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LINEAR WAVES AND STABILITY IN IDEAL MAGNETOHYDRODYNAMICS

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Abstract. Linear waves superimposed on an arbitrary basic state in ideal magnetohydrodynamics are studied by an asymptotic expansion valid for short wavelengths. It has not been necessary to introduce any assumption beyond the usual regularity assumptions on the arbitrarily given solution which represents our basic state, it may even be time dependent. The theory also allows for a gravitational potential; it may therefore be applied both in astrophysics and in problems related to thermonuclear fusion. The linearized equations for the perturbations of the basic state are found in the form of a symmetric hyperbolic system. This symmetric hyperbolic system is shown to possess characteristics of nonuniform multiplicity, which implies that waves of different types may interact. In particular it is shown that the mass waves, the Alfvén waves, and the slow magnetoacoustic waves will persistently interact in the exceptional case where the local wave number vector is perpendicular to the magnetic field. The equations describing this interaction are found in the form of a weakly coupled hyperbolic system. This weakly coupled hyperbolic system is studied in a number of special cases, and detailed analytic results are obtained for some such cases. The results show that the interaction of the waves may be one of the major causes of instability of the basic state. It seems beyond doubt that the interacting waves contain the physically relevant parts of the waves, which often are referred to as ballooning modes, including Suydam modes and Mercier modes.

I. INTRODUCTION

In a very wide spectrum of applications the problems of linear wave propagation and stability play a central role. There is a vast literature available in these fields, but still the problems are solved only in special cases. The conventional approaches to these problems are the normal mode method and the energy principle. These methods are not without difficulties and limitations, this author has therefore looked for other methods [1,2], which have been shown to be useful in fluid mechanics[3,4]. At a conference in 1984 in Trieste, Italy Dr.E. Hameiri and the author realized that there were certain similarities between the latters approach in fluid mechanics and the approach Hameiri had applied in magnetohydrodynamics (MHD)[5,6]. Since the methods applied were clearly different, it was decided that it might be worthwhile to try to apply the author's approach in MHD and Hameiri's approach in fluid mechanics in order to see if it is possible to improve upon the results obtained earlier. The first part of a contribution to the problems of linear wave propagation and stability in MHD will be outlined in this paper.

We shall work entirely within the framework of ideal MHD where we shall be concerned with linear waves superimposed on a given basic state. The given basic state may be <u>any</u> given solution of the fundamental MHD equations, it may be with or without flow, and it may be stationary or time dependent. The linearized equations for the perturbations of this given basic state are found in Sec.II in the form of a symmetric hyperbolic system. In this paper we limit our study of solutions of this hyperbolic system to asymptotic expressions valid for short wavelengths and/or high frequencies. The approach applied is the generalized progressing wave expansion method involving the characteristic equation and transport equations along the characteristics. The method is briefly outlined in Sec.III and follows the approach given by Eckhoff[1]. In contrast to fluid mechanics, the equations governing ideal MHD do not have characteristics with constant multiplicity. In order to study all the relevant modes it is therefore necessary to extend the approach described by Eckhoff[1] to cases where the multi-'icity assumptions are not satisfied. In fact, after having looked at the different modes in the nonsingular cases in Secs.V-VII without detecting any instabilites, we turn to the singular cases in Secs.VIII and IX. In Sec.IX we show that the singular case where the local wavenumber vector $\underline{k}$ is perpendicular to the magnetic field, $\underline{B}_0$, always constitutes a persistent property along the rays. As a consequence of this, the mass waves, the Alfvén waves, and the slow magnetoacoustic waves may persistently interact along the rays in this case. The transport equations describing this interaction are derived in Sec. IX and are seen to constitute a weakly coupled hyperbolic system.

The system of transport equations derived in Sec.IX may serve as a starting point for extensive studies of linear waves with short wavelengths in ideal MHD. In particular, Suydam-modes, Mercier-modes, and ballooning modes must be properly described by this system of transport equations[7-10], as well as analogous waves in more general geometries than have previously been studied. Such waves may contain important information about the stability of the given basic state, giving both the growth rates and the structure of the unstable modes. In this paper we restrict our study of the system of transport equations to some special cases which are especially attractive analytically in Secs.X-XII. In particular we are able to obtain detailed analytic results for slabs and screw pinches with constant pitch. It is not yet known to what extent it is possible to derive analytic results for more general cases, but it is clear that the system of transport equations obtained is very attractive for numerical methods. Thus at least by numerical methods it may be possible to obtain necessary conditions for stability of far more general basic states by our approach than by conventional approaches.

II. THE BASIC EQUATIONS

The fundamental ideal magnetohydrodynamic equations are

$$\begin{aligned}
&\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v} = -\rho^{-1}\nabla p + \rho^{-1}(\nabla\times\underline{B})\times\underline{B} + \nabla V, \\
&\frac{\partial \underline{B}}{\partial t} + \underline{v}\cdot\nabla\underline{B} - \underline{B}\cdot\nabla\underline{v} + \underline{B}\nabla\cdot\underline{v} = \underline{0}, \\
&\frac{\partial \rho}{\partial t} + \underline{v}\cdot\nabla\rho + \rho\nabla\cdot\underline{v} = 0, \\
&\frac{\partial}{\partial t}(p\rho^{-\gamma}) + \underline{v}\cdot\nabla(p\rho^{-\gamma}) = 0,
\end{aligned} \tag{1}$$

where $\underline{v}$ denotes the velocity, ρ the density, p the pressure, $\underline{B}$ the magnetic field, V a given potential for the external forces acting on the plasma, and γ is a constant. The initial conditions associated with (1) always include the equation

$$\nabla\cdot\underline{B} = 0, \tag{2}$$

but otherwise it is not necessary to specify any initial or boundary conditions at this stage.

We shall consider an arbitrarily given basic flow for the plasma,

$$\begin{aligned}
&\underline{v} = \underline{v}_0(\underline{x},t), \quad \rho = \rho_0(\underline{x},t), \quad p = p_0(\underline{x},t), \\
&\underline{B} = \underline{B}_0(\underline{x},t),
\end{aligned} \tag{3}$$

satisfying the fundamental equations (1) and (2). We want to study the linear waves which can be superimposed on this basic flow (3). We therefore perturb it by introducing into (1) the following expressions:

$$\underline{v} = \underline{v}_0 + \varrho_0^{-\frac{1}{2}}\underline{u}, \quad \varrho = \varrho_0 + c_0^{-1}\,\varrho_0^{\frac{1}{2}}\,(\eta + \zeta),$$
$$p = p_0 + c_0\,\varrho_0^{\frac{1}{2}}\,\zeta, \quad \underline{B} = \underline{B}_0 + \underline{b}. \tag{4}$$

Here $c_0 = (\gamma p_0/\varrho_0)^{\frac{1}{2}}$ denotes the local sound speed and the eightdimensional vector

$$\underline{w} = \{\underline{u},\underline{b},\eta,\zeta\} \tag{5}$$

represents the perturbations superimposed on the basic flow (3). The transformation (4) is analogous to transformations considered earlier in fluid mechanics[3].

