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NEW APPLICATIONS OF BOSON'S COHERENT STATES OF DOUBLE MODES AT REGULAR PRODUCT *

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ABSTRACT

After Fan et al. [1], this paper presents a series of new applications of boson's coherent states of double modes by means of the technique of regular products. They include non-coupled double oscillator solutions at two time dependent extra-sources; coupled double oscillator solutions at two time dependent extra-sources; some applications to angular momentum theory; an explicit expression for time-reversal operator.

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I. INTRODUCTION

Since 1963 [2], the coherent state has changed not only into an important concept but also into an important method in quantum theory. In 1984, Fan et al. had presented a new method of calculating boson's coherent state of single mode. The main point of this method is that the technique of regular product (normal ordering) of operators is used in the processes of the calculations and in the expressions. As is known, inside the signal of regular product the operators which cannot originally commute may always commute. Therefore, inside the signal of regular product we may carry out various integral and differential calculations to all operators as if they are some numbers. Due to this convenience, many calculations may now be carried out when they cannot be carried out originally because of the complication of calculation. But Fan's paper presented the calculations only for a single mode. In this paper we present the calculations for two modes.

In Sec.II, we simply give the fundamental aspects of boson's coherent states of double modes with the form of regular product, and then discuss its applications to a non-coupled double oscillator at two time dependent extra-sources. We have obtained some results, including the rigorous solution of time-development operator for this case, transformation matrix elements between coherent states and Feynman's matrix elements. In Sec.III, the correspondent solutions of a coupled double oscillator at the above sources are given. In Sec.IV, some applications for the angular momentum theory are presented, including the expression of rotation operators in form of regular product, integral calculations contained some rotation operators, several expressions for class-operator of O_3 group. In Sec.V an explicit form of the expression for time-reversal operator is given. Finally, there is an appendix about some boson's commutators of double modes.

II. THE COHERENT STATES OF DOUBLE MODES AND RIGOROUS SOLUTIONS FOR A NON-COUPLED DOUBLE OSCILLATOR AT TWO TIME DEPENDENT EXTRA-SOURCES

We have two sets of boson's operators corresponding to double modes,

$$[a, a^+] = [b, b^+] = 1 ,$$
$$[a, b] = [a, b^+] = 0 .$$

Therefore, we may define a set of boson's coherent states for double modes as follows

$$|z_1 z_2\rangle = \exp\left\{-\frac{1}{2}(|z_1|^2+|z_2|^2)+z_1 a^+ + z_2 b^+\right\}|00\rangle$$

where z is, generally, a complex number and the vacuum state of double modes has the following characteristics (note $|00\rangle$ as $|0\rangle$)

$$a|0\rangle = b|0\rangle = 0 \; ,$$

$$\langle 0|0\rangle = 1 \; , \qquad |0\rangle\langle 0| = :e^{-a^+a-b^+b}:$$

and we have a relation for completeness,

$$1 = \int \frac{d^2z_1 d^2z_2}{\pi^2}|z_1 z_2\rangle\langle z_1 z_2|$$

$$= \int \frac{d^2z_1 d^2z_2}{\pi^2} e^{-\frac{1}{2}(|z_1|^2+|z_2|^2)+z_1 a^+ + z_2 b^+} :e^{-a^+a-b^+b}: e^{-\frac{1}{2}(|z_1|^2+|z_2|^2)+z_1^* a + z_2^* b}$$

$$= \int \frac{d^2z_1 d^2z_2}{\pi^2} :e^{-(|z_1|^2+|z_2|^2)+z_1 a^+ + z_2 b^+ + z_1^* a + z_2^* b - a^+a - b^+b}:$$

These formulas have directly extended Fan's results to two modes.

By means of some integral and differential calculations, we may obtain some useful formulas which will be used later. Their deductions may refer to the appendix.

$$e^{-\alpha a^+a-\beta b^+b} = :\exp\left\{(e^{-\alpha}-1)a^+a+(e^{-\beta}-1)b^+b\right\}: \; ,$$

$$e^{\beta ab}e^{\alpha a^+b^+} = \frac{1}{1-\alpha\beta}e^{\frac{\alpha}{1-\alpha\beta}a^+b^+}\left(\frac{1}{1-\alpha\beta}\right)^{a^+a+b^+b}e^{\frac{\beta}{1-\alpha\beta}ab} \; ,$$

$$f(a,b)e^{-\lambda a^+-\delta b^+} = e^{-\lambda a^+-\delta b^+}f(a-\lambda, b-\delta) \; ,$$

$$f(a,b)e^{-\lambda a^+a-\delta b^+b} = e^{-\lambda a^+a-\delta b^+b}f(ae^{-\lambda}, be^{-\delta}) \; ,$$

$$e^{-\lambda(a^+b-b^+a)}a^+e^{\lambda(a^+b-b^+a)} = a^+\cos\lambda + b^+\sin\lambda \; .$$

Here f is defined as

$$f(a,b) = \sum_{m,n} C_{mn}a^m b^n \; ,$$

where C_{nm} are some numbers.

