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GENERAL RELATIVITY INVARIANCE AND STRING FIELD THEORY *

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ABSTRACT

The general covariance principle in the string field theory is considered. The algebraic properties of the string Lie derivative are discussed. The string vielbein and spin connection are introduced and an action invariant under general co-ordinate transformation is proposed.

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I. INTRODUCTION

Recently, important results have been obtained in the gauge invariant field theory of free [1] and interacting [2]-[6] open strings. Closed strings are also considered in some papers [7], however, the field theory of closed strings is now in a rather unsatisfactory state (see the discussion in Witten's papers [2]), in particular, closed string theory pretending to describe the general relativity starts from a linearized approximation.

In this paper we discuss the general principles which are behind the theory of closed strings and in particular the fundamental general covariance principle in the string field theory. We also consider a formulation of the string field theory in terms of vielbein and spin connection string fields.

The string theories in some sense generalize the previously known gauge theories, namely the open string and the closed string generalize the Yang-Mills theory and the general relativity theory, respectively. Therefore, it is natural to think that at the base of string field theories lies a gauge invariance principle. Such a principle does lie at the basis of the interacting open string field theory. In Ref.[8] we have demonstrated that the consideration of the Yang-Mills type string gauge invariance permits to introduce the string field algebra and we have described the purely algebraic method to introduce interactions. An algebraic method of construction of interactions of strings has also been discussed in Ref.[9]. For the formulation of this principle and the construction of the gauge theory of open strings, the notion of the string analogue of the Yang-Mills covariant derivative is important. The role of connection was played by Siegel's string field $\Phi[X(\sigma),c(\sigma),\bar{c}(\sigma)]$, depending on the string configuration $X^{\mu}(\sigma)$, as well as Faddeev-Popov ghosts $c(\sigma),\bar{c}(\sigma)$, and taking value in some Lie algebra.

In general relativity, we deal with general covariance principle. Note that the gauge groups of general relativity and the Yang-Mills theory have essentially different features. The gauge transformations in gravity are generated by the Lie derivative $\mathcal{L}_{\xi} = i_{\xi}d + di_{\xi}$ and they are homogeneous in contrast to the non-homogeneous transformations in the Yang-Mills theory, which are given by the covariant derivative $D = d + A$.

There is also another reason why the gauge group for string theory with Yang-Mills type of invariance and the general co-ordinate gauge group should have different features. The string field can be considered as an infinite set of usual local fields. Let us suppose that we have the set of local gauge fields $A_{\mu}, A_{\mu\nu},\dots$ If the gauge transformations of these fields are described by the set of gauge parameters $\lambda, \lambda_{\mu},\dots$ namely, $\delta A_{\mu} = \partial_{\mu}\lambda$, $\delta A_{\mu\nu} = \partial_{[\mu}\lambda_{\nu]},\dots$ then we will name this gauge invariance the Yang-Mills type one. In particular, in the string field theory the gauge parameter is a string field, to which the set of local fields $\lambda, \lambda_{\mu},\dots$ corresponds.

A different picture arises when we deal with the general co-ordinate transformations. In this case for any given set of fields on manifold M (for example, on $\mathbb{R}^D$) co-ordinate transformations are generated by only one vector function, connected with the change of co-ordinates, $\delta X^\mu = \xi^\mu(X)$. This fact is closely connected with the universality of gravity and it also concerns the string theory. A general co-ordinate transformation in string field theory $\phi[X(\sigma) \to \phi[X(\sigma) + \xi(X(\sigma))]$ will not be specified by a string functional but only by one usual vector field ξ^μ. So, we see the fundamental difference between a gauge group of the Yang-Mills type and the group of general co-ordinate transformation in string field theories.

Will general relativity transformations appear as a subgroup of a gauge group of a string gauge group connected with chordal transformation $\delta\phi = L_{-n}\Omega_n$ [1]? It follows from Banks and Peskin's paper [1] that on the lowest levels the chordal group does not contain the general coordinate transformations for the open string nor for the closed one. For example, the tachyon field is invariant under chordal transformation but it must transform as scalar under general coordinate transformation, i.e. $\delta \varphi = \xi^\mu \partial_\mu \varphi$. The reason for this is that a chordal transformation is a reparametrization of a string sheet but a general relativity transformation is a transformation of the space-time coordinates.

