interested in the possibility that the "Ultimate Theory" contains a parameter β such that at $\beta=0$ the Pauli principle is absolutely exact, but at nonzero β the Pauli principle violating processes are possible, but suppressed, say, by the powers of β. An example of such process is the transition of an atomic electron to the state which is already occupied by another electron.

The salient feature of the case under consideration is that the question of the choice of the parameter β is far from being trivial. This is so because in order to answer this question one has to construct some model describing the violation of the Pauli principle which vanishes when $\beta \to 0$. We know of no published attempts in this direction.(Here we should like to mention once more that here we are not concerned with alternative types of statistics such as parastatistics, interpolating statistics etc.).

These models of such kind are interesting not only because of the traditional "why not" motivation. The point is that the violation of the Pauli principle, if it really exists, could be seen in the same type of experiments as are designed for searches of electron instability (in both cases there occurs the radiation of a hard γ-quantum after the transition of an electron from a higher level to the K-shell.

Let us make clear why a possibility of a small violation of the Pauli principle is not forbidden by the general theorems of the quantum field theory which describe the connection between spin and statistics. In the axiomatic field theory (see, e.g. Ref [8]) people have proved the following spin-statistics theorem. Let $\psi(x)$ be a complex quantum field transforming according to an arbitrary irreducible representation $\mathcal{D}\left(\frac{j}{2}, \frac{k}{2}\right)$ of $SL(2,C)$ group. Then, if

$$\psi_\beta(x)\,\psi_\beta^*(y) = -(-1)^{j+k}\,\psi_\beta^*(y)\,\psi_\beta(x) \quad \text{for} \quad (x-y)^2 < 0$$

then $\psi_\beta(x) = 0$.

Thus we see that this and analogous theorems forbid the Bose quantization of the spin-$\frac{1}{2}$ fields. However, one can think (at least, in principle) of many other ways of quantizing the spin-$\frac{1}{2}$ fields beside the Bose way, and all these ways are not excluded by the spin-statistic theorems. In particular, the question of whether there exists a method of quantization which would lead to a small violation of the Pauli principle is left open by this theorems.

The validity of the Pauli principle follows automatically from the form of the anticommutation relations between the operators of the electron-positron fields $\psi(x)$, $\bar{\psi}(x)$ or, which is equivalent, between the creation and annihilation operators of the electrons and positrons $a_{k\sigma}$, $a^+_{k\sigma}$, $b_{k\sigma}$, $b^+_{k\sigma}$.

Therefore in a theory, describing the violation of the principle the anticommutation relations should be certainly changed. Apriori it is not at all clear in what form should one take new commutation relations. Generally speaking, to solve this problem one could reason as follows. Let us consider various sets of the commutation relations between the fields $\psi(x)$, $\bar{\psi}(x)$ of the most general form:

$$\sum_{ij} C^{(q)}_{i_1 \dots i_n j_1 \dots j_m}\, \psi(x_{i_1}) \dots \psi(x_{i_n})\,\bar{\psi}(y_{j_1}) \dots \bar{\psi}(y_{j_m}) = 0 \qquad q = 1, 2 \dots k \tag{1}$$

One should consider not only bilinear relations, but also trilinear ones and so on, each set containing not necessarily one but, in general, several (k) independent relations. Furthermore, one should require that these relations satisfy a number of general principles of the quantum field theory: the positivity of energy, Lorentz invariance, relativistic causality, the electric charge (fermionic number) conservation, C-parity. As for the last four principles, one should not require that these principles be absolutely valid, it suffices to require that their possible violations were at most of order $O(\beta)$ (where β is the Pauli principle violating parameter) so that when $\beta \to 0$ all these violations were unobservably small.

In this way one could in principle find all the allowed commutation relations (or prove that they do not exist) containing a small parameter β and resulting in the usual fermi-statistics of the electrons when $\beta \to 0$.