We suppose that the Hamiltonian of the system (Schrödinger's picture) is

$$H = H_0 + H'(t) \; ,$$

where

$$H_0 = \omega_1(a^+a+\tfrac{1}{2}) + \omega_2(b^+b+\tfrac{1}{2}) \; ,$$

$$H'(t) = \gamma_1(t)a^+ + \gamma_1^*(t)a + \gamma_2(t)b^+ + \gamma_2^*(t)b \; .$$

Then, we have the H_I' of the interaction picture

$$H_I'(t) = e^{iH_0t}H'(t)e^{-iH_0t}$$

$$= \gamma_1(t)a^+e^{i\omega_1t} + \gamma_1^*(t)ae^{-i\omega_1t} + \gamma_2(t)b^+e^{i\omega_2t} + \gamma_2^*(t)be^{-i\omega_2t} \; .$$

The relation between the two time-development operators is

$$U_S(t,t_0) = e^{-iH_0t}U_I(t,t_0)e^{iH_0t} \; ,$$

where $U_I(t,t_0)$ satisfies the following equation

$$i\frac{\partial U_I(t,t_0)}{\partial t} = H_I'(t)U_I(t,t_0) \; , \qquad U_I(t_0,t_0) = 1 \; .$$

It is not difficult to discover that the solution of this equation has the following form,

$$U_I(t,t_0) = e^{-i\xi_1(t)a^+}e^{-i\xi_2(t)b^+}e^{-i\xi_1^*(t)a}e^{-i\xi_2^*(t)b}e^{-A_1-A_2} \; ,$$

where

$$\xi_j(t) = \int_{t_0}^{t}\gamma_j(s)e^{i\omega_j s}ds \; ,$$

$$A_j(t) = \int_{t_0}^{t}\gamma_j^*(s)\xi_j(s)e^{-i\omega_j s}ds \; .$$

This fact may be checked by directly substituting this solution into the above U_I equation. In checking, it is necessary to use the following two commutators

$$[e^{-i\xi_1 a^+}, \gamma_1^* e^{-i\omega_1 t} a] = i\xi_1 \gamma_1^* e^{-i\omega_1 t} e^{-i\xi_1 a^+}$$

and the other for b which has a similar form to this.

From the above expression of U_I, we may immediately obtain the transformation matrix elements in a coherent state picture as follows

$$\begin{aligned}
\langle z_1 z_2 t | z_1^0 z_2^0 t_0 \rangle &= \langle z_1 z_2 | U_S(t,t_0) | z_1^0 z_2^0 \rangle \\
&= \exp\left\{ -\frac{1}{2}\left(|z_1|^2 + |z_2|^2 + |z_1^0|^2 + |z_2^0|^2\right) - A_1 - A_2 - \frac{i}{2}(\omega_1 + \omega_2)T \right\} \langle 0 | e^{z_1^* a + z_2^* b} \cdot \\
&\quad \cdot e^{-i\omega_1 t a^+ a - i\omega_2 t b^+ b} e^{-i\xi_1 a^+} e^{-i\xi_2 b^+} e^{-i\xi_1^* a} e^{-i\xi_2^* b} e^{i\omega_1 t_0 a^+ a + i\omega_2 t_0 b^+ b} e^{z_1^0 a^+ + z_2^0 b^+} | 0 \rangle \\
&= \exp\left\{ -\frac{1}{2}\left(|z_1|^2 + |z_2|^2 + |z_1^0|^2 + |z_2^0|^2\right) - A_1 - A_2 - \frac{i}{2}(\omega_1 + \omega_2)T + z_1^0 z_1^* e^{-i\omega_1 T} - i z_1^* \xi_1 e^{-i\omega_1 t} \right. \\
&\quad \left. + z_2^0 z_2^* e^{-i\omega_2 T} - i z_2^* \xi_2 e^{-i\omega_2 t} - i z_1^0 \xi_1^* e^{+i\omega_1 t_0} - i z_2^0 \xi_2^* e^{+i\omega_2 t_0} \right\},
\end{aligned}$$

where $T = t - t_0$.

In addition, if we suppose that the coordinate eigenstates are $|q_1 q_2\rangle$, i.e.

$$|q_1 q_2\rangle = \frac{1}{\pi^{1/2}} e^{-\frac{1}{2}(q_1^2 + q_2^2) + \sqrt{2} q_1 a^+ + \sqrt{2} q_2 b^+ - \frac{(a^+)^2}{2} - \frac{(b^+)^2}{2}} |0\rangle,$$

Feynman's matrix elements may be calculated as follows

$$\begin{aligned}
\langle q_1 q_2 t | q_1^0 q_2^0 t_0 \rangle &= \langle q_1 q_2 | U_S(t,t_0) | q_1^0 q_2^0 \rangle \\
&= \int \langle q_1 q_2 | z_1 z_2 \rangle \frac{d^2 z_1 d^2 z_2}{\pi^2} \langle z_1 z_2 | U_S(t,t_0) | z_1^0 z_2^0 \rangle \frac{d^2 z_1^0 d^2 z_2^0}{\pi^2} \langle z_1^0 z_2^0 | q_1^0 q_2^0 \rangle
\end{aligned}$$

where we have

$$\begin{aligned}
\langle q_1 q_2 | z_1 z_2 \rangle &= \frac{1}{\pi^{1/2}} \langle 0 | e^{-\frac{1}{2}(q_1^2 + q_2^2) + \sqrt{2} q_1 z_1 + \sqrt{2} q_2 z_2 - \frac{z_1^2}{2} - \frac{z_2^2}{2}} e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2) + z_1 a^+ + z_2 b^+} | 0 \rangle \\
&= \frac{1}{\pi^{1/2}} \exp\left\{ -\frac{1}{2}\left(q_1^2 + q_2^2 + |z_1|^2 + |z_2|^2\right) + \sqrt{2}(q_1 z_1 + q_2 z_2) - \frac{1}{2}(z_1^2 + z_2^2) \right\}.
\end{aligned}$$