Here it is natural to ask whether it would be possible to get the general relativity transformations by means of inner automorphisms of string field algebra, i.e. $\phi \to \phi \circ \Omega$ (compare with Witten's discussion [10]).

In trying to analyze this question we need an invariant string field theory. Recall that to construct the usual general relativity we ought to develop the tensor analysis on a manifold. We cannot pursue this line in string field theory because the vector symbols are hidden in the string functionals, so at first sight it is unclear how the tensor analysis in the language of string functionals can be developed. However, general relativity gives a hint as to what we have to do. Let us remember that vector indices referred to coordinates on a manifold do not enter explicitly into the Cartan forms, the frame form and the spin connection form, in terms of which the general relativity can be formulated.

There also exists another reason for the search for a new field theory formulation of interacting closed strings, which would not be the naive generalization of the theory of open string. It is well-known that closed superstrings in the limit $\alpha' \to 0$ have to describe supergravity. Recall that the main object in supergravity [11] is not the metric itself but the Cartan variables, the vielbein $e_\mu{}^a$ and the spin connection $\omega^a{}_{b\mu}$. These local fields cannot be described as zero-mass-level components of some scalar Siegel string field. This is the reason why fields E^a and $\Omega^a{}_b$ in the field theory of closed superstrings as well as heterotic string have to be introduced, providing a natural appearance of local fields $e^a{}_\mu$ and $\omega^a{}_{b\mu}$ in mode expansion.

It is well-known that gravity can be formulated as gauge theory with the Yang-Mills type of covariant derivative only if one deals with the spin connection which transforms as Yang-Mills connection under the Lorentz rotation of local frame. So, the straightforward generalization of the formalism to closed string should necessarily lead to the Siegel field taking values in the Lie algebra of the Lorentz group.

The string vielbein E^a and the string spin connection $\Omega^a{}_b$ were considered in Ref.[12], where the action in terms of E^a and $\Omega^a{}_b$ was proposed. This action is different from the usual Chern-Simons type action. Note that a string spin connection has also been considered in Ref.[13], where an interpretation of the Chern-Simons action in terms of the spin connection has been discussed.

As just mentioned, in general relativity there is also another type of invariance, namely the invariance under general coordinate transformations. The transformation properties of the vielbein and the

Lorentz connection under general coordinate transformations are given with the help of the Lie derivative. So, it is natural to believe that some string analogue of the Lie derivative must play the crucial role in gauge invariant field theory formulation of closed strings. In this paper, we introduce the string Lie derivative which will possess the same axiomatic properties as the usual Lie derivative. Then we consider formal algebraic properties of the algebra of fields, E^a and $\Omega^a{}_b$, needed to get the general relativity invariant action.

The paper is organized as follows. In Sec. II we shall start with the discussion of the general coordinate transformations in string field theories. In Sec.III we discuss the algebraic properties of string Lie derivative. Then in Sec.IV we introduce string vielbein and spin connection fields and present a pure algebraic mechanism of the construction of gauge invariant theory. This theory is gauge invariant under the general coordinate transformations parametrized by one vector field as well as under string generalization of the Lorentz group parametrized by string functionals.

II. GENERAL RELATIVITY GAUGE GROUP

Let us recall the general coordinate transformations in usual local field theory. General coordinate invariance is the invariance under the diffeomorphism group of a manifold M, Diff(M). The infinitesimal action of Diff(M) is essentially equivalent to the Lie derivative along a vector field ξ^μ. Under the infinitesimal change of variables

$$X'^\mu = X^\mu - \xi^\mu(X) \quad , \quad \text{i.e.} \quad \delta_G X^\mu = -\xi^\mu(X) \tag{2.1}$$

the scalar field transforms as $\varphi'(X') = \varphi(X)$ providing

$$\delta_G \varphi(X) = \varphi'(X) - \varphi(X) = \xi^\mu \partial_\mu \varphi = \mathcal{L}_\xi \varphi . \tag{2.2}$$