By substituting (4) into (1), the linearized equations for the perturbations are found to be

$$\begin{aligned}\underline{u}_t &+ \underline{v}_0\cdot\nabla\underline{u} + \varrho_0^{-\frac{1}{2}}\underline{B}_0\times(\nabla\times\underline{b}) + c_0\nabla\zeta + \underline{u}\cdot\nabla\underline{v}_0\\ &+ \frac{1}{2}\,\underline{u}\nabla\cdot\underline{v}_0 - \varrho_0^{-\frac{1}{2}}(\nabla\times\underline{B}_0)\times\underline{b} + \frac{\gamma}{2}\,(c_0\varrho_0)^{-1}\,\nabla p_0\zeta\\ &+ (c_0\varrho_0)^{-1}[(\nabla\times\underline{B}_0)\times\underline{B}_0 - \nabla p_0](\eta+\zeta) = \underline{0},\end{aligned} \tag{6}$$

$$\begin{aligned}\underline{b}_t &+ \underline{v}_0\cdot\nabla\underline{b} + \varrho_0^{-\frac{1}{2}}(\underline{B}_0\nabla\cdot\underline{u} - \underline{B}_0\cdot\nabla\underline{u})\\ &+ \varrho_0^{-\frac{1}{2}}\underline{u}\cdot\nabla\underline{B}_0 - \frac{1}{2}\,\varrho_0^{-\frac{3}{2}}(\underline{u}\cdot\nabla\varrho_0)\underline{B}_0\\ &+ \frac{1}{2}\,\varrho_0^{-\frac{3}{2}}(\underline{B}_0\cdot\nabla\varrho_0)\underline{u} + \underline{b}\nabla\cdot\underline{v}_0 - \underline{b}\cdot\nabla\underline{v}_0 = \underline{0},\end{aligned} \tag{7}$$

$$\eta_t + \underline{v}_0\cdot\nabla\eta + \varrho_0^{-1}(c_0\nabla\varrho_0 - c_0^{-1}\nabla p_0)\cdot\underline{u} + \frac{\gamma}{2}\,\nabla\cdot\underline{v}_0\eta = 0, \tag{8}$$

$$\zeta_t + \underline{v}_0\cdot\nabla\zeta + c_0\nabla\cdot\underline{u} + \varrho_0^{-1}(c_0^{-1}\nabla p_0 - \frac{1}{2}\,c_0\nabla\varrho_0)\cdot\underline{u} + \frac{\gamma}{2}\nabla\cdot\underline{v}_0\zeta = 0. \tag{9}$$

If we use Cartesian space-coordinates (x,y,z), say, we may write the system of equations (6)-(9) on matricial form

$$L\underline{w} = \frac{\partial \underline{w}}{\partial t} + A^1 \frac{\partial \underline{w}}{\partial x} + A^2 \frac{\partial \underline{w}}{\partial y} + A^3 \frac{\partial \underline{w}}{\partial z} + D\underline{w} = \underline{0}. \tag{10}$$

The 8×8 matrices A^1, A^2, A^3 in (10) obtained from (6)-(9) are

$$A^1 = \begin{bmatrix} v_1 & 0 & 0 & 0 & \rho_0^{-\frac{1}{2}}B_2 & \rho_0^{-\frac{1}{2}}B_3 & 0 & c_0 \\ 0 & v_1 & 0 & 0 & -\rho_0^{-\frac{1}{2}}B_1 & 0 & 0 & 0 \\ 0 & 0 & v_1 & 0 & 0 & -\rho_0^{-\frac{1}{2}}B_1 & 0 & 0 \\ 0 & 0 & 0 & v_1 & 0 & 0 & 0 & 0 \\ \rho_0^{-\frac{1}{2}}B_2 & -\rho_0^{-\frac{1}{2}}B_1 & 0 & 0 & v_1 & 0 & 0 & 0 \\ \rho_0^{-\frac{1}{2}}B_3 & 0 & -\rho_0^{-\frac{1}{2}}B_1 & 0 & 0 & v_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v_1 & 0 \\ c_0 & 0 & 0 & 0 & 0 & 0 & 0 & v_1 \end{bmatrix}, \tag{11}$$

$$A^2 = \begin{bmatrix} v_2 & 0 & 0 & -\rho_0^{-\frac{1}{2}}B_2 & 0 & 0 & 0 & 0 \\ 0 & v_2 & 0 & \rho_0^{-\frac{1}{2}}B_1 & 0 & \rho_0^{-\frac{1}{2}}B_3 & 0 & c_0 \\ 0 & 0 & v_2 & 0 & 0 & -\rho_0^{-\frac{1}{2}}B_2 & 0 & 0 \\ -\rho_0^{-\frac{1}{2}}B_2 & \rho_0^{-\frac{1}{2}}B_1 & 0 & v_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_2 & 0 & 0 & 0 \\ 0 & \rho_0^{-\frac{1}{2}}B_3 & -\rho_0^{-\frac{1}{2}}B_2 & 0 & 0 & v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v_2 & 0 \\ 0 & c_0 & 0 & 0 & 0 & 0 & 0 & v_2 \end{bmatrix}, \tag{12}$$

$$A^3 = \begin{bmatrix} v_3 & 0 & 0 & -\rho_0^{-\frac{1}{2}}B_1 & 0 & 0 & 0 & 0 \\ 0 & v_3 & 0 & 0 & -\rho_0^{-\frac{1}{2}}B_3 & 0 & 0 & 0 \\ 0 & 0 & v_3 & \rho_0^{-\frac{1}{2}}B_1 & \rho_0^{-\frac{1}{2}}B_2 & 0 & 0 & c_0 \\ -\rho_0^{-\frac{1}{2}}B_3 & 0 & \rho_0^{-\frac{1}{2}}B_1 & v_3 & 0 & 0 & 0 & 0 \\ 0 & -\rho_0^{-\frac{1}{2}}B_3 & \rho_0^{-\frac{1}{2}}B_2 & 0 & v_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & v_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v_3 & 0 \\ 0 & 0 & c_0 & 0 & 0 & 0 & 0 & v_3 \end{bmatrix}, \qquad (13)$$

From (11)-(13) we see that A^1, A^2, A^3 are symmetric, hence the basic system of equations for our study of linear waves and stability (6)-(9) is a symmetric hyperbolic system.