And then in calculation we may pay attention to

$$\int \frac{d^2 z}{\pi} \exp\left\{ -|z|^2 - \frac{z^2}{2} + \alpha z + \beta z^* \right\} = \exp\left\{ \beta\left(\alpha - \frac{\beta}{2}\right) \right\}.$$

(N.B. when $\beta = 0$, this integral is independent of α), and

$$\int \frac{d^2 z}{\pi} \exp\left\{ -|z|^2 - \frac{z^{*2}}{2} - \frac{z^2}{2} e^{-2i\omega T} + \gamma z + \delta z^* \right\} = \frac{1}{(1 - e^{-2i\omega T})^{1/2}} \exp\left\{ \frac{-(\gamma - \delta)^2}{2(1 - e^{-i2\omega T})} + \frac{\delta^2}{2} \right\}$$

(when $T = -i\infty$, this formula shall transform into the previous formula substituting γ and δ, respectively, for β and α). Finally, we may obtain

$$\begin{aligned}
\langle q_1 q_2 t | q_1^0 q_2^0 t_0 \rangle &= \sqrt{\frac{\omega_1 \omega_2}{\pi^2 \cdot 2i\sin\omega_1 T \cdot 2i\sin\omega_2 T}} \exp\left\{ \frac{i\omega_1}{2\sin\omega_1 T}\left((q_1^2 + q_1^{02})\cos\omega_1 T - 2q_1 q_1^0\right) + \frac{i\omega_2}{2\sin\omega_2 T}\left((q_2^2 + q_2^{02})\cos\omega_2 T - 2q_2 q_2^0\right) \right. \\
&\quad + \frac{1}{2i\sin\omega_1 T}\left(\frac{\xi_1^2}{2} e^{-i\omega_1(t+t_0)} + \frac{\xi_1^{*2}}{2} e^{i\omega_1(t+t_0)} - |\xi_1|^2 e^{-i\omega_1 T} \right) + \frac{1}{2i\sin\omega_2 T}\left(\frac{\xi_2^2}{2} e^{-i\omega_2(t+t_0)} + \frac{\xi_2^{*2}}{2} e^{i\omega_2(t+t_0)} \right. \\
&\quad \left. - |\xi_2|^2 e^{-i\omega_2 T} \right) - \frac{\sqrt{\omega_1}}{\sqrt{2}\sin\omega_1 T}\left(q_1 \xi_1 e^{-i\omega_1 t_0} - q_1 \xi_1^* e^{i\omega_1 t_0} - q_1^0 \xi_1 e^{-i\omega_1 t} + q_1^0 \xi_1^* e^{i\omega_1 t} \right) - \\
&\quad \left. - \frac{\sqrt{\omega_2}}{\sqrt{2}\sin\omega_2 T}\left(q_2 \xi_2 e^{-i\omega_2 t_0} - q_2 \xi_2^* e^{i\omega_2 t_0} - q_2^0 \xi_2 e^{-i\omega_2 t} + q_2^0 \xi_2^* e^{i\omega_2 t} \right) - A_1 - A_2 \right\}.
\end{aligned}$$

Here according to [1], we have substituted $q_j^0 \to \sqrt{\omega_j}\, q_j^0$, $q_j \to \sqrt{\omega_j}\, q_j$, $j = 1,2$ and multiplied with $\sqrt{\omega_1 \omega_2}$. When there is only one oscillator, this formula transforms into formula (64) of Ref.[1] or Ref.[4]. Further, if we take a real extra-source $\gamma(t) = \gamma^*(t) = -\frac{J(t)}{\sqrt{2\omega}}$, Feynman's results may be obtained [3], [4].

$$\langle q t | q_0 t_0 \rangle = \sqrt{\frac{\omega}{2\pi i \sin\omega T}}\, e^{iW}.$$

$$\begin{aligned}
W &= \frac{\omega}{2\sin\omega T}\left[(q^2 + q_0^2)\cos\omega T - 2q q_0\right] + \frac{q}{\sin\omega T}\int_{t_0}^{t} du\, J(u)\sin\omega(u - t_0) + \frac{q_0}{\sin\omega T}\int_{t_0}^{t} du\, J(u)\sin\omega(t - u) \\
&\quad - \frac{1}{\omega\sin\omega T}\int_{t_0}^{t} du \int_{t_0}^{u} ds\, J(u) J(s) \sin\omega(t - u)\sin\omega(s - t_0).
\end{aligned}$$

In this place, our calculation and result are more universal than [1], [4].

III. THE RIGOROUS SOLUTIONS FOR COUPLED DOUBLE OSCILLATOR AT TWO TIME DEPENDENT EXTRA-SOURCES

In this case the Hamiltonian is

$$H = H_0 + H'(t)\,, \qquad H'(t) = H_1'(t) + H_2'(t)\,,$$

$$H_1'(t) = \gamma_1(t)a^+ + \gamma_1^* a + \gamma_2(t) b^+ + \gamma_2^* b\,,$$

$$H_2'(t) = \alpha(t)a^+b + \beta(t)ab^+ + \delta(t)a^+b^+ + \lambda(t)ab\,.$$

Therefore, in the interaction picture the interaction part of the Hamiltonian is

$$\begin{aligned} H_I'(t) &= e^{iH_0t}H'(t)e^{-iH_0t} \\ &= \gamma_1 a^+ e^{i\omega_1 t} + \gamma_1^* a e^{-i\omega_1 t} + \gamma_2 b^+ e^{i\omega_2 t} + \gamma_2^* b e^{-i\omega_2 t} + \alpha a^+ b e^{i(\omega_1-\omega_2)t} \\ &\quad + \beta a b^+ e^{-i(\omega_1-\omega_2)t} + \delta a^+ b^+ e^{i(\omega_1+\omega_2)t} + \lambda a b e^{-i(\omega_1+\omega_2)t}\,, \end{aligned}$$

and the time-development equation is

$$i\frac{\partial U_I(t,t_0)}{\partial t} = H_I'(t)U_I(t,t_0)\,, \qquad U_I(t_0,t_0) = 1\,.$$