In the same way, we have for a vector field

$$\delta_G A_\mu(X) = \mathcal{L}_\xi A_\mu(X) = A'_\mu(X) - A_\mu(X) = \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu A_\nu \tag{2.3}$$

and so on. Here $\mathcal{L}_\xi$ is the Lie derivative. Let us mention that (2.1) and (2.2) are written without referring to the metric on the manifold. An

action $S = S(\psi)$ depending on a set of fields $\psi^i = (\varphi, A_\mu, B_{\mu\nu}, \dots)$ is invariant under the general coordinate transformation if under the transformation $\delta_G \psi = \mathcal{L}_\xi \psi$ the variation of action

$$\delta_G S = \int \frac{\delta S}{\delta \psi^i(X)} \delta_G \psi^i(X)\, dX$$

is equal to zero. To describe the general coordinate invariant theories it is necessary to develop the tensor analysis. It is worth noting that among general coordinate invariant theories there exist the theories formulated without a metric, for example, the characteristic Chern-Simons classes.

Let us now consider a string field theory. If the string is specified by the equation $X^\mu = X^\mu(\sigma)$, then under general coordinate transformation (2.1) the string equation transforms into $X'^\mu = X'^\mu(\sigma)$, i.e.

$$\delta_G X^\mu(\sigma) = -\xi^\mu(X(\sigma)) \quad . \tag{2.4}$$

This means that a scalar functional $\Phi[X(\sigma)]$ transforms under general coordinate transformations as

$$\delta_G \Phi[X] = \mathcal{L}_\xi \Phi[X] = \int \xi^\mu(X(\sigma)) \frac{\delta \Phi[X]}{\delta X^\mu(\sigma)} \tag{2.5}$$

This is nothing but the string analogue of the Lie derivative for scalar fields (2.1). It can be presented also in the following form:

$$\mathcal{L}_\xi = \int : \xi^\mu(X(\sigma)) \mathcal{P}_\mu(\sigma) : d\sigma \tag{2.6}$$

where $\mathcal{P}_\mu(\sigma)$ is a canonical conjugate momenta.

For the usual Lie derivative the following relations hold:

$$[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{[\xi_1, \xi_2]} \tag{2.7}$$

Here

$$[\xi_1, \xi_2]^\mu = \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu \tag{2.8}$$

and the Leibnitz rule

$$\mathcal{L}_\xi \left(T_1 \otimes T_2 \right) = \left(\mathcal{L}_\xi T_1 \right) \otimes T_2 + T_1 \otimes \mathcal{L}_\xi T_2 \qquad (2.9)$$

where T_1 and T_2 are elements of the tensor algebra.

It is obvious that for the string Lie derivative (2.6) the relations (2.7) hold up to the possible anomalies which can, in principle, appear caused by a normal ordering operation. As to the string analogue of Eq.(2.9) here the situation is more complicated. Firstly, we dare to say that string fields are the counterparts of differential forms more than tensors. And secondly, on the string differential forms we have the product operation and it is desirable to have for the string Lie derivative the Leibnitz rule with respect to a given multiplication.

In this context it is useful to note that to provide the general coordinate invariance it is sufficient to restrict ourselves in the expansion

$$\xi^\mu \left(X(\sigma) \right) = C^\mu + C^\mu_\nu X^\nu(\sigma) + C^\mu_{\nu\lambda} X^\nu(\sigma) X^\lambda(\sigma) + \dots$$

to the same special form of transformations (2.1), namely, dimensional affine and conformal transformations, see [14].

The Leibnitz rule for the string Lie derivative with special form of vector field ξ^μ, namely with $\xi^\mu = c^\mu$ and $\xi^\mu = c^\mu{}_\nu X^\nu(\sigma)$ with anti-symmetric matrix $c_{\mu\nu}$ follows immediately from the conservation of the corresponding two-dimensional currents, $\mathcal{P}^\mu_\alpha = \partial_\alpha X^\mu$ and $M_\alpha^{\mu\nu} = X^{[\mu} \partial_\alpha X^{\nu]}$ (these currents are conserved even in the presence of world sheet curvature).

Let us explicitly demonstrate that the transformation (2.5) at lower levels indeed leads to the usual non-linearized general coordinate transformation of the scalar field φ and the vector field A_μ.