III. THE METHOD OF STUDY

The conventional approaches to the problems of linear wave propagation and stability are the normal mode method and the energy principle. These methods are only applicable for special basic flows (3), and they are not without difficulties and limitations. Since the basic system of equations (6)-(9) is a symmetric hyperbolic system, however, there are also other methods available. One such method is the generalized progressing wave expansion method where families of solutions of the following type are studied

$$\underline{w}_\omega(\underline{x},t) = \underline{a}_0(\underline{x},t)\exp\{i\omega\varphi(\underline{x},t)\} + \underline{R}(\underline{x},t;\omega). \tag{14}$$

In (14) the phase function φ and the amplitude $\underline{a}_0$ are determined such that the remainder

$$\underline{R}(\underline{x},t;\omega) = O(\frac{1}{\omega}) \quad \text{when} \quad \omega \to \infty. \tag{15}$$

If we compare (14) with a conventional plane normal mode, we see that $i\omega\varphi$ is analogous to $i(\underline{k}\cdot\underline{x} - qt)$. Thus we see that $\omega\nabla\varphi$ is analogous to the wavenumber vector $\underline{k}$ and $-\omega\varphi_t$ is analogous to the angular frequency q. With this background ω is called a frequency parameter in (14), and $\underline{a}_0\exp(i\omega\varphi)$ is in view of (15) seen to be an asymptotic expression for the family of solutions $\underline{w}_\omega$ valid for short wavelengths and/or high frequencies.

Rewriting (10) in the following way:

$$L\underline{w} = \underline{w}_t + \sum_{\nu=1}^{3} A^\nu \underline{w}_{x_\nu} + D\underline{w} = \underline{0}, \tag{16}$$

we obtain by substituting (14)

$$L\underline{W}_{\omega} = \{i\omega(\varphi_t I + \sum_{\nu=1}^{3} \varphi_{x_\nu} A^\nu)\underline{a}_0 + L\underline{a}_0\}\exp(i\omega\varphi) + L\underline{R} = \underline{0}. \tag{17}$$

Letting $\omega \to \infty$, (17) implies in view of (15)

$$(\varphi_t I + \sum_{\nu=1}^{3} \varphi_{x_\nu} A^\nu)\underline{a}_0 = \underline{0}. \tag{18}$$

Since we assume that $\underline{a}_0 \neq \underline{0}$, (18) can only be satisfied when the phase function φ satisfies the characteristic equation

$$\det (\varphi_t I + \sum_{\nu=1}^{3} \varphi_{x_\nu} A^\nu) = 0. \tag{19}$$

Introducing the notations

$$\lambda = -\varphi_t, \quad \underline{k} = (\xi^1, \xi^2, \xi^3) = \nabla\varphi, \quad E = \sum_{\nu=1}^{3} \xi^\nu A^\nu, \tag{20}$$

(19) shows that λ must be an eigenvalue of the symmetric matrix E. If

$$\lambda = \Omega(\underline{x}, t, \underline{k}) \tag{21}$$

is an eigenvalue of E, we see that (19) is satisfied when

$$\varphi_t + \Omega(\underline{x}, t, \nabla\varphi) = 0. \tag{22}$$

The eigenvalues of the matrix E are called <u>the characteristic roots</u> associated with the symmetric hyperbolic system (16). To the different characteristic roots there correspond different families of phase functions, which again correspond to different families of solutions of the form (14).

In our case where the matrices A^1, A^2, A^3 are given by (11) - (13), the characteristic roots are found to be given by $\Omega_1, \ldots, \Omega_7$ where

$$\Omega_1 = \underline{k}\cdot\underline{v}_0, \tag{23}$$

$$\Omega_2 = \underline{k}\cdot\underline{v}_0 + \rho_0^{-\frac{1}{2}}\,\underline{k}\cdot\underline{B}_0, \qquad (24)$$

$$\Omega_3 = \underline{k}\cdot\underline{v}_0 - \rho_0^{-\frac{1}{2}}\,\underline{k}\cdot\underline{B}_0, \qquad (25)$$

$$\Omega_4 = \underline{k}\cdot\underline{v}_0 + k(P+Q)^{\frac{1}{2}}, \qquad (26)$$

$$\Omega_5 = \underline{k}\cdot\underline{v}_0 - k(P+Q)^{\frac{1}{2}}, \qquad (27)$$

$$\Omega_6 = \underline{k}\cdot\underline{v}_0 + k(P-Q)^{\frac{1}{2}}, \qquad (28)$$

$$\Omega_7 = \underline{k}\cdot\underline{v}_0 - k(P-Q)^{\frac{1}{2}}, \qquad (29)$$

$$P = \frac{1}{2}\,(\rho_0^{-1}\,B_0^{\,2} + c_0^{\,2}), \qquad (30)$$

$$Q = \{P^2 - \rho_0^{-1}(\underline{k}\cdot\underline{B}_0)^2\,c_0^{\,2}\,k^{-2}\}^{\frac{1}{2}}. \qquad (31)$$

From (23)-(31) we see that (10) does not have characteristics with constant multiplicity. In fact, in order that the multiplicity assumptions in Eckhoff (Ref.1, Sec.5) will be satisfied for all the characteristic roots (23) - (29), it is necessary to assume that

$$\underline{k}\times\underline{B}_0 \neq \underline{0} \ \ \& \ \ \underline{k}\cdot\underline{B}_0 \neq 0. \qquad (32)$$

When (32) is satisfied, we see that Ω_1 is a double root while all the other roots $\Omega_2,\ldots,\Omega_7$ are simple roots in the characteristic equation (19). The characteristic root Ω_1 corresponds to the mass waves (internal gravity waves), Ω_2 and Ω_3 correspond to the Alfvén waves, Ω_4 and Ω_5 correspond to the fast magnetoacoustic waves, and Ω_6 and Ω_7 correspond to the slow magnetoacoustic waves.