It may be discovered that the solution of this equation, i.e. the time-development operator of system, is

$$U_I(t,t_0) = e^{-i\xi_1 a^+}e^{-i\xi_2 b^+}e^{-i\xi_1^* a}e^{-i\xi_2^* b}e^{-iBa^+b}e^{-iCab^+}e^{-iDa^+b^+}e^{-iEab}e^{-A_1-A_2-K}\,,$$

where

$$B(t) = \int_{t_0}^{t}\alpha(s)e^{i(\omega_1-\omega_2)s}ds\,, \qquad C(t) = \int_{t_0}^{t}\beta(s)e^{-i(\omega_1-\omega_2)s}ds\,,$$

$$D(t) = \int_{t_0}^{t}\delta(s)e^{i(\omega_1+\omega_2)s}ds\,, \qquad E(t) = \int_{t_0}^{t}\lambda(s)e^{-i(\omega_1+\omega_2)s}ds\,,$$

and K is some function of t which is arranged to cancel those commutators produced by the terms containing B,C,D and E.

To put an end to this calculation, we suppose several cases.

First case: $H_2'(t) = \alpha(t)a^+b$.

In this case,

$$H_I'(t) = \gamma_1 a^+ e^{i\omega_1 t} + \gamma_1^* a e^{-i\omega_1 t} + \gamma_2 b^+ e^{i\omega_2 t} + \gamma_2^* b e^{-i\omega_2 t} + \alpha a^+ b e^{i(\omega_1-\omega_2)t}\,,$$

and the solution of the time-development operator is

$$U_I(t,t_0) = e^{-i\xi_1 a^+}e^{-i\xi_2 b^+}e^{-i\xi_1^* a}e^{-i\xi_2^* b}e^{-iBa^+b}e^{-A_1-A_2-F}\,,$$

where

$$F(t) = -i\int_{t_0}^{t}\alpha(s)\xi_1^*(s)\xi_2(s)e^{i(\omega_1-\omega_2)s}ds\,.$$

This expression for $F(t)$ may be obtained as long as we note the following commutator

$$\left[e^{-i\xi_2 b^+}e^{-i\xi_1^* a},\, -ia^+b\alpha e^{i(\omega_1-\omega_2)t}\right] = -i\alpha\xi_2\xi_1^* e^{i(\omega_1-\omega_2)t}e^{-i\xi_2 b^+}e^{-i\xi_1^* a}\,.$$

The term containing this commutator should be cancelled by the term containing F(t).

$$\begin{aligned} U_S(t,t_0) &= e^{-iH_0t}U_I(t,t_0)e^{iH_0t} \\ &= e^{-i\omega_1 Ta^+a - i\omega_2 Tb^+b}e^{-i\xi_1 a^+ e^{-i\omega_1 t_0}}e^{-i\xi_2 b^+ e^{-i\omega_2 t_0}}e^{-i\xi_1^* a e^{i\omega_1 t_0}}e^{-i\xi_2^* b e^{i\omega_2 t_0}}e^{-iBa^+be^{-i(\omega_1+\omega_2)t_0}}e^{-A_1-A_2-F-\frac{i(\omega_1+\omega_2)T}{2}} \end{aligned}$$

It is easy to obtain a general expression for transformation matrix elements in a coherent state picture. The result is

$$\begin{aligned} \langle z_1z_2t|z_1^0z_2^0t_0\rangle &= \langle z_1z_2|U_S(t,t_0)|z_1^0z_2^0\rangle \\ &= \exp\Big\{-A_1-A_2-F-\frac{i}{2}(\omega_1+\omega_2)T-\frac{1}{2}\left(|z_1|^2+|z_2|^2+|z_1^0|^2+|z_2^0|^2\right)-i\xi_1 z_1^* e^{-i\omega_1 t}-i\xi_2 z_2^* e^{-i\omega_2 t} \\ &\quad -i\xi_1^* z_1^0 e^{i\omega_1 t_0}-i\xi_2^* z_2^0 e^{i\omega_2 t_0}+z_1^* z_1^0 e^{-i\omega_1 T}+z_2^* z_2^0 e^{-i\omega_2 T}-iz_2^0 z_1^* B e^{-i\omega_1 t+i\omega_2 t_0}-z_2^0\xi_1^* B e^{i\omega_2 t_0}\Big\}\,. \end{aligned}$$

Second case: $H_2'(t) = \beta(t)ab^+$.

$$H_I'(t)=\gamma_1 a^+ e^{i\omega_1 t}+\gamma_1^* a e^{-i\omega_1 t}+\gamma_2 b^+ e^{i\omega_2 t}+\gamma_2^* b e^{-i\omega_2 t}+\beta a b^+ e^{-i(\omega_1-\omega_2)t} .$$