In practice, however, this way seems intractable. A more simple method consists in the following. Instead of searching for the algebra of the operators, let us try to construct first the representation of this algebra possessing the necessary property and after that try to find the commutation relations themselves. It is more convenient to work with the creation and annihilation operators than with the field operators. Let us further simplify the problem by discarding the momentum and spin variables and considering only electron (but not positron) operators.*)

2. The construction of an algebra incorporating a small violation of the Pauli principle.

Thus, we should find representation of the creation and annihilation operators a, a^+ depending on the parameter β so that when β tends to zero this representation transforms into the usual Fermi representation

$$a_F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a_F^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2}$$

(As the orthonormal base here we take the vacuum $|0\rangle$ and one-particle state $|1\rangle$). Evidently, the minimal dimensionality of the state space we are looking for is three, choose as the basis of that space the states $|0\rangle$ (vacuum), $|1\rangle$ (one-particle state) and $|2\rangle$ (two-particle state).**) Suppose that the action of the creation and annihilation operators is defined as follows (the parameter β is supposed to be real):

$$\begin{aligned} a^+|0\rangle &= |1\rangle & a|0\rangle &= 0 \\ a^+|1\rangle &= \beta|2\rangle & a|1\rangle &= |0\rangle \\ a^+|2\rangle &= 0 & a|2\rangle &= \beta|1\rangle \end{aligned} \tag{3}$$

Then the matrices of these operators in the chosen basis take the form:

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix} \qquad a^+ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix} \tag{4}$$

The Hilbert state space H can be decomposed into the direct sum of the subspaces H_2 (stretched over the vectors $|0\rangle$, $|1\rangle$) and H_1 (stretched over the vector $|2\rangle$). It is clear that if $\beta = 0$ the transitions between the states in H_2 and H_1 become forbidden so that the space H_1 gets completely decoupled from H_2.

Now, let us construct the commutation relations (i.e., algebra) which is satisfied by the operators a, a^+. To do that, one should calculate various products of the operators a, a^+ of the form a^2, a^+a, aa^+, a^3 etc and then find the relations between them (such relations should certainly exist because there are only 9 independent 3x3 matrices). It is convenient to choose 9 basic matrices M_{ij} (i, j = 1, 2, 3) as follows

$$M_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5}$$

and so on. These matrices are not Hermitean: $M^+_{ij} = M_{ji}$. Calculat-

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*) In the present paper the problem will be considered only in such simplified framework, so that the problems of causality, Lorentz invariance and the electric charge conservation do not arise. A more complete treatment will be given in a separate publication.

**) These names will be justified further after the construction of the particle number operator N.

ing the products of the operators a, a^+ up to triple products and decomposing them with respect to M_{ij}, we find

$$\begin{aligned} a &= M_{12} + \beta M_{23} & a^3 &= 0 \\ a^2 &= \beta M_{13} & a^2 a^+ &= \beta^2 M_{12} \\ a^+ a &= M_{22} + \beta^2 M_{33} & a a^+ a &= M_{12} + \beta^3 M_{23} \\ a a^+ &= M_{11} + \beta^2 M_{22} & a^+ a^2 &= \beta M_{23} \end{aligned} \tag{6}$$

Hermitean conjugate relations are not written down. Using (6), let us try first to find bilinear commutation relations, i.e. the relations of the form

$$C_1 a a^+ + C_2 a^+ a + C_3 \cdot \mathbb{I} + C_4 a + C_5 a^+ + C_6 a^2 + C_7 (a^+)^2 = 0 \tag{7}$$

The first three terms in Eq. (7) are diagonal; the last four terms have zero diagonal elements and are linearly independent, therefore $C_4 = C_5 = C_6 = C_7 = 0$. The coefficients $C_1 - C_3$ should obey the following set of equations

$$\left.\begin{aligned} &C_1 + C_3 = 0 \\ &C_1\beta^2 + C_2 + C_3 = 0 \\ &C_2\beta^2 + C_3 = 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} &C_2 + (1-\beta)C_3 = 0 \\ &C_2\beta^2 + C_3 = 0 \end{aligned}\right\} \tag{8}$$

The determinant of the last system is equal to $1-\beta^2+\beta^4$ i.e., it does not vanish with any β. Therefore, we conclude that the operators a, a^+ and their bilinear products are linearly independent, i.e. the bilinear commutation relations are absent in the model under consideration.