Now let Ω be one of the seven characteristic roots (23)-(29) and let $\varphi(\underline{x},t)$ be a real-valued solution of (22) which is such that $\nabla\varphi \neq \underline{0}$. Suppose that $\Omega(\underline{x},t,\nabla\varphi)$ for this solution φ is an eigenvalue of fixed multiplicity μ, say, in the considered domain. Equation (18) then shows that

$$\underline{a}_0 = \sum_{\iota=1}^{\mu} \sigma_\iota \underline{r}_\iota, \tag{33}$$

where $\underline{r}_1,\ldots,\underline{r}_\mu$ are orthonormal eigenvectors associated with the eigenvalue Ω and $\sigma_1,\ldots,\sigma_\mu$ are scalar functions to be determined. In order to do that, we write the remainder in the following way

$$\underline{R} = \frac{1}{i\omega}\,[\ \underline{a}_1(\underline{x},t)\exp(i\omega\varphi(\underline{x},t)) + \underline{u}(\underline{x},t;\omega)]. \tag{34}$$

In view of (18), (17) then becomes

$$(L\underline{a}_0 + (\varphi_t I + \sum_{\nu=1}^{3} \varphi_{x_\nu} A^\nu)\underline{a}_1 + \frac{1}{i\omega} L\underline{a}_1)\exp(i\omega\varphi) + \frac{1}{i\omega} L\underline{u} = \underline{0}, \tag{35}$$

which is satisfied if

$$L\underline{a}_0 + (\varphi_t I + \sum_{\nu=1}^{3} \varphi_{x_\nu} A^\nu)\underline{a}_1 = \underline{0}, \tag{36}$$

$$L\underline{u} + (L\underline{a}_1)\exp(i\omega\varphi) = \underline{0}. \tag{37}$$

Equation (36) may be considered as a system of algebraic equations for $\underline{a}_1$. When φ has the assumed properties, (36) therefore has a solution if and only if

$$\underline{r}_\iota \cdot L\underline{a}_0 = 0 \quad (\iota=1,\ldots,\mu). \tag{38}$$

Substituting (33) into (38) yields

$$(\sigma_\iota)_t + \sum_{\nu=1}^{3} \sum_{m=1}^{\mu} (\underline{r}_\iota \cdot A^\nu \underline{r}_m)(\sigma_m)_{x_\nu} + \sum_{m=1}^{\mu} (\underline{r}_\iota \cdot L\underline{r}_m)\ \sigma_m = 0, \tag{39}$$

$$(\iota=1,\ldots,\mu),$$

which is a symmetric hyperbolic system for $\underline{\sigma} = (\sigma_1,\ldots,\sigma_\mu)$. Properties of this system (39) are studied in Eckhoff[1]. In particular it is shown that when $\underline{\sigma}$ is determined by (39), such that $\underline{a}_0\exp(i\omega\varphi)$ is smooth, then (15) holds. As a consequence of this Eckhoff[1] also shows that in order for the trivial solution $\underline{w} = \underline{0}$ of (10) to be stable it is necessary for the trivial solution $\underline{a} = \underline{0}$ of (39) to be stable.

By studying the system of transport equations (39) corresponding to the possible phase functions φ we may therefore be able to obtain asymptotic wave solutions and get information about the stability properties of the basic flow (3). This is the purpose of the present paper, and we start in the next section by calculating the eigenvectors corresponding to the different characteristic roots.

IV. THE EIGENVECTORS

In order to be able to calculate the amplitude (33) in the asymptotic solution and the associated transport equations (39), we have to calculate the eigenvectors corresponding to the different characteristic roots (23)-(29). In this calculation we have to distinguish between the nonsingular case where (32) is satisfied and the singular cases where either $\underline{k} \times \underline{B}_0 = \underline{0}$ or $\underline{k}\cdot\underline{B}_0 = 0$.

When (32) is satisfied, we may choose the eigenvectors associated with the double root Ω_1 to be

$$\underline{r}_{11} = (\underline{0},\underline{0},1,0), \quad \underline{r}_{12} = k^{-1}(\underline{0},\underline{k},0,0). \tag{40}$$

The eigenvectors associated with the roots Ω_2 and Ω_3 may be chosen to be, respectively,

$$\underline{r}_2 = 2^{-\frac{1}{2}}|\underline{k}\times\underline{B}_0|^{-1}(\underline{k}\times\underline{B}_0,\ -\underline{k}\times\underline{B}_0,\ 0,\ 0), \tag{41}$$

$$\underline{r}_3 = 2^{-\frac{1}{2}}|\underline{k}\times\underline{B}_0|^{-1}(\underline{k}\times\underline{B}_0,\ \underline{k}\times\underline{B}_0,\ 0,\ 0). \tag{42}$$

The eigenvectors associated with the roots Ω_κ, $\kappa = 4,5,6,7$ may be chosen to be, respectively,

$$\underline{r}_\kappa = \varepsilon_\kappa(\underline{a}_\kappa,\ \underline{d}_\kappa,\ 0,\ e_\kappa), \tag{43}$$

where

$$\varepsilon_\kappa = \frac{1}{2}\, k^{-1} V_\kappa\, c_0\{(V_\kappa{}^2 - P)(V_\kappa{}^2 - \varrho_0{}^{-1}B_0{}^2)\}^{-\frac{1}{2}}, \tag{44}$$

$$\underline{a}_\kappa = \underline{k} - V_\kappa{}^{-2}\,\varrho_0{}^{-1}(\underline{k}\cdot\underline{B}_0)\underline{B}_0, \tag{45}$$

$$\underline{d}_\kappa = V_\kappa^{-1} \varrho_0^{-\frac{1}{2}}(k\underline{B}_0 - k^{-1}(\underline{k}\cdot\underline{B}_0)\underline{k}), \tag{46}$$

$$e_\kappa = c_0 V_\kappa^{-1}(k - V_\kappa^{-2}\varrho_0^{-1}k^{-1}(\underline{k}\cdot\underline{B}_0)^2). \tag{47}$$

Here we have introduced the notation

$$V_\kappa = (-1)^\kappa\{P + (-1)^\mu Q\}^{\frac{1}{2}} \; ; \; \kappa = 4,5,6,7, \tag{48}$$

where $\mu = 0$ when $\kappa = 4,5$ and $\mu = 1$ when $\kappa = 6,7$.