We have

$$U_I(t,t_0)=e^{-i\xi_1 a^+}e^{-i\xi_2 b^+}e^{-i\xi_1^* a}e^{-i\xi_2^* b}e^{-iCab^+}e^{-A_1-A_2-G} ,$$

$$G(t) = -i\int_{t_0}^{t}\beta(s)\xi_1(s)\xi_2^*(s)e^{-i(\omega_1-\omega_2)s}\,ds .$$

And

$$U_S(t,t_0)=\exp\left\{-A_1-A_2-G-\frac{i}{2}(\omega_1+\omega_2)T\right\}e^{-i\omega_1 T a^+a-i\omega_2 T b^+b}e^{-i\xi_1 a^+ e^{-i\omega_1 t_0}}e^{-i\xi_2 b^+ e^{-i\omega_2 t_0}} .$$
$$\cdot e^{-i\xi_1^* a e^{i\omega_1 t_0}}e^{-i\xi_2^* b e^{i\omega_2 t_0}}e^{-iCab^+ e^{i(\omega_1-\omega_2)t_0}} ,$$

$$\langle z_1 z_2 t|z_1^\circ z_2^\circ t_0\rangle=\exp\Big\{-A_1-A_2-G-\frac{i}{2}(\omega_1+\omega_2)T-\frac{1}{2}\left(|z_1|^2+|z_2|^2+|z_1^\circ|^2+|z_2^\circ|^2\right)-i\xi_1 z_1^* e^{-i\omega_1 t}$$
$$-i\xi_2 z_2^* e^{-i\omega_2 t}-i\xi_1^* z_1^\circ e^{i\omega_1 t_0}-i\xi_2^* z_2^\circ e^{i\omega_2 t_0}+z_1^* z_1^\circ e^{-i\omega_1 T}+z_2^* z_2^\circ e^{-i\omega_2 T}$$
$$-iCz_1^\circ z_2^* e^{-i\omega_2 t+i\omega_1 t_0}-Cz_1^\circ\xi_2^* e^{i\omega_1 t_0}\Big\} .$$

Third case: $H_2'(t) = \delta(t)a^+b^+$.

$$H_I'(t)=\gamma_1 a^+ e^{i\omega_1 t}+\gamma_1^* a e^{-i\omega_1 t}+\gamma_2 b^+ e^{i\omega_2 t}+\gamma_2^* b e^{-i\omega_2 t}+\delta a^+ b^+ e^{i(\omega_1+\omega_2)t} .$$

We have

$$U_I(t,t_0)=e^{-i\xi_1 a^+}e^{-i\xi_2 b^+}e^{-i\xi_1^* a}e^{-i\xi_2^* b}e^{-iDa^+b^+}e^{-A_1-A_2-I}$$

$$I(t) = i\int_{t_0}^{t}\delta(s)\xi_1^*(s)\xi_2^*(s)e^{i(\omega_1+\omega_2)s}\,ds$$

$$U_S(t,t_0)=e^{-A_1-A_2-I-\frac{i}{2}(\omega_1+\omega_2)T}e^{-i\omega_1 T a^+a-i\omega_2 T b^+b}e^{-i\xi_1 a^+ e^{-i\omega_1 t_0}}e^{-i\xi_2 b^+ e^{-i\omega_2 t_0}} .$$
$$\cdot e^{-i\xi_1^* a e^{i\omega_1 t_0}}e^{-i\xi_2^* b e^{i\omega_2 t_0}}e^{-iDa^+b^+ e^{-i(\omega_1+\omega_2)t_0}} ,$$

$$\langle z_1 z_2 t|z_1^\circ z_2^\circ t_0\rangle=\exp\Big\{-A_1-A_2-I-\frac{i}{2}(\omega_1+\omega_2)T-\frac{1}{2}\left(|z_1|^2+|z_2|^2+|z_1^\circ|^2+|z_2^\circ|^2\right)-i\xi_1 z_1^* e^{-i\omega_1 t}-i\xi_2 z_2^* e^{-i\omega_2 t}-iz_1^\circ\xi_1^* e^{i\omega_1 t_0}$$
$$-iz_2^\circ\xi_2^* e^{i\omega_2 t_0}+z_1^* z_1^\circ e^{-i\omega_1 T}+z_2^* z_2^\circ e^{-i\omega_2 T}-iDz_1^* z_2^* e^{-i(\omega_1+\omega_2)t}+D\left(-\xi_1^* z_2^* e^{-i\omega_2 t}-\xi_2^* z_1^* e^{-i\omega_1 t}+i\xi_1^*\xi_2^*\right)\Big\} .$$

Fourth case: $H_2'(t) = \lambda(t)ab$.

$$H_I'(t)=\gamma_1 a^+ e^{i\omega_1 t}+\gamma_1^* a e^{-i\omega_1 t}+\gamma_2 b^+ e^{i\omega_2 t}+\gamma_2^* b e^{-i\omega_2 t}+\lambda a b e^{-i(\omega_1+\omega_2)t} .$$