The nine-dimensional linear space of 3x3 matrices L can be decomposed into the direct sum of the three-dimensional space L_d (diagonal matrices) and six-dimensional space L_o (matrices

with zero diagonal elements). The space L_d contains two independent operators a^+a and aa^+ whereas the space L_o contains ten (independent) operators (a, a^2, a^2a^+, aa^+a, a^+a^2 plus their Hermitean conjugates). Therefore, in the space L_o there should exist four independent linear relations between the operators. These relations can be written down for example, in the following form:

$$a^2 a^+ + \beta^2 a^+ a^2 = \beta^2 a \tag{9}$$

$$a^2 a^+ + \beta^4 a^+ a^2 = \beta^2 a a^+ a \tag{10}$$

plus their Hermitean conjugate relations. To these relations one should add the equalities

$$a^3 = 0, \quad (a^+)^3 = 0. \tag{11}$$

Now, the equalities (9) - (11) form the looked for algebra which is obeyed by the annihilation and creation operators.

Now, let us construct the particle number operator N in this model. In the chosen representation the operator N has the form

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{12}$$

so that the usual commutation relations hold true

$$[N, a] = -a, \quad [N, a^+] = a^+ \tag{13}$$

Let us try to find the expression for the operator N in terms of the creation and annihilation operators and their bilinear products of the form

$$N = A_1 a^+ a + A_2 aa^+ + A_3 \tag{14}$$

Using the decompositions (6), we find the values of the coefficients A_i:

$$A_1 = \frac{-1+2\beta^2}{1-\beta^2+\beta^4}$$
$$A_2 = \frac{-2+\beta^2}{1-\beta^2+\beta^4} \tag{15}$$
$$A_3 = \frac{2-\beta^2}{1-\beta^2+\beta^4}$$

Thus we have completed the construction of the algebra of the creation and annihilation operators and also have found the expression for the particle number operator. For the sake of reader's convenience, let us write down all the basic relations together. The following resume is the basic result of the present work:

$$a^2 a^+ + \beta^2 a^+ a^2 = \beta^2 a \tag{9}$$
$$a^2 a^+ + \beta^4 a^+ a^2 = \beta^2 aa^+ a \tag{10}$$
$$a^3 = 0 \tag{11}$$
$$N = \frac{1}{1-\beta^2+\beta^4}\left[(-1+2\beta^2)a^+a + (-2+\beta^2)aa^+ + (2-\beta^2)\cdot \mathbb{I}\right]$$

The constructed algebra proves to be <u>trilinear</u> in the creation and annihilation operators. It can be shown (see the Appendix) that there are no <u>bilinear</u> algebra with the required

property (i.e. small violation of the Pauli principle).

3. The representation of the algebra

Now, one should consider various representation of the algebra under consideration. When constructing the representations, we shall extensively use the commutation relations between the particle number operator and creation and annihilation operators. Therefore let us now show that the commutation relations (13) are in fact a consequence of the algebra (9) - (11) and are valid in any representation of the algebra, not only in a specific representation (4).

Let us show that

$$A \equiv [N, a] + a = 0 \tag{16}$$

We have

$$A = A_1 a^+ a^2 - A_2 a^2 a^+ + (A_2 - A_1) aa^+ a + a \tag{17}$$

Multiplying this equality by β^2 and expressing the operators aa^+a and a through a^+a^2 and a^2a^+ and using the algebra (9) - (10), one obtains

$$A = [(\beta^2 - \beta^4)A_1 + \beta^4 A_2 + \beta^2]\, a^+ a^2 + \\ + [(1-\beta^2)A_2 - A_1 + 1]\, a^2 a^+ \tag{18}$$

Substituting here the expressions for A_i from (15), we obtain that A = 0, Q.E.D.

Consider the representations of the algebra (9) - (11) for $\beta \neq 0$. Let us show that in the case $\beta \neq 0$ there are no two-dimensional representations.