In the singular case where $\underline{k}\cdot\underline{B}_0=0$ we see that

$$Q = P, \tag{49}$$

hence all the above eigenvectors except $\underline{r}_6$, $\underline{r}_7$ are well defined when $\underline{B}_0 \neq \underline{0}$. Furthermore we see that Ω_4 and Ω_5 are simple roots, while

$$\Omega_1 = \Omega_2 = \Omega_3 = \Omega_6 = \Omega_7 = \underline{k}\cdot\underline{v}_0 \tag{50}$$

in this case. Thus the fast magnetoacoustic waves are the only waves that can be studied by the theory in Eckhoff (Ref.1, Secs.5-7) in the singular case $\underline{k}\cdot\underline{B}_0 = 0$. A modification of the theory is needed in order to study the other waves which in view of (50) may all be coupled in this case. In this modified study we shall replace the eigenvectors $\underline{r}_6$, $\underline{r}_7$ by the following

$$\underline{r}_{6\perp} = 2^{-\frac{1}{2}}B_0^{-1}(\underline{B}_0, -c_0(2P)^{-\frac{1}{2}}\underline{B}_0, 0, (2P\varrho_0)^{-\frac{1}{2}}B_0^2), \tag{51}$$

$$\underline{r}_{7\perp} = 2^{-\frac{1}{2}}B_0^{-1}(\underline{B}_0, c_0(2P)^{-\frac{1}{2}}\underline{B}_0, 0, -(2P\varrho_0)^{-\frac{1}{2}}B_0^2). \tag{52}$$

In the singular case where $\underline{k} \times \underline{B}_0 = \underline{0}$ we obtain

$$Q = \frac{1}{2} |\varrho_0^{-1} B_0^2 - c_0^2|. \qquad (53)$$

From (23)-(30) we therefore see that in this case we have to distinguish between the following three cases:

$$\begin{aligned} &a) \quad \varrho_0^{-1} B_0^2 > c_0^2, \\ &b) \quad \varrho_0^{-1} B_0^2 < c_0^2, \\ &c) \quad \varrho_0^{-1} B_0^2 = c_0^2. \end{aligned} \qquad (54)$$

In case a) we have

$$\Omega_2 = \Omega_4 \ \& \ \Omega_3 = \Omega_5, \qquad (55)$$

and the corresponding eigenvectors $\underline{r}_2$, $\underline{r}_3$, $\underline{r}_4$, $\underline{r}_5$ are not well defined. In this case the waves corresponding to the roots Ω_1, Ω_6, Ω_7 may be studied by the theory in Eckhoff (Ref.1, Secs.5-7), while a modification is needed for the waves corresponding to the roots (55). In this modified study we may replace the eigenvectors $\underline{r}_2$, $\underline{r}_3$, $\underline{r}_4$, $\underline{r}_5$ by the following:

$$\underline{r}_{2\parallel} = (\underline{a}, -\underline{a}, 0, 0), \qquad (56)$$

$$\underline{r}_{3\parallel} = (\underline{a}, \underline{a}, 0, 0), \qquad (57)$$

$$\underline{r}_{4\parallel} = (\underline{d}, -\underline{d}, 0, 0), \qquad (58)$$

$$\underline{r}_{5\parallel} = (\underline{d}, \underline{d}, 0, 0), \qquad (59)$$

where $\underline{a}, \underline{d}$ *are arbitrary vectors satisfying*

$$\begin{aligned} &\underline{a}\cdot\underline{d} = \underline{a}\cdot\underline{B}_0 = \underline{d}\cdot\underline{B}_0 = 0, \\ &a^2 = d^2 = \frac{1}{2}. \end{aligned} \qquad (60)$$

In case b) we have

$$\Omega_2 = \Omega_6 \quad \& \quad \Omega_3 = \Omega_7, \tag{61}$$

and the corresponding eigenvectors $\underline{r}_2$, $\underline{r}_3$, $\underline{r}_6$, $\underline{r}_7$ are not well defined. In this case the waves corresponding to the roots Ω_1, Ω_4, Ω_5 may be studied by the theory in Eckhoff (Ref.1, Secs.5-7) when $\underline{B}_0 \neq \underline{0}$, while a modification is needed for the waves corresponding to the roots (61). In this modified study we may replace the eigenvectors $\underline{r}_2$, $\underline{r}_3$ by (56), (57) and the eigenvectors $\underline{r}_6$, $\underline{r}_7$ by

$$\underline{r}^{b}_{6\parallel} = (\underline{d},\ -\underline{d},\ 0,\ 0), \tag{62}$$

$$\underline{r}^{b}_{7\parallel} = (\underline{d},\ \underline{d},\ 0,\ 0), \tag{63}$$

where $\underline{a}, \underline{d}$ are still arbitrary vectors satisfying (60).

In case c) we have

$$\Omega_2 = \Omega_4 = \Omega_6 \quad \& \quad \Omega_3 = \Omega_5 = \Omega_7 \tag{64}$$

and the corresponding eigenvectors $\underline{r}_2$, $\underline{r}_3$, $\underline{r}_4$, $\underline{r}_5$, $\underline{r}_6$, $\underline{r}_7$ are not well defined. In this case only the waves corresponding to the characteristic root Ω_1 can be studied by the theory in Eckhoff (Ref.1, Secs.5-7), while a modification is needed for all the other waves. In this modified study we may replace the eigenvectors $\underline{r}_2$, $\underline{r}_3$, $\underline{r}_4$, $\underline{r}_5$ by (56), (57), (58), (59) and the eigenvectors $\underline{r}_6$, $\underline{r}_7$ by

$$r^{c}_{6\parallel} = 2^{-\frac{1}{2}}\, B_0^{-1} (\underline{B}_0,\ \underline{0},\ 0,\ B_0), \tag{65}$$

$$r^{c}_{7\parallel} = 2^{-\frac{1}{2}}\, B_0^{-1} (\underline{B}_0,\ \underline{0},\ 0, -B_0). \tag{66}$$

V. THE MASS WAVES

In this section we shall use the procedure developed by Eckhoff[1] to study the linear waves superimposed on the basic flow (3) that are associated with the characteristic root Ω_1. The multiplicity assumptions in Eckhoff (Ref.1, Sec.5) are satisfied if and only if $\underline{k}\cdot\underline{B}_0 \neq 0$, we shall therefore limit our discussion to that case here. The singular case $\underline{k}\cdot\underline{B}_0 = 0$ will be studied in Sec.IX.