For simplicity we will perform the calculation for open string where

$$X^\mu(\sigma) = x^\mu + i \sum_{n>0} \frac{1}{n} \left(\alpha_n^\mu - \alpha_n^{\mu +} \right) \cos n\sigma$$

$$\mathcal{P}_\mu(\sigma) = p_\mu + \sum_{n>0} \left(\alpha_{n\mu} + \alpha_{n\mu}^+ \right) \cos n\sigma$$

and the string field has a form

$$\Phi = \left(\varphi(x) + \alpha_{-1}^\mu A_\mu(x) + \dots \right) \Phi_0 \quad .$$

We have

$$\mathcal{L}_\xi \Phi = \frac{i}{\pi} \int_0^\pi d\sigma : \xi^\mu \Big(x^\mu + i \sum_{n>0} \frac{1}{n} \left(\alpha_n^\mu - \alpha_n^{\mu +} \right) \cos n\sigma \Big) \Big(p_\mu + \sum_{n>0} \left(\alpha_{n\mu} + \alpha_{n\mu}^+ \right) \cos n\sigma \Big) : \Phi$$

and therefore

$$\mathcal{L}_\xi \Phi = \frac{i}{\pi} \int_0^\pi d\sigma : \xi^\mu \Big(x^\mu + i \sum_{n>0} \frac{1}{n} \left(\alpha_n^\mu - \alpha_n^{\mu +} \right) \cos n\sigma \Big) \Big(p_\mu + \sum_{n>0} \left(\alpha_{n\mu} + \alpha_{n\mu}^+ \right) \cos n\sigma \Big) \cdot$$

$$\left(\varphi(x) + \alpha_{-1}^\nu A_\nu(x) + \dots \right) \Phi_0 =$$

$$= \Big\{ \xi^\mu \partial_\mu \varphi + \frac{1}{2} \left(\xi^\mu \partial_\mu A_\lambda + \partial_\lambda \xi^\mu A_\mu \right) \alpha_{-1}^\nu +$$

$$+ \frac{1}{2} \left(\xi^\nu \partial_\nu A^\mu - \partial_\nu \xi^\mu A^\nu \right) \alpha_{-1\mu} + \dots \Big\} \Phi_0$$

So, we get the correct general coordinate transformations (2.2) and (2.3). Analogous results can be obtained for the closed string.

As to the invariance under the general coordinate transformation of any given string action, the more simple way *) to ensure that invariance consists in the construction of a realization of the string Lie derivative as inner differential operator in the $\mathfrak{B}$ algebra .

$$\mathcal{L}_\xi \Phi = J_\xi \circ \Phi \quad .$$

The Lie derivative (2.6) in the particular case corresponding to a constant vector field ξ^μ can be realized as inner derivative in $\mathfrak{B}_1$ algebra with Witten's vertex. This statement follows directly from the results obtained in a recent paper by Horowitz and Strominger [15].

*) Of course, in principle, it can happen that an action has general coordinate invariance, but the string Lie derivative cannot be presented as an inner derivative.

One can expect the following general formula:

$$J_\xi = \mathcal{L}_\xi^R I \quad ,$$

where I is the unit element of the Witten star algebra and

$$\mathcal{L}_\xi^R = \int_{\pi/2}^{\pi} \xi^\mu(x(\sigma)) \mathcal{P}_\mu(x(\sigma))\, d\sigma$$

Formally, it is easy to show that the Lie derivative (2.6) with special form of $\xi^\mu(X)$ corresponding to Lorentz rotations, i.e. $\xi^\mu = c^\mu{}_\nu X^\nu$, $c^\mu{}_\nu$ is antisymmetrix matrix in an inner derivative.

III. LIE DERIVATIVES

3.1 Lie Derivatives of Forms

The Lie derivative $\mathcal{L}_\xi$ acting on the differential form algebra $\mathcal{D}(M)$ is a derivative on this algebra of zero degree which commutes with the external differential d, and conversely, any derivative of zero degree acting on the algebra $\mathcal{D}(M)$ and commuting with d is equal to $\mathcal{L}_\xi$ with some vector field ξ. So, the Lie derivative has the following properties:

$$\mathcal{L}_\xi(\omega_p \wedge \lambda_q) = (\mathcal{L}_\xi \omega_p)\wedge \lambda_q + (-1)^p \omega_p \wedge \mathcal{L}_\xi \lambda_q \tag{3.1}$$

where ω_p and λ_q are p-form and q-form

$$[\mathcal{L}_\xi, d] = 0 \tag{3.2}$$

and

$$[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{[\xi_1,\xi_2]} \tag{3.3}$$

where $[\xi_1,\xi_2]^\mu = \xi_1{}^\nu \partial_\nu \xi_2{}^\mu - \xi_2{}^\nu \partial_\nu \xi^\mu{}_1$, i.e. $\mathcal{L}_\xi$ generates a Lie algebra structure on the space of vector fields over M. The Lie derivative is also a derivative on the tensor algebra over M.