We have

$$U_I(t,t_0)=e^{-i\xi_1 a^+}e^{-i\xi_2 b^+}e^{-i\xi_1^* a}e^{-i\xi_2^* b}e^{-iEab}e^{-A_1-A_2-J} .$$

$$J(t)= i\int_{t_0}^{t}\lambda(s)\xi_1(s)\xi_2(s)e^{-i(\omega_1+\omega_2)s}\,dS ,$$

$$U_S(t,t_0)= e^{-A_1-A_2-J-\frac{i}{2}(\omega_1+\omega_2)T}e^{-i\omega_1 T a^+a-i\omega_2 T b^+b}e^{-i\xi_1 a^+ e^{-i\omega_1 t_0}}e^{-i\xi_2 b^+ e^{-i\omega_2 t_0}} .$$
$$\cdot e^{-i\xi_1^* a e^{i\omega_1 t_0}}e^{-i\xi_2^* b e^{i\omega_2 t_0}}e^{-iEab e^{i(\omega_1+\omega_2)t_0}} .$$

$$\langle z_1 z_2 t|z_1^\circ z_2^\circ t_0\rangle=\exp\Big\{-A_1-A_2-J-\frac{i}{2}(\omega_1+\omega_2)T-\frac{1}{2}\left(|z_1|^2+|z_2|^2+|z_1^\circ|^2+|z_2^\circ|^2\right)-i\xi_1^* z_1^\circ e^{i\omega_1 t_0}-i\xi_2^* z_2^\circ e^{i\omega_2 t_0}$$
$$-i\xi_1 z_1^* e^{-i\omega_1 t}-i\xi_2 z_2^* e^{-i\omega_2 t}+z_1^* z_1^\circ e^{-i\omega_1 T}+z_2^* z_2^\circ e^{-i\omega_2 T}-iEz_1^\circ z_2^\circ e^{i(\omega_1+\omega_2)t_0}\Big\} .$$

According to this method we may also calculate the cases that $H_2'(t)$ consists of several terms containing α, β, or λ.

IV. SOME APPLICATIONS FOR THE ANGULAR MOMENTUM THEORY

At first, we will express the space-rotation operator as the form of a regular product.

We note the rotation operator rotated an angle ψ about the direction $\vec{n}(\theta, \varphi)$ as

$$R_{\vec{n}}(\psi)= e^{-i\psi\vec{n}\cdot\vec{J}} ,$$

where $\vec{n}$ = ($\sin\theta\cos\varphi$, $\sin\theta\sin\varphi$, $\cos\theta$). We know that it may be divided into

$$e^{-i\psi\vec{n}\cdot\vec{J}} = e^{-iJ_z\varphi}e^{-iJ_y\theta}e^{-i\psi J_z}e^{iJ_y\theta}e^{iJ_z\varphi} .$$

We introduce the Schwinger representation of angular momentum [5]

$$J_i=\frac{1}{2}(a^+,b^+)\sigma_i\begin{pmatrix}a\\ b\end{pmatrix}, \qquad i=x,y,z,$$

i.e.

$$J_x=\frac{1}{2}(a^+b+b^+a),\ J_y=\frac{1}{2i}(a^+b-b^+a),\ J_z=\frac{1}{2}(a^+a-b^+b),$$
$$J^2=\frac{1}{4}S(S+2),$$

where $s = a^+a + b^+b$. Using the identities of some commutators in Sec.II, we have

$$e^{-i\psi\vec{n}\cdot\vec{J}}\begin{pmatrix}a\\ b\end{pmatrix}e^{i\psi\vec{n}\cdot\vec{J}}=u^+\begin{pmatrix}a\\ b\end{pmatrix},$$
$$e^{-i\psi\vec{n}\cdot\vec{J}}(a^+,b^+)e^{i\psi\vec{n}\cdot\vec{J}}=(a^+,b^+)u .$$

Here

$$u=\begin{pmatrix}\cos\frac{\psi}{2}-i\sin\frac{\psi}{2}\cos\theta, & -i\sin\frac{\psi}{2}\sin\theta e^{-i\varphi}\\ -i\sin\theta\sin\frac{\psi}{2}e^{i\varphi}, & \cos\frac{\psi}{2}+i\sin\frac{\psi}{2}\cos\theta\end{pmatrix}.$$

Using these equations, we may get

$$e^{-i\psi\vec{n}\cdot\vec{J}}=\int\frac{d^2z_1d^2z_2}{\pi^2}e^{-i\psi\vec{n}\cdot\vec{J}}e^{z_1a^++z_2b^+}e^{i\psi\vec{n}\cdot\vec{J}}e^{-i\psi\vec{n}\cdot\vec{J}}|0\rangle\langle z_1z_2|e^{-\frac{1}{2}(|z_1|^2+|z_2|^2)}$$
$$=\int\frac{d^2z_1d^2z_2}{\pi^2}:\exp\Big\{-|z_1|^2-|z_2|^2+\Big[z_1\Big(\cos\frac{\psi}{2}-i\sin\frac{\psi}{2}\cos\theta\Big)-iz_2\sin\frac{\psi}{2}\sin\theta e^{i\varphi}\Big]a^++$$
$$+\Big[-iz_1\sin\theta\sin\frac{\psi}{2}e^{i\varphi}+z_2\Big(\cos\frac{\psi}{2}+i\sin\frac{\psi}{2}\cos\theta\Big)\Big]b^++z_1^*a+z_2^*b-a^+a-b^+b\Big\}:$$
$$=:\exp\Big\{\Big(\cos\frac{\psi}{2}-1\Big)S-2i\sin\frac{\psi}{2}\cos\theta J_z-i\sin\frac{\psi}{2}\sin\theta\big(e^{i\varphi}J_-+e^{-i\varphi}J_+\big)\Big\}:$$

or

$$e^{-i\psi\vec{n}\cdot\vec{J}}=:\exp\Big\{(a^+,b^+)(u-1)\begin{pmatrix}a\\ b\end{pmatrix}\Big\}: .$$

This formula is a general expression for the rotation operator with a form of normal ordering and it is different to that expressed with the three Euler-angles in Ref.[6].