The bicharacteristic equations associated with the characteristic root Ω_1 are

$$\frac{d\underline{x}}{dt} = \underline{v}_0 \quad , \qquad \frac{d\underline{k}}{dt} = -(\nabla\underline{v}_0)\cdot\underline{k}. \tag{67}$$

The transport equations are found to be along the rays

$$\frac{d\sigma_1}{dt} = -\frac{\gamma}{2}(\nabla\cdot\underline{v}_0)\sigma_1, \tag{68}$$

$$\frac{d\sigma_2}{dt} = (k^{-2}\underline{k}\cdot(\nabla\underline{v}_0)\cdot\underline{k} - \nabla\cdot\underline{v}_0)\sigma_2. \tag{69}$$

Along the rays governed by (67) we have in view of (1) the relation

$$\nabla\cdot\underline{v}_0 = -\rho_0^{-1}\frac{d\rho_0}{dt}. \tag{70}$$

The equations (68) and (69) may therefore be written in the following way:

$$\sigma_1^{-1}\frac{d\sigma_1}{dt} = \frac{\gamma}{2}\rho_0^{-1}\frac{d\rho_0}{dt}, \tag{71}$$

$$\sigma_2^{-1}\frac{d\sigma_2}{dt} = \rho_0^{-1}\frac{d\rho_0}{dt} - \frac{1}{2}k^{-2}\frac{d}{dt}(k^2). \tag{72}$$

These equations are directly integrable giving

$$\sigma_1 = c_1\,\rho_0^{\frac{\gamma}{2}} \quad , \quad \sigma_2 = c_2 k^{-1}\,\rho_0, \tag{73}$$

where c_1, c_2 are arbitrary constants along the rays.

The stability equations are found to be

$$\frac{d}{dt} S_1 = \frac{1-\gamma}{2} (\nabla \cdot \underline{v}_0) S_1, \tag{74}$$

$$\frac{d}{dt} S_2 = (k^{-2} \underline{k} \cdot (\nabla \underline{v}_0) \cdot \underline{k} - \frac{1}{2} \nabla \cdot \underline{v}_0) S_2. \tag{75}$$

These equations are directly integrable in a completely analogous way to the transport equations above, giving

$$S_1 = c_1 \rho_0^{\frac{\gamma-1}{2}} , \quad S_2 = c_2 k^{-1} \rho_0^{\frac{1}{2}}, \tag{76}$$

where c_1, c_2 are again arbitrary constants along the rays.

From the expressions obtained above we first notice that to leading order we do not get any local freqency for the MHD-mass waves resembling the Brunt-Väisälä frequency in fluid mechanics[3,4]. Second, we see that the amplitude only depends on ρ_0 and k along the rays which from (67) are seen to coincide with the path lines for the plasma particles in the basic flow (3). On physical grounds ρ_0 must at least for a steady basic flow be a bounded function. Hence the only necessary condition for stability that we obtain from the theory in Eckhoff[1] for the mass waves is that k must not tend to zero along the rays. This condition is trivially seen to be satisfied in the static case $\underline{v}_0 \equiv \underline{0}$ and also in the more general case, where $\underline{v}_0$ is independent of $\underline{x}$, since $\underline{k}$ then is seen from (67) to be conserved along the rays. In the case of a sheared basic flow of the plasma, $\underline{k}$ is no longer conserved and we have to look more carefully at the bi-characteristic equations (67) in order to settle the stability problem for the mass waves. We shall confine our study to two special cases, namely slabs and screw pinches.

Slabs are possible basic flows for a plasma if the potential for the external forces takes the form

$$V = V(z) \tag{77}$$

in a Cartesian space-coordinate system (x,y,z). With (77) a slab is a basic flow for a plasma of the following type:

$$\begin{aligned} \underline{v}_0 &= (v_1(z),\ v_2(z),\ 0),\ \varrho_0 = \varrho_0(z), \\ p_0 &= p_0(z),\ \underline{B}_0 = (B_1(z),\ B_2(z),\ 0). \end{aligned} \tag{78}$$

Here v_1, v_2, ϱ_0, B_1, B_2 may be arbitrarily given and p_0 is then determined to within an arbitrary additive constant by

$$(p_0 + \frac{1}{2} B_0^2\ )' = \varrho_0\ V', \tag{79}$$

where a prime denotes differentiation with respect to z. With (78) the solutions of the bicharacteristic equations (67) are readily found to be

$$\begin{aligned} x &= x_0 + v_1(z_0)t,\ y = y_0 + v_2(z_0)t,\ z = z_0, \\ \xi^1 &= \xi_0^1,\ \xi^2 = \xi_0^2,\ \xi^3 = \xi_0^3 - (\xi_0^1 v_1'(z_0) + \xi_0^2 v_2'(z_0))t, \end{aligned} \tag{80}$$

where the subscript 0 refers to the initial values of the bicharacteristics at t = 0. From (80) we see that $\underline{k}$ is conserved along the bicharacteristics satisfying

$$\xi_0^1 v_1'(z_0) + \xi_0^2 v_2'(z_0) = 0, \tag{81}$$

while on all other bicharacteristics

$$k \to \infty \quad \& \quad k \geq ((\xi_0^1)^2 + (\xi_0^2)^2)^{\frac{1}{2}}. \tag{82}$$

Thus for a slab S_1, S_2 are clearly seen from (76) to be bounded along each ray.

Screw pinches are possible basic flows for a plasma if the potential for the external forces takes the form

$$V = V(r) \tag{83}$$

in a cylindricial space-coordinate system (r,θ,z). With (83) a screw pinch is a basic flow for a plasma of the following type

$$\begin{aligned} \underline{v}_0 &= v_\theta(r)\hat{\underline{\theta}} + v_z(r)\hat{\underline{z}}, \quad \rho_0 = \rho_0(r), \\ p_0 &= p_0(r), \quad \underline{B}_0 = B_\theta(r)\hat{\underline{\theta}} + B_z(r)\hat{\underline{z}}. \end{aligned} \tag{84}$$

Here $v_\theta, v_z, \rho_0, B_\theta, B_z$ may be arbitrarily given and p_0 is then determined to within an arbitrary additive constant by

$$(p_0 + \frac{1}{2} B_0^2)' + \frac{1}{r} B_\theta^2 = \frac{\rho_0}{r} v_\theta^2 + \rho_0 V' , \tag{85}$$

where a prime denotes differentiation with respect to r. With (84) the solutions of the bicharacteristic equations (67) are readily found to be[3]

$$\begin{aligned} & r = r_0, \quad \theta = \theta_0 + r_0^{-1} v_\theta(r_0)t, \quad z = z_0 + v_z(r_0)t, \\ & \xi^1 = \xi_0^1 + (r_0^{-1} \xi_0^2[r_0^{-1} v_\theta(r_0) - v_\theta'(r_0)] - \xi_0^3 v_z'(r_0))t, \\ & \xi^2 = \xi_0^2, \quad \xi^3 = \xi_0^3, \quad \underline{k} = \xi^1\hat{\underline{r}} + r^{-1}\xi^2\hat{\underline{\theta}} + \xi^3\hat{\underline{z}}, \end{aligned} \tag{86}$$

where again the subscript 0 refers to the initial values of the bicharacteristics at $t = 0$. From (86) we see that $\underline{k}$ is conserved along the bicharacteristics satisfying

$$r_0^{-1}\,\xi_0^2[r_0^{-1}v_\theta(r_0) - v_\theta'(r_0)] - \xi_0^3 v_z'(r_0) = 0, \tag{87}$$

while on all other bicharacteristics

$$k \to \infty \quad \& \quad k > ((r_0^{-1}\,\xi_0^2)^2 + (\xi_0^3)^2)^{\frac{1}{2}}. \tag{88}$$

Thus S_1, S_2 are also seen to be bounded along each ray for a screw pinch.