Suppose on the contrary that the algebra has such a representation. Without loss of generality, we can assume that in this representation the particle number operator is diagonal:

$$N = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{19}$$

Let the operator a has the form:

$$a = \begin{pmatrix} \alpha & \mu \\ \delta & \gamma \end{pmatrix} \tag{20}$$

Then, from the commutation relation $[N, a] = -a$ it follows that

$$\begin{pmatrix} 0 & (\lambda_1 - \lambda_2)\mu \\ (\lambda_2 - \lambda_1)\delta & 0 \end{pmatrix} = \begin{pmatrix} -\alpha & -\mu \\ -\delta & -\gamma \end{pmatrix} \tag{21}$$

This equation has two (unitary equivalent) solutions:

$$a_1 = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix}$$

$$N_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda+1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda-1 \end{pmatrix} \tag{22}$$

In any case, we have $a^2 = 0$ which contradicts the commutation relation (9) (provided $\beta \neq 0$).

Thus we have shown that the lowest dimensionality of the representations of the algebra (9) with $\beta \neq 0$ is equal to three.

Now consider the case $\beta = 0$. Then the algebra (9) - (11) takes the form

$$a^2 a^+ = 0, \quad a^3 = 0 \tag{23}$$

plus the Hermitean conjugate relations. The particle number operator is equal to

$$N = -a^+a - 2aa^+ + 2 \tag{24}$$

To find the representations of the algebra (23) one can use the solutions (22). In our case we have:

$$a = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} -2\mu^2 + 2 & 0 \\ 0 & -\mu^2 + 2 \end{pmatrix} \tag{25}$$

$$\Rightarrow \mu^2 = 1 .$$

Therefore, the algebra (23) has the unique (up to the unitary equivalence) representation

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{26}$$

As was anticipated, this representation coincides with the two-dimensional representation of the usual Fermi algebra

$$aa^+ + a^+a = 1, \quad a^2 = 0, \quad N = a^+a \tag{27}$$

4. A toy model describing small violations of the Pauli principle

To introduce some dynamics into our model we should choose a Hamiltonian. Of course, in our case this procedure is rather arbitrary. Let us, however, make the following simple and seemingly natural choice just to get a general view of what events can happen in models based on our algebra:

$$H = H_0 + H_{int} = EN + \varepsilon V \tag{28}$$

where N is the particle number operator, V is the interaction energy

$$V = a^2 a^+ + a^+ a^2 + a a^+ a + h.c. \tag{29}$$

E and ε are parameters with the dimension of energy which have the meanings of the energy of one-particle state and the coupling constant, respectively. For the sake of convenient application of the perturbation theory, let us assume that $\varepsilon \ll E$. Note that apart from the ratio ε/E the Hamiltonian (28) does not contain any small parameter.

In the representation where the operator N is diagonal our Hamiltonian takes the form

$$H = \begin{pmatrix} 0 & \varepsilon & 0 \\ \varepsilon & E & \beta\varepsilon \\ 0 & \beta\varepsilon & 2E \end{pmatrix} \tag{30}$$

(When deriving (30) we have used Eq. (6)).

Up to the terms of order $O(\varepsilon^2)$ the eigenvalues of the Hamiltonian are equal to

$$\tilde{E}_0 = \varepsilon^2 \sum_{i\neq 0} \frac{|V_{i0}|^2}{E_0 - E_i} = -\frac{\varepsilon^2}{E}$$

$$\tilde{E}_1 = E + \varepsilon^2 \sum_{i\neq 1} \frac{|V_{i1}|^2}{E_1 - E_i} = E + (1-\beta^2)\frac{\varepsilon^2}{E}$$

$$\tilde{E}_2 = 2E + \varepsilon^2 \sum_{i\neq 2} \frac{|V_{i2}|^2}{E_2 - E_i} = 2E + \frac{\beta^2\varepsilon^2}{E} \tag{31}$$

The normalized eigenfunctions of the Hamiltonian, up to $O(\varepsilon)$ terms are equal to

$$|\tilde{0}\rangle = |0\rangle + \varepsilon \sum_{i\neq 0} \frac{V_{i0}}{E_0 - E_i} |i\rangle = |0\rangle - \frac{\varepsilon}{E}|1\rangle$$

$$|\tilde{1}\rangle = |1\rangle + \varepsilon \sum_{i\neq 1} \frac{V_{i1}}{E_1 - E_i} |i\rangle = |1\rangle + \frac{\varepsilon}{E}|0\rangle - \frac{\beta\varepsilon}{E}|2\rangle$$