For differential forms we can write the Lie derivative in the form

$$\mathcal{L}_\xi \omega_p = (d i_\xi + i_\xi d)\omega_p \tag{3.4}$$

where ω_p is a p-form $\omega_p = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p} dX^{\mu_1}\wedge\ldots\wedge dX^{\mu_p}$, the inner multiplication i_ξ is defined by

$$i_\xi \omega_p = \frac{1}{(p-1)!}\xi^\mu \omega_{\mu\mu_1\ldots\mu_{p-1}} dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_{p-1}}. \tag{3.5}$$

The properties of i_ξ are

$$\{ i_{\xi_1}, i_{\xi_2} \} = 0, \tag{3.6}$$

$$[i_\xi, \mathcal{L}_\xi] = 0, \tag{3.7}$$

$$i_\xi(\omega_p \wedge \lambda_q) = (i_\xi \omega_p)\wedge\lambda_q + (-1)^p \omega_p \wedge i_\xi \lambda_q . \tag{3.8}$$

3.2 String Field Algebra

Let us now concentrate on the string field theory. We will briefly describe the mathematical structure which appears in this context.

The requirement of the closure of the gauge transformation algebra permits to introduce [8] the algebraic structure (i.e. to introduce the multiplication of two functionals) on the space denoted by the symbol $\circ$. The space of functionals with given multiplication operation, we call the string $\mathfrak{B}$-algebra. Introducing an inner product and a derivative Q one obtains the action

$$S = (\phi, Q\phi) + \tfrac{1}{3}(\phi, \phi\circ\phi) \tag{3.9}$$

which is invariant under the gauge transformations

$$\delta\phi = Q\Lambda + \phi\circ\Lambda \quad .$$

Let us mention that there are two possible ways to realize the closure of the gauge transformation algebra $\mathfrak{B}_I$ and $\mathfrak{B}_{II}$. In the first case we have to deal with the situation where $\mathfrak{B}_I$ is nothing but a Lie super-algebra and in the second case $\mathfrak{B}_{II}$ is a supercommutative non-associative algebra. The connection of $\mathfrak{B}_I$ with Witten's associative non-commutative algebra is given in the same way as in the Lie algebra theory, i.e. $\phi \circ \psi = \phi * \psi - (-1)^{|\phi||\psi|} \psi * \phi$, where $*$ is Witten's associative multiplication and $|\phi|$ is the Grassmannian parity and we have the equality

$$S = (\phi, Q\phi) + \frac{1}{3}(\phi, \phi\circ\phi) = \int \phi * Q\phi + \frac{2}{3}\int \phi * \phi * \phi$$

So in fact Witten's approach deals with the Lie superalgebra.

Note that the action for the open string is written in terms of Siegel's field ϕ. The straightforward generalization of this approach was also considered for the closed string [6],[7].

In the following section we will describe a possible action of Lie derivative on the $\mathfrak{B}$-algebra using the analogue with the algebra of usual differential forms, i.e. we will try to tie together the set $(\mathfrak{B}, Q, (.,.))$ and the string Lie derivative $\mathcal{L}_\xi$.

3.3 String Lie Derivative

The consideration of Siegel's string field as string differential form [16] was very instructive. In this interpretation the BRST charge Q was considered as the exterior differential operator. Let us try to understand the string analogue of the Lie derivative. Having in mind the usual differential geometry formula (3.5) we have at first to introduce the notion of string inner multiplication I_ξ. Suppose that the action on string fields of I_ξ being itself some string field can be represented as

$$I_\xi\phi = I_\xi \cdot \phi \qquad (3.10)$$

with some kind of multiplication operator, and, moreover, we postulate the string analogue of the properties (3.6) and (3.8),

$$I_{\xi_1}\cdot(I_{\xi_2}\cdot\phi) + I_{\xi_2}\cdot(I_{\xi_1}\cdot\phi) = 0, \qquad (3.11)$$

$$I_\xi\cdot(\Omega\circ\Lambda) = (I_\xi\cdot\Omega)\circ\Lambda + (-1)^{|\Omega|}\Omega\circ(I_\xi\cdot\Lambda) \qquad (3.12)$$

Here the string forms ϕ, Ω and Λ can have some extra indices while I_ξ does not carry any indices.