In view of the above discussion it seems natural to conjecture that to leading order the mass waves never give rise to any instabilities in MHD, at least not in the case of stationary basic flows (3). For the slab and the screw pinch we see that the density-perturbation part of the mass wave, i.e., $\sigma_1\underline{r}_{11}$, is carried unaltered along each streamline, while the perturbation of the magnetic field, $\sigma_2\underline{r}_{12}$, dies out along most bicharacteristics. Only along the special bicharacteristics satisfying (81) and (87), respectively, i.e., for special choices of the phase function, is the perturbation of the magnetic field conserved along each streamline for the mass waves to leading order in the nonsingular case where $\underline{k}\cdot\underline{B}_0 \neq 0$.

VI. THE ALFVEN WAVES

In this section we shall use the same procedure as in the preceding section to study the linear waves superimposed on the basic flow (3) which are associated with the two characteristic roots Ω_2 and Ω_3, respectively. The multiplicity assumptions in Eckhoff (Ref.1, Sec.5) are satisfied if and only if (32) is satisfied. We shall therefore limit our discussion to that case here. The singular cases where (32) is not satisfied will be studied in Secs.VIII and IX.

The bicharacteristic equations associated with the characteristic root Ω_2 are

$$\frac{d\underline{x}}{dt} = \underline{v}_0 + \rho_0^{-\frac{1}{2}} \underline{B}_0, \tag{89}$$

$$\frac{d\underline{k}}{dt} = -(\nabla \underline{v}_0)\cdot \underline{k} - \rho_0^{-\frac{1}{2}}(\nabla \underline{B}_0)\cdot \underline{k} + \frac{1}{2} \rho_0^{-\frac{3}{2}}(\underline{k}\cdot\underline{B}_0)\nabla\rho_0 . \tag{90}$$

By a direct calculation of $\underline{r}_2 \cdot L\underline{r}_2$, the transport equation is found to be along the rays

$$\frac{d\sigma}{dt} = (\frac{1}{2} \rho_0^{-\frac{1}{2}} |\underline{k}\times\underline{B}_0|^{-1} \; [(\underline{k}\times\underline{B}_0)\times\underline{B}_0]\cdot\nabla\times(|\underline{k}\times\underline{B}_0|^{-1}\underline{k}\times\underline{B}_0)$$

$$+ \frac{1}{2} \rho_0^{-\frac{1}{2}} |\underline{k}\times\underline{B}_0|^{-2}(\underline{k}\times\underline{B}_0)\cdot(\nabla\underline{B}_0)\cdot(\underline{k}\times\underline{B}_0)$$

$$+ \frac{1}{4} \rho_0^{-\frac{3}{2}} \underline{B}_0\cdot\nabla\rho_0 - \frac{3}{4}\nabla\cdot\underline{v}_0)\sigma. \tag{91}$$

Using (2) and vector identities including the following one, which is valid whenever $\underline{k}\times\underline{B}_0 \neq \underline{0}$,

$$\nabla\cdot\underline{B}_0 \equiv B_0^{-2}\underline{B}_0\cdot(\nabla\underline{B}_0)\cdot\underline{B}_0 + |\underline{k}\times\underline{B}_0|^{-2}(\underline{k}\times\underline{B}_0)\cdot(\nabla\underline{B}_0)\cdot(\underline{k}\times\underline{B}_0)$$

$$+ B_0^{-2}|\underline{k}\times\underline{B}_0|^{-2}((\underline{k}\times\underline{B}_0)\times\underline{B}_0)\cdot(\nabla\underline{B}_0)\cdot((\underline{k}\times\underline{B}_0)\times\underline{B}_0), \tag{92}$$

it is possible to show that (91) reduces to

$$\frac{d\sigma}{dt} = (\frac{1}{4} \rho_0^{-\frac{3}{2}} \underline{B}_0 \cdot \nabla \rho_0 - \frac{3}{4} \nabla \cdot \underline{v}_0)\sigma. \tag{93}$$

The stability equation takes the form

$$\frac{d}{dt} S = - \frac{1}{4} \nabla \cdot \underline{v}_0 \; S. \tag{94}$$

In a completely analogous way we find that the bicharacteristic equations associated with the characteristic root Ω_3 are

$$\frac{d\underline{x}}{dt} = \underline{v}_0 - \rho_0^{-\frac{1}{2}} \underline{B}_0, \tag{95}$$

$$\frac{d\underline{k}}{dt} = -(\nabla \underline{v}_0) \cdot \underline{k} + \rho_0^{-\frac{1}{2}} (\nabla \underline{B}_0) \cdot \underline{k} - \frac{1}{2} \rho_0^{-\frac{3}{2}} (\underline{k} \cdot \underline{B}_0) \nabla \rho_0, \tag{96}$$

and that the transport equation and the stability equation take the following forms, respectively, along the rays:

$$\frac{d\sigma}{dt} = -(\frac{1}{4} \rho_0^{-\frac{3}{2}} \underline{B}_0 \cdot \nabla \rho_0 + \frac{3}{4} \nabla \cdot \underline{v}_0)\sigma, \tag{97}$$

$$\frac{d}{dt} S = - \frac{1}{4} \nabla \cdot \underline{v}_0 \; S. \tag{98}$$

From the above expressions we note that it is not necessary to take into account the equations for $\underline{k}$, i.e., (90) and (96), respectively, in order to solve the transport equations and the stability equations. For the slab and the screw pinch we immediately see that both σ and S are conserved along the rays, and S is in fact conserved whenever the basic flow is incompressible. Hence there seems to be no reason to believe that the Alfvén waves ever can give rise to any instabilities in MHD to leading order in the nonsingular case where (32) is satisfied, at least not in the case of stationary basic flows (3).