$$|\tilde{2}\rangle = |2\rangle + \varepsilon \sum_{i\neq 3} \frac{V_{i3}}{E_3 - E_i} |i\rangle = |2\rangle + \frac{\beta\varepsilon}{E}|1\rangle \tag{32}$$

where the states $|i\rangle$ are the eigenstates of the particle number operator N. In other words, the transition matrix from the states $|i\rangle$ to the states $|\tilde{j}\rangle$ has the form

$$|\tilde{j}\rangle = M_{ji}|i\rangle, \quad M = \begin{pmatrix} 1 & -\frac{\varepsilon}{E} & 0 \\ \frac{\varepsilon}{E} & 1 & -\beta\frac{\varepsilon}{E} \\ 0 & \beta\frac{\varepsilon}{E} & 1 \end{pmatrix} \tag{33}$$

The matrix M is orthogonal up to $O(\varepsilon^2)$ terms:

$$MM^T = \mathbb{1} + O(\varepsilon^2) \tag{34}$$

Now, let us find the probabilities of the oscillations between the various states $|i\rangle, |j\rangle$. To do this, let us decompose them along the stationary states $|\tilde{j}\rangle$

$$|i\rangle = C_{ij}|\tilde{j}\rangle, \quad \text{where} \quad C = M^{-1} = M^T \tag{35}$$

By the time t the state $|i\rangle$ will develop to the state:

$$|i(t)\rangle = C_{ij}|\tilde{j}\rangle e^{-i\tilde{E}_j t} \tag{36}$$

We are interested in the amplitude of the transition from this state to the state $\langle k|$:

$$\langle k| = C_{k\ell}\langle\tilde{\ell}| \tag{37}$$

i.e. the quantity

$$A_{ik}(t) = \langle k|i(t)\rangle \tag{38}$$

We have

$$A_{ik}(t) = \langle\tilde{\ell}|\tilde{j}\rangle C_{ij}C_{k\ell}e^{-i\tilde{E}_j t} = C_{i\ell}C_{k\ell}e^{-i\tilde{E}_\ell t} \tag{39}$$

Consequently, for the oscillation probability W_{ik} we obtain

$$W_{ik}(t) = A_{\underline{ik}}\cdot A^*_{\underline{ik}} =$$

$$= \sum_{j,\ell=0,1,2} C_{\underline{i}\ell}C_{\underline{k}\ell}C_{\underline{i}j}C_{\underline{k}j}e^{-i(\tilde{E}_j-\tilde{E}_\ell)t} =$$

$$= \sum_\ell (C_{i\ell})^2(C_{k\ell})^2 + 2\sum_{\substack{(j\ell)=(01),\\(02),(12)}} C_{\underline{i}\ell}C_{\underline{k}\ell}C_{\underline{i}j}C_{\underline{k}j}\cos(\tilde{E}_\ell-\tilde{E}_j)t \tag{40}$$

(the underlined indices are not summed over). From this we see that the matrix W_{ik} is symmetric: $W_{ik} = W_{ki}$. Substituting in (40) the values C_{ij} which are equal to

$$C = \begin{pmatrix} 1 & \frac{\varepsilon}{E} & 0 \\ -\frac{\varepsilon}{E} & 1 & \frac{\beta\varepsilon}{E} \\ 0 & -\frac{\beta\varepsilon}{E} & 1 \end{pmatrix} \tag{41}$$

we find

$$W_{01} = 2\,\frac{\varepsilon^2}{E^2}\,(1-\cos Et)$$

$$W_{02} = 0 \qquad (42)$$

$$W_{12} = 2\beta^2 \left(\frac{\varepsilon}{E}\right)^2 (1-\cos Et)$$

From Eq. (42) it is clearly seen that the probabilities of the oscillations violating the Pauli principle (W_{02} and W_{12}) are suppressed at least by the factor β^2 and vanish if $\beta=0$.

Let us note the characteristic feature of the result obtained: although the interaction Hamiltonian does not contain any small parameter*), the transitions between certain states (namely, those transitions which are forbidden by the Pauli principle) are strongly suppressed. The reason is that not the Hamiltonian, but the commutation relations contain a small parameter.

*) The small ratio ε/E is irrelevant here because it is introduced only to make possible the application of the perturbation theory.