One can satisfy (3.11) and (3.12) simultaneously by supposing that the Jacobi identities hold for different kinds of multiplication

$$(-1)^{|I||\Lambda|} I_\xi\cdot(\Omega\circ\Lambda) + (-1)^{|\Omega||I|}\Omega\circ(\Lambda\cdot I_\xi) + (-1)^{|\Lambda||\Omega|}\Lambda\circ(I_\xi\cdot\Omega) = 0, \qquad (3.13)$$

$$(-1)^{|I||\phi|} I_{\xi_1}\cdot(I_{\xi_2}\cdot\phi) + (-1)^{|I|} I_{\xi_2}\cdot(\phi\cdot I_{\xi_1}) + (-1)^{|\phi||I|}\phi\cdot(I_{\xi_1}\cdot I_{\xi_2}) = 0 \qquad (3.14)$$

and that I_ξ is the subject of

$$I_{\xi_1}\cdot I_{\xi_2} = 0 \quad . \qquad (3.15)$$

Indeed, let us consider for the definiteness the $\mathfrak{B}_I$ case, then the physical fields are odd ones, so if I_ξ is an odd one too, then

$$I_\xi\cdot(\Omega\circ\Lambda) = (I_\xi\cdot\Omega)\circ\Lambda + (-1)^{|\Omega|}\Omega\circ(I_\xi\cdot\Lambda) \qquad (3.16)$$

in accordance with (3.12), and the condition (3.11) follows immediately from Eqs.(3.14) and (3.15).

With I_ξ operation in hand in the line of the usual differential geometry we can define the string Lie derivative as

$$\mathcal{L}_\xi\phi = Q(I_\xi\cdot\phi) + I_\xi\cdot(Q\phi) \qquad (3.17)$$

which is equal to $QI_\xi\cdot\phi$, if the Leibnitz rule takes place, i.e.

$$\mathcal{L}_\xi\phi = (QI_\xi)\cdot\phi, \qquad (3.18)$$

If the string Lie derivative acts as in Eq.(3.18) then it is obvious that the string analogues of Eqs.(3.1) and (3.2)

$$\mathcal{L}_\xi(\phi\circ\psi)=\mathcal{L}_\xi\phi\circ\psi+(-1)^{|\phi|}\phi\circ\mathcal{L}_\xi\psi \tag{3.19}$$

$$[\mathcal{L}_\xi, Q]=0 \tag{3.20}$$

hold. Indeed,

$$(QI_\xi)\bullet(\phi\circ\psi)=\phi\circ(QI_\xi\bullet\psi)+\psi\circ(QI_\xi\bullet\phi)$$

Here we suppose that ϕ and ψ are odd ones and we take into account the Jacobi identity. Further,

$$Q(\mathcal{L}_\xi\phi)-\mathcal{L}_\xi(Q\phi)=Q(QI_\xi\bullet\phi)-QI_\xi\bullet(Q\phi)=$$
$$=QI_\xi\bullet Q\phi-QI_\xi\bullet Q\phi=0,$$

Here the Leibnitz rule for $\bullet$ multiplication is supposed to hold. Under these assumptions the following condition also holds:

$$[\mathcal{L}_{\xi_1},\mathcal{L}_{\xi_2}]=\mathcal{L}_{[\xi_1,\xi_2]} \tag{3.21}$$

where

$$\mathcal{L}_{[\xi_1,\xi_2]}=Q\left(\tfrac{1}{2}\left(QI_{\xi_1}\cdot I_{\xi_2}-QI_{\xi_2}\cdot I_{\xi_1}\right)\right)$$

Let us say that the string Lie derivative can be defined more abstractly without referring to string analogue of the inner multiplication i_ξ. We call the operation $\mathcal{L}$ the string Lie derivative if it satisfies Eqs.(3.19) and (3.20). As the string Lie derivative we can consider the operation given by the following formula

$$\mathcal{L}_\Xi\,\phi=(Q\,\Xi)\circ\phi \tag{3.22}$$

where Ξ is the string functional and $\circ$ is the usual multiplication.