Secondly, we calculate the class-operator $C(\psi)$ of the O_3 group. Its definition is

$$C(\psi)=\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\sin\theta\, e^{-i\psi\vec{n}\cdot\vec{J}}$$

According to the above results, we may obtain

$$C(\psi)=:e^{(\cos\frac{\psi}{2}-1)S}\int_0^{\pi}d\theta\sin\theta e^{-2i\sin\frac{\psi}{2}\cos\theta J_z}\oint_{|z|=1}\frac{dz}{iz}e^{-i\sin\frac{\psi}{2}\sin\theta(zJ_-+\frac{J_+}{z})}:$$

From the Cauchy theorem, we may get

$$C(\psi)=:e^{(\cos\frac{\psi}{2}-1)S}\int_0^{\pi}d\theta\sin\theta e^{-2i\sin\frac{\psi}{2}\cos\theta J_z}2\pi\sum_{k=0}^{\infty}\frac{1}{(2k)!}\Big(-i\sin\theta\sin\frac{\psi}{2}\Big)^{2k}C_{2k}^{k}(J_-J_+)^k:$$
$$=4\pi:e^{(\cos\frac{\psi}{2}-1)S}\sum_{k,n=0}^{\infty}\frac{(-1)^{k+n}2^{k+n}(\sin\frac{\psi}{2})^{2k+2n}}{n!\,k!\,(2k+2n+1)!!}(J_-J_+)^kJ_z^{2n}:$$

In the above calculations, we have used the following formula

$$\int_0^{\pi}\cos^{2n}\theta\sin^{2k+1}\theta\,d\theta=B\Big(k+1,n+\frac{1}{2}\Big)=\frac{2^{k+1}k!\,(2n-1)!!}{(2k+2n+1)!!}$$

Let $m = k + n$ and note that

$$\sum_{k,n=0}^{\infty}A_kB_n=\sum_{m,n=0}^{\infty}A_{m-n}B_n, \qquad (m\geqslant n)$$

and

$$:J_-J_++J_z^2:=:\Big(\frac{S}{2}\Big)^2:$$

And from the binomial theorem, we can finally get

$$C(\psi)=4\pi : e^{(\cos\frac{\psi}{2}-1)S}\ \frac{\sin\left(S\sin\frac{\psi}{2}\right)}{S\sin\frac{\psi}{2}} :$$

This formula for $C(\psi)$ may also change into another form. In order to do this we write $C(\psi)$ as

$$C(\psi)=4\pi : \frac{e^{(e^{i\frac{\psi}{2}}-1)S}-e^{(e^{-i\frac{\psi}{2}}-1)S}}{2iS\sin\frac{\psi}{2}} :$$

and use

$$: e^{xS} : = \sum_{k=0}^{\infty}\frac{x^k}{k!} : S^k : = \sum_{k=0}^{\infty}\frac{x^k}{k!}S(S-1)\cdots\cdots(S-k+1) = (1+x)^S,$$

and its integral

$$: \frac{e^{xS}-1}{S} : = \frac{1}{S+1}\left\{(1+x)^{S+1}-1\right\},$$

where x is a parameter. Then, let $t = s + 1$ and note that $t^2 - 1 = 4J^2$ (J is a total angular momentum operator). We will finally obtain

$$C(\psi) = 4\pi\frac{\sin(\frac{\psi}{2}t)}{t\sin\frac{\psi}{2}}$$

or

$$C(\psi)=4\pi\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(4J^2+1)^k\left(\frac{\psi}{2}\right)^{2k}\Bigg/\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+1)!}\left(\frac{\psi}{2}\right)^{2l}$$

$$=4\pi\left\{1-\frac{1}{3!}\psi^2J^2+\frac{1}{180}\psi^4(12J^4-14J^2-5)-\cdots\cdots\right\}.$$

From this formula we know that $C(\psi)$ may be expressed as an infinite series of J^2. Therefore, $C(\psi)$ is independent to some special direction of space. From this we know being that $C(\psi)$ is a periodic function (one period is equal to 4π).

For $\psi = 2\pi, 4\pi$, we have

$$\overline{C(4\pi)} = \frac{C(4\pi)}{\int_{4\pi}d\Omega} = 1$$

$$\overline{C(2\pi)} = : e^{-2S} : = (-1)^S.$$

According to the definition of s, we have $J^2 = \frac{s}{2}(\frac{s}{2}+1)$. When s acts upon the eigenstate of the angular momentum j, we obtain an eigenvalue $2j$. Then we get

$$\overline{C(2\pi)} = (-1)^{2j}.$$

We also have the matrix elements of $C(\psi)$ between two coherent states,

$$\langle z_1z_2|C(\psi)|z_1^\circ z_2^\circ\rangle=4\pi\exp\left\{(\cos\frac{\psi}{2}-1)(z_1^*z_1^\circ+z_2^*z_2^\circ)-\frac{1}{2}(|z_1|^2+|z_2|^2+|z_1^\circ|^2+|z_2^\circ|^2)\right\}\frac{\sin\left((z_1^*z_1^\circ+z_2^*z_2^\circ)\sin\frac{\psi}{2}\right)}{(z_1^*z_1^\circ+z_2^*z_2^\circ)\sin\frac{\psi}{2}}.$$

Based on these results, we can further calculate the C-matrix elements in the coordinate representation and the angular momentum representation. For the methods of calculating these matrix elements the reader may refer to Ref.[6]. They are omitted in this paper.