Thus, our model demonstrates a new way of violating the fundamental principles: through the smallness contained in the commutation relations.

5. Summary and conclusions

To summarize, we have considered the problem of quantum-mechanical description of a possibility of small violations of the Pauli exclusion principle. We have found the simplest form of the commutation relations for the creation and annihilation operators which allows one to describe small deviations from the exclusion principle. It is shown that these commutation relations cannot be bilinear in the creation and annihilation operators but should be at least trilinear. Our method of constructing the simplest trilinear operator algebra can be used also for a generalization to more realistic cases. The dynamic models based on our algebra have the following common feature although there are no small parameter in the interaction Hamiltonian, the transitions in which Pauli principle is violated (i.e. the transitions to the state with two identical particles) prove to be strongly suppressed. This suppression is controlled by a small parameter which enters the commutation relations.

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APPENDIX

Let us show that a small violation of the Pauli principle cannot be described by any algebra defined by <u>bilinear</u> commutation relations. To be more exact, we shall prove the following theorem.

<u>Theorem</u>. Let the operators a, a^+ obey a bilinear commutation relation of the form

$$C_1 a^+ a + C_2 a a^+ + C_3 \cdot \mathbb{1} + C_4 a + C_5 a^+ + C_6 a^2 + C_7 (a^+)^2 = 0 \qquad \text{(A1)}$$

In addition to it, let there exists an Hermitean operator N, satisfying the commutation relations of the form

$$[N, a] = -a, \quad [N, a^+] = a^+ \qquad \text{(A2)}$$

and having as its eigenvalues the numbers 0, 1, ... n-1.

Then, for any finite-dimensional representation of the operator a, the following equality holds true:

$$a^2 = 0 \qquad \text{(A3)}$$

<u>Proof.</u> Let us work in a representation where the operator N is diagonal

$$N = diag\,(\lambda_1 \dots \lambda_n), \quad \lambda_k = k-1 \qquad \text{(A4)}$$

n being the dimensionality of the representation. Let us find the commutator $[N, a]$ in the representation:

$$(Na)_{ij} = \sum_k N_{ik} a_{kj} = \lambda_{\underline{i}} a_{\underline{i}j}$$

$$(aN)_{ij} = \sum_k a_{ik} N_{kj} = a_{i\underline{j}} \lambda_{\underline{j}} \qquad \text{(A5)}$$

$$[N, a]_{ij} = a_{\underline{i}\underline{j}} (\lambda_{\underline{i}} - \lambda_{\underline{j}})$$

The underlined indices are not summed over.

Substituting (A5) into (A2), we obtain the equation

$$a_{\underline{i}\underline{j}} (\lambda_{\underline{i}} - \lambda_{\underline{j}}) = -a_{ij} \qquad \text{(A6)}$$

From Eq. (A6) it follows that the only nonvanishing elements of matrix a can be $a_{i,i+1}$ ($i = 1, \dots n-1$), that is, the elements, standing along the over-the-main diagonal (we shall call this diagonal the first one).

Calculating the matrices a^+a, aa^+, a^2, we find

$$a^+a = diag\,(0, |a_{12}|^2, |a_{23}|^2, \dots |a_{n-1,n}|^2)$$

$$aa^+ = diag\,(|a_{12}|^2, |a_{23}|^2, \dots 0) \qquad \text{(A7)}$$

$$(a^2)_{i,i+2} = a_{i,i+1} \cdot a_{i+1,i+2}, \quad i = 1, \dots, n-2$$

the other $(a^2)_{ij}$ being equal to zero. Thus, all the nonvanishing elements of the matrix a^2 stand along the second diagonal (that is the diagonal lying over the first one).