It is obvious that in this case the relations (3.19) and (3.20) are satisfied and

$$[\mathcal{L}_{\Xi_1},\mathcal{L}_{\Xi_2}]=\mathcal{L}_{\frac{1}{2}(Q\Xi_1\circ\Xi_2-Q\Xi_2\circ\Xi_1)} \tag{3.23}$$

In this section we have presented the general properties of the Lie derivative defined by Eq.(3.22) or (3.18). Can the Lie derivative introduced in the previous section, or better to say its generalization including ghosts, be presented in the form of (3.19) or (3.22)? If so, a theory invariant under transformations (3.18) is invariant under general coordinate transformations. For a special kind of vector field ξ^μ one can give a positive answer and in this case the multiplication coincides with the multiplication.

The following remark about Eq.(3.20) is in order. The conventional BRST invariant action for the bosonic string in arbitrary background space-time metric $G_{\mu\nu}$ is [17]

$$S=\frac{1}{2}\int d^2\sigma\left[G_{\mu\nu}(x)\partial_\alpha x^\mu\partial^\alpha x^\nu+2b_{++}\partial_-c^++2b_{--}\partial_+c^-\right]$$

where

$$\sigma^\pm=(\sigma^0\pm i\sigma^1)/\sqrt{2}\ ,\quad \partial_\pm=\tfrac{1}{\sqrt{2}}(\partial_0\pm i\partial_1)\ ,\quad \sigma^\pm=e^{2(\tau\pm i\sigma)}$$

The action is invariant under the general relativity transformation (we can start with the Minkowski metric $G_{\mu\nu}$ in flat coordinates) and the corresponding BRST charge commutes with the Lie derivative

$$[\mathcal{L}_\xi, Q]=0\ .$$

Therefore for the free action, we have

$$\delta_G(\Phi,Q\Phi)=(\mathcal{L}_\xi\Phi,Q\Phi)+(\Phi,Q\mathcal{L}_\xi\Phi)=(\Phi,[Q,\mathcal{L}_\xi]\Phi)=0\ .$$

Let us now go on to describe invariant action in the framework of theory $(\mathcal{B},Q,(\cdot,\cdot),\mathcal{L}_\xi)$.

IV. STRING VIELBEIN AND SPIN CONNECTION

4.1 The Vielbein Formalism in the General Relativity

The metric $G_{\mu\nu}$ on a manifold M can be expressed as $G_{\mu\nu} = e^a{}_\mu \, e^b{}_\nu \, \eta_{ab}$, where $e^a{}_\mu$ is a vielbein and η_{ab} is the Minkowski metric. Geometrically, the $e^a{}_\nu$ form an orthonormal set of vectors in the tangent space of M at a given point X. Given $G_{\mu\nu}$ the vielbein is not uniquely determined. Any local Lorentz transformation of the frame one-form $e^a = e^a{}_\mu \, dX^\mu$

$$\delta_L e^a = \lambda^a{}_b \, e^b \tag{4.1}$$

leads to an equally good vielbein. Thus we extend the principle of general covariance, by requiring that the equations of a gravitation theory be covariant not only under changes of coordinate bases of M, but also under local Lorentz transformations of orthonormal bases in the tangent space of M.

The local Lorentz invariance is implemented like Yang-Mills gauge invariance with gauge group $SO(D-1,1)$. In fact, one can define the spin connection $\omega^a{}_b$ as a matrix valued one-form which transforms as

$$\delta_L \, \omega = d\lambda + [\omega, \lambda] \quad . \tag{4.2}$$

The torsion and curvature of a connection are related to the frame one-form by the Cartan structure equations

$$de^a + \omega^a_b \wedge e^b = T^a \tag{4.3}$$

$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = R^a_b \tag{4.4}$$

where T^a and $R^a{}_b$ are the torsion two-form and the curvature two-form, respectively.

The action for pure gravity in the first-order formalism is

$$S[e,\omega] = \int dX \, eR \quad , \tag{4.5}$$

where $R = R^a{}_{b\mu\nu} \, e^\mu{}_a \, e^{b\nu}$, $e = \det(e^a{}_\mu)$. Here $\omega^a{}_{b\mu}$ and $e^a{}_\mu$ are independent field variables, and the dynamics will determine the relation between them. The equation of motion for $e_{a\mu}$ is a non-symmetric form of the Einstein equation

$$R_{a\mu} - \frac{1}{2} e_{a\mu} R = 0 \quad . \tag{4.6}$$

From the equation for the connection we get the torsion-free expression $\omega = \omega(e)$. If we insert $\omega(e)$ in Eq.(4.6), we get the conventional Einstein equation.