V. AN EXPLICIT EXPRESSION FOR TIME-REVERSAL OPERATOR

Based on the above results, we may easily obtain an explicit expression for time-reversal operator.

We know that the time-reversal operator may be expressed as [7]

$$T = e^{-i\pi S_y}K$$

where s_y is the second component of the spin angular momentum, and K is an operator of complex conjugation.

We may then carry out a similar calculation by using the technique of boson's coherent states of double modes at normal ordering

$$T=\int\frac{d^2z_1d^2z_2}{\pi^2}e^{-i\pi S_y}|z_1z_2\rangle\langle z_1z_2|K$$

$$=\int\frac{d^2z_1d^2z_2}{\pi^2}e^{-\frac{1}{2}(|z_1|^2+|z_2|^2)}e^{-i\pi S_y}e^{z_1a^++z_2b^+}e^{i\pi S_y}e^{-i\pi S_y}|0\rangle\langle z_1z_2|K$$

$$=\int\frac{d^2z_1d^2z_2}{\pi^2}e^{-\frac{1}{2}(|z_1|^2+|z_2|^2)}\cdot e^{-i\pi S_y}e^{z_1a^+}e^{i\pi S_y}\cdot e^{-i\pi S_y}e^{z_2b^+}e^{i\pi S_y}\cdot e^{-i\pi S_y}|0\rangle\langle z_1z_2|K$$

$$=\int\frac{d^2z_1d^2z_2}{\pi^2}e^{-(|z_1|^2+|z_2|^2)}e^{z_1b^+}e^{-z_2a^+}|0\rangle\langle 0|e^{z_1^*a+z_2^*b}K,$$

where the first step of equality is due to $\int \frac{d^2 z_1 d^2 z_2}{\pi^2} |z_1 z_2 \rangle\langle z_1 z_2| = 1$;
the fourth step is due to $e^{-i\pi S_y}|0\rangle = |0\rangle$ and previous transformation formulas of a^+, b^+ at rotation.

After completing the integral, we obtain a normal ordering form for time-reversal operator under the Schwinger boson representation

$$T = :e^{-a^+a - b^+b + b^+a - a^+b}: K \ .$$

In fact, this expression may be obtained also from the normal ordering expression of $e^{-i\psi \vec{n}\cdot\vec{J}}$, provided that $\psi\vec{n} = \pi\vec{e}_y$.

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APPENDIX

We prove the following formula

$$e^{\beta ab} e^{\alpha a^+ b^+} = \frac{1}{1-\alpha\beta} e^{\frac{\alpha}{1-\alpha\beta} a^+ b^+} \left(\frac{1}{1-\alpha\beta}\right)^{a^+a + b^+b} e^{\frac{\beta}{1-\alpha\beta} ab} \ .$$

we have

$$\begin{aligned} e^{\beta ab} e^{\alpha a^+ b^+} &= \int \frac{d^2 z_1 d^2 z_2}{\pi^2} e^{\beta ab} |z_1 z_2\rangle\langle z_1 z_2| e^{\alpha a^+ b^+} \\ &= \int \frac{d^2 z_1 d^2 z_2}{\pi^2} :e^{-|z_1|^2 - |z_2|^2 + \beta z_1 z_2 + z_1 a^+ + z_2 b^+ + z_1^* a + z_2^* b + \alpha z_1^* z_2^* - a^+a - b^+b}: \\ &= \int \frac{d^2 z_2}{\pi} :e^{-(1-\beta\alpha)|z_2|^2 + z_2(b^+ + \beta a) + z_2^*(b + \alpha a^+) - b^+b}: \\ &= \frac{1}{1-\alpha\beta} e^{\frac{\alpha}{1-\alpha\beta} a^+ b^+} :e^{\frac{\alpha\beta}{1-\alpha\beta}(a^+a + b^+b)}: e^{\frac{\beta}{1-\alpha\beta} ab} \ . \end{aligned}$$

Due to

$$:e^{\gamma s}: = (1+\gamma)^s$$

we finally get

$$e^{\beta ab} e^{\alpha a^+ b^+} = \frac{1}{1-\alpha\beta} e^{\frac{\alpha}{1-\alpha\beta} a^+ b^+} \left(\frac{1}{1-\alpha\beta}\right)^{a^+a + b^+b} e^{\frac{\beta}{1-\alpha\beta} ab} \ .$$

Other four formulas may be deduced from the following formulas [1] (and their conjugations)

$$\begin{aligned} f(a) e^{-\lambda a^+} &= e^{-\lambda a^+} f(a-\lambda) , \\ f(a) e^{-\lambda a^+ a} &= e^{-\lambda a^+ a} f(a e^{-\lambda}) , \\ e^{A} B e^{-A} &= B + [A,B] + \frac{1}{2}[A,[A,B]] + \cdots\cdots \ . \end{aligned}$$

REFERENCES

[1] Fan Hongyi and Ruan Tunan, Scientia Sinica (A), No.1, (1984) 61.

[2] R.G. Glauber, Phys. Rev. 131 (1963) 2766.

[3] R.P. Feynman et al., Quantum Mechanics and Path Integral (McGraw-Hill, New York, 1965).

[4] L.D. Faddeev, Methods in Field Theory (North-Holland Amsterdam, 1976) 16.

[5] J. Schwinger, Quantum Theory of Angular Momentum (Academic Press, 1965).

[6] Fan Hongyi, Ruan Tunan, KuXue TongBao (Chinese), No.14 (1985) 1063.

[7] P. Roman, Advanced Quantum Theory (Addison-Wesley Publishing Co. Inc., 1965).