Now consider the bilinear commutation relation of the most general form

$$C_1 a^+ a + C_2 a a^+ + C_3 \cdot \mathbb{1} + C_4 a + C_5 a^+ + C_6 a^2 + C_7 (a^+)^2 = 0 \qquad \text{(A8)}$$

All the terms in (A8) are diagonal or quasidiagonal: the terms $C_1 - C_3$ are diagonal; the terms C_4 and C_5 are situated on the first diagonal and its symmetric one; the terms C_6 and C_7 are situated on the second diagonal and its symmetric one. From this fact it follows that when $n \geqslant 3$ the following equalities should

hold: $$C_1 a^+ a + C_2 a a^+ + C_3 \cdot \mathbb{I} = 0,$$
$$C_4 a = 0, \quad C_5 a^+ = 0, \tag{A9}$$
$$C_6 a^2 = 0, \quad C_7 (a^+)^2 = 0,$$

We are, of course, interested in the case where a and a^2 are not equal to zero, therefore the coefficients $C_{4,5,6,7}$ should be equal to zero. Thus, provided a, $a^2 \neq 0$, the commutation relation (A8) is equivalent to the Eq. (A9).

Substituting into the Eq. (A9) the explicit forms of the operators a^+a and aa^+, we obtain the set of n equations for the determination of the coefficients $C_1 - C_3$ and matrix elements $a_{i,i+1}$:

$$\begin{aligned} C_2 b_1 + C_3 &= 0 \\ C_1 b_1 + C_2 b_2 + C_3 &= 0 \\ C_1 b_2 + C_2 b_3 + C_3 &= 0 \\ \cdots \\ C_1 b_{n-1} + C_3 &= 0 \end{aligned} \tag{A10}$$

where $b_i = a_{i,i+1}{}^2$, $b_n \equiv 0$.

This set of equations can be considered as the set of n linear equations with three unknowns $C_1 - C_3$, the coefficients b_i being nonnegative and containing at least one pair of neighbouring nonzero numbers: $b_k b_{k+1} \neq 0$ (the latter condition is equivalent to the requirement $a^2 \neq 0$). In order that this set had a nonzero solution (i.e. not all C_i were vanishing) it is necessary that the rank of the coefficient matrix M_n was less than three (i.e. was equal to 1 or 2), where

$$M_n = \left| \begin{matrix} 0 & b_1 & 1 \\ b_1 & b_2 & 1 \\ b_2 & b_3 & 1 \\ \cdots & \cdots & \cdots \\ b_{n-1} & b_n & 1 \end{matrix} \right| \tag{A11}$$

Let us make two transformations of the matrix M_n which will not change its rank, namely: subtract the first line from all the other lines and then subtract the third column multiplied by b_1 from the second column. Then the matrix will take the form

$$M_n' = \left| \begin{matrix} 0 & 0 & 1 \\ b_1 & b_2 - b_1 & 0 \\ b_2 & b_3 - b_1 & 0 \\ \cdots & \cdots & \cdots \\ b_{n-1} & b_n - b_1 & 0 \end{matrix} \right| \tag{A12}$$

The rank of this matrix is evidently larger by one than the rank of the matrix

$$M_n'' = \left| \begin{matrix} b_1 & b_2 - b_1 \\ b_2 & b_3 - b_1 \\ \cdots & \cdots \\ b_{n-1} & b_n - b_1 \end{matrix} \right| \tag{A13}$$

For the following considerations it is convenient to exclude first three special cases: $b_1 = 0$; $b_1 \neq 0$; $b_2/b_1 = 1$; $b_1 \neq 0$; $b_2/b_1 = 2$.

<u>The case $b_1 = 0$.</u> Then the matrix M_n'' takes the form

$$M_n'' = \left| \begin{matrix} 0 & b_2 \\ b_2 & b_3 \\ \cdots & \cdots \\ b_{n-1} & b_n \end{matrix} \right| \tag{A14}$$

We require the linear dependence of the second and first lines and thus we obtain $b_2 = 0$; then, analogously, $b_3 = 0$ etc., and, finally, $b_{n-1} = 0$ which means that all b_i are vanishing. Therefore, the case $b_1 = 0$ is excluded.

__The case $b_1 \neq 0$, $b_2/b_1 = 1$.__ Acting in analogy with the previous case, one can prove, that the linear dependence of all the lines implies that all b_k (including b_n) are equal to b_1. This result, however, contradicts the condition $b_n = 0$.

__The case $b_1 \neq 0$, $b_2/b_1 = 2$.__ In this case for all b_k (including b_n) it is true that $b_k = kb_1$ which again contradicts the condition $b_n = 0$.