Every field referred to local Lorentz frames transforms as a coordinate scalar. The frame one-form and the spin connection form transform under the coordinate transformation generated by the vector field ξ^μ with the help of the Lie derivative

$$\delta_G \, e^a_\mu = \mathcal{L}_\xi \, e^a_\mu = \xi^\nu \partial_\nu e^a_\mu + \partial_\mu \xi^\nu \, e^a_\nu \, , \tag{4.7}$$

$$\delta_G \, \omega^a_{b\mu} = \mathcal{L}_\xi \, \omega^a_{b\mu} = \xi^\nu \partial_\nu \, \omega^a_{b\mu} + \partial_\mu \xi^\nu \, \omega^a_{b\nu} \tag{4.8}$$

4.2 String Vielbein Formalism

Let us consider the string vielbein fields E_a and E^a and a string spin connection field $\Omega^a{}_b$ depending on the string position $X(\sigma)$ as well as on the ghost coordinates. The indices $a,b = 1,\ldots,D$ are indices referring to the Lorentz frame, and the functional $\Omega^a{}_b$ takes values in the Lie algebra of the Lorentz group.

To generalize the transformations (4.1) and (4.2) for the string functionals E_a, E^a and $\Omega^a{}_b$ it is natural to suppose that

$$\delta_L \, E^a = \Lambda^a_b \circ E^b , \tag{4.9}$$

$$\delta_L \, E_a = E_b \circ \Lambda^b_a \, , \tag{4.10}$$

$$\delta_L \, \Omega^a_b = Q \Lambda^a_b + (\Omega \circ \Lambda)^a_b \tag{4.11}$$

Hence we suppose that string fields E_a, E^a are transformed under string counterpart of the usual Lorentz gauge transformation as matter fields in accordance with the homogeneous transformation (4.1) and the string field $\Omega^a{}_b$ as a connection, i.e. as a usual Lie algebra valued string field.

The action of the general coordinate transformation on string Cartan variables we suppose is given by the Lie derivative

$$\delta_G E^a = \mathcal{L}_\xi E^a \tag{4.12}$$

$$\delta_G \Omega^a_b = \mathcal{L}_\xi \Omega^a_b \tag{4.13}$$

where $\mathcal{L}_\xi$ is given by the formula (2.6) or (3.22).

Let us introduce the string curvature

$$R^a_b = Q\Omega^a_b + \frac{1}{2}(\Omega \circ \Omega)^a_b \tag{4.14}$$

which transforms homogenously under the string gauge transformation (4.11) as well as under general coordinate transformation (4.13),

$$\delta_G R^a_b = \mathcal{L}_\Xi R^a_b = (Q\Xi)\cdot R^a_b \tag{4.15}$$

We believe that it is natural to introduce the following string analogue of the action (4.5)

$$S = (R^a{}_b, E_a \circ E^b) \tag{4.16}$$

Here $(\cdot,\cdot)$ is the scalar product on the algebra $\mathcal{B}$ of string fields.

It is easy to verify that under natural assumptions on the multiplication, the action (4.16) is gauge invariant and also invariant under the transformations (4.12) and (4.13).

The usual vielbein and spin connection appear at the first mass level of the expansions

$$E^a(\alpha) = e^a_\mu \alpha^\mu_1 + \dots$$
$$E_a(\bar\alpha) = e^a_\mu \bar\alpha^\mu_1 + \dots$$
$$\omega^{ab}(\alpha,\bar\alpha) = \omega^{ab}_\mu(\alpha^\mu_1 + \bar\alpha^\mu_1) + \dots$$

A detailed investigation of the mode expansions is found in Refs.[12],[18].

V. CONCLUSION

We have discussed the general covariance principle in the context of string field theory. The main ingredient of our consideration is the string Lie derivative. The string vielbein and spin connection are introduced and an action invariant under general coordinate transformation is proposed. We hope that all these objects will become particularly useful in the context of closed superstrings.

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