Having excluded the above special cases, consider now the most general case, when $b_1 \neq 0$ and $b_1/b_2 \neq 1, 2$. Write down the condition of linear dependence of the k-th and the first lines of the matrix M''_n (using the condition $b_1 \neq 0$).

$$\det \begin{vmatrix} b_1 & b_2 - b_1 \\ b_k & b_{k+1} - b_1 \end{vmatrix} = 0 \qquad \text{(A15)}$$

$$\Rightarrow b_{k+1} - b_k \left(\frac{b_2 - b_1}{b_1} \right) - b_1 = 0$$

In order to eliminate the inhomogeneity in this recurrent equation, let us shift $k \rightarrow k + 1$:

$$b_{k+2} - b_{k+1} \left(\frac{b_2 - b_1}{b_1} \right) - b_1 = 0 \qquad \text{(A16)}$$

and, subtracting (A15) from (A16), obtain

$$b_{k+2} - b_{k+1} \frac{b_2}{b_1} + b_k \left(\frac{b_2 - b_1}{b_1} \right) = 0 \qquad \text{(A17)}$$

The characteristic equation for this recurrent relation has the form

$$X^2 - \frac{b_2}{b_1} X + \frac{b_2 - b_1}{b_1} = 0 \qquad \text{(A18)}$$

Its solutions are

$$x_1 = \frac{b_2 - b_1}{b_1}, \quad x_2 = 1 \qquad \text{(A19)}$$

Therefore, the general solution of the equation (A17) can be written in the form

$$b_k = C_1 x_1^k + C_2 x_2^k = C_1 \left(\frac{b_2 - b_1}{b_1} \right)^k + C_2 \qquad \text{(A20)}$$

The constants C_1 and C_2 can be determined from the initial conditions: the first two terms of the sequence (A20) are equal to b_1 and b_2, hence, using the conditions $b_1/b_2 \neq 1, 2$ we find

$$C_1 = \frac{b_1 b_2}{b_1 - 2b_2}$$

$$C_2 = - \frac{b_1 b_2}{b_1 - 2b_2} \qquad \text{(A21)}$$

So, the n-th term of the sequence (A20), which we are interested in, has the form

$$b_n = \frac{b_1 b_2}{b_1 - 2b_2} \left[\left(\frac{b_2}{b_1} - 1 \right)^n - 1 \right] \qquad \text{(A22)}$$

Equating b_1 to zero, we find that (taking into account the condition $b_1/b_2 \neq 2$) for odd n the equation $b_n = 0$ has no solution whereas for even n there is the unique solution $b_2/b_1 = 0$, therefore

$$b_{even} = 0, \quad b_{odd} = b_1. \qquad \text{(A23)}$$

This sequence, however, does not contain a pair of neighbouring nonzero terms. Thus the last case under consideration is also excluded. Q.E.D.

REFERENCES

[1] A. Yu. Ignatiev, V.A. Kuzmin and M.E. Shaposhnikov, preprint, IYaI P-0101, (1978); Phys. Lett. 84B, 315 (1979); Preprint IYaI P-0142 (1980).

[2] L.B. Okun and Ya. B. Zeldovich, Phys. Lett. B78, 597 (1978).

[3] T. Banks, SLAC-PUB-3376 (1984).

[4] H.B. Nielsen and M. Ninomiya, Nucl. Phys. B141, 153 (1978).

[5] V.A. Kuzmin, in: Proc. of 3rd Seminar on Quantum Gravity, Eds. M.A. Markov, V.A. Berezin and V.P. Frolov (World Scientific, Singapore, 1984) p. 270; Nordita preprint 85/4 (1985).

[6] H.S. Green, Phys. Rev. 90, 270 (1953);
D.V. Volkov, ZhETF 36, 1560 (1959);
A.B. Govorkov, Int. J. Theor. Phys. 7, 49 (1973);
I.G. Kaplan, Usp. Fyz. Nauk 117, 691 (1975).

[7] F. Wilczek and A. Zee, Phys. Rev. Lett. 51, 2250 (1983).

[8] N.N. Bogolubov, A.A. Logunov and I.T. Todorov, The Fundamentals of Axiomatic Approach in the Quantum Field Theory (M. Nauka, 1969) (in Russian).