VII. THE MAGNETOACOUSTIC WAVES

In this section we shall use the same procedure as in the preceding sections to indicate how the linear waves superimposed on the basic flow (3) and associated with the characteristic roots Ω_κ, $\kappa = 4,5,6,7$, may be studied. The multiplicity assumptions in Eckhoff (Ref.1, Sec.5) are satisfied for all these waves if (32) holds. If $\underline{k}\cdot\underline{B}_0 = 0$, the multiplicity assumptions will only be satisfied for the fast waves, i.e., for the waves associated with Ω_4 and Ω_5. If $\underline{k}\times\underline{B}_0 = \underline{0}$, the multiplicity assumptions will be violated for all the magnetoacoustic waves in the case (54c), while they will hold for the fast waves only in case (54b) and for the slow waves only in case (54a). Here we shall limit our discussion to the cases where the multiplicity assumptions hold; the other cases will be studied in Secs.VIII and IX.

Consistent with the notations introduced in Secs.III and IV, we may write (for $\kappa = 4,5,6,7$)

$$\Omega_\kappa = \underline{k}\cdot\underline{v}_0 + kV_\kappa, \qquad V_\kappa = (-1)^\kappa (P + (-1)^\mu Q)^{\frac{1}{2}}, \tag{99}$$

where $\mu = 0$ when $\kappa = 4,5$ and $\mu = 1$ when $\kappa = 6,7$. The bicharacteristic equations associated with the characteristic roots Ω_κ, $\kappa = 4,5,6,7$ are then easily found to be

$$\frac{d\underline{x}}{dt} = \underline{v}_0 + k^{-1}V_\kappa\,\underline{k} - (-1)^\mu \frac{1}{2} k^{-2} Q^{-1} \rho_0^{-\frac{1}{2}} c_0^2 (\underline{k}\cdot\underline{B}_0)\underline{d}_\kappa, \tag{100}$$

$$\begin{aligned}\frac{d\underline{k}}{dt} = &- (\nabla\underline{v}_0)\cdot\underline{k} - (-1)^\mu \frac{1}{2} kQ^{-1}V_\kappa \nabla P \\ &- (-1)^\mu \frac{1}{2} k^{-1}Q^{-1}V_\kappa^{-1}\rho_0^{-2}(\underline{k}\cdot\underline{B}_0)^2 (c_0^2\,\nabla\rho_0 - \frac{\gamma}{2}\nabla p_0) \\ &+ (-1)^\mu \frac{1}{2} k^{-1}Q^{-1}V_\kappa^{-1}\rho_0^{-1}c_0^{\,2}(\underline{k}\cdot\underline{B}_0)(\nabla\underline{B}_0)\cdot\underline{k}.\end{aligned} \tag{101}$$

After some manipulations the transport equation along the bicharacteristics determined by (100) and (101) is found to be

$$\frac{d\sigma}{dt} = (D_\kappa + F_\kappa)\sigma, \tag{102}$$

where

$$\begin{aligned} D_\kappa = {} & \varepsilon_\kappa \rho_0^{-\frac{1}{2}} \underline{B}_0 \cdot (\nabla(\varepsilon_\kappa \underline{a}_\kappa)) \cdot \underline{d}_\kappa - \varepsilon_\kappa e_\kappa c_0 \nabla \cdot (\varepsilon_\kappa \underline{a}_\kappa) \\ & - \varepsilon_\kappa \rho_0^{-\frac{1}{2}} (\underline{d}_\kappa \cdot \underline{B}_0) \nabla \cdot (\varepsilon_\kappa \underline{a}_\kappa) - \varepsilon_\kappa c_0 \underline{a}_\kappa \cdot \nabla(\varepsilon_\kappa e_\kappa) \\ & - \varepsilon_\kappa \rho_0^{-\frac{1}{2}} (\underline{k} \times \underline{B}_0) \cdot \nabla \times (\varepsilon_\kappa \underline{d}_\kappa) - \varepsilon_\kappa^2 \rho_0^{-\frac{1}{2}} \underline{a}_\kappa \cdot (\nabla \underline{B}_0) \cdot \underline{d}_\kappa \\ & + \varepsilon_\kappa^2 \underline{d}_\kappa \cdot (\nabla \underline{v}_0) \cdot \underline{d}_\kappa - \varepsilon_\kappa^2 \underline{a}_\kappa \cdot (\nabla \underline{v}_0) \cdot \underline{a}_\kappa \\ & - \varepsilon_\kappa^2 (\frac{1}{2} a_\kappa^2 + d_\kappa^2 + \frac{\gamma}{2} e_\kappa^2) \nabla \cdot \underline{v}_0 \\ & + \frac{1}{2} \varepsilon_\kappa^2 \rho_0^{-\frac{3}{2}} (\underline{d}_\kappa \cdot \underline{B}_0) \underline{k} \cdot \nabla \rho_0 + \frac{1}{2} \varepsilon_\kappa^2 \rho_0^{-1} c_0 e_\kappa \underline{a}_\kappa \cdot \nabla \rho_0 \\ & - \frac{\gamma}{2} \varepsilon_\kappa^2 \rho_0^{-1} c_0^{-1} e_\kappa \underline{a}_\kappa \cdot \nabla p_0, \end{aligned} \tag{103}$$

and F_κ is the focusing coefficient which is given by

$$F_\kappa = - \frac{1}{2} \sum_{j=1}^{3} \sum_{v=1}^{3} \varphi_{x_j x_v} \frac{\partial^2 \Omega_k}{\partial \xi^j \partial \xi^v} \tag{104}$$

Since the characteristic roots Ω_κ, $\kappa = 4,5,6,7$ are nonlinear with respect to $\underline{k}$, the focusing coefficient (104) will in general be a complicated nonvanishing expression involving the unknown quantities $\varphi_{x_j x_v}$. These quantities may also be determined by transport equations along the rays, but in order to close the system of equations (100), (101), and (102), we then have to introduce six additional complicated equations in general. We shall not pursue that approach here.

As discussed in Eckhoff[1], the difficulty connected with closing the system of transport equations does not exist if we look at the stability equation instead. That equation takes the form

$$\frac{d}{dt} S = (D_\kappa + K_\kappa) S, \qquad (105)$$

where D_κ still is given by (103) and the compression coefficient K_κ is given by

$$K_\kappa = \frac{1}{2} \sum_{p=1}^{3} \frac{\partial^2 \Omega_\kappa}{\partial x_p \partial \xi^p}$$

$$= \frac{1}{2} \nabla \cdot \underline{v}_0 + \frac{1}{2} k^{-1} \underline{k} \cdot \nabla V_\kappa - (-1)^\mu \frac{1}{4} k^{-2} \nabla \cdot (Q^{-1} \varrho_0^{-1} c_0^2 (\underline{k} \cdot \underline{B}_0) \underline{d}_\kappa) \quad (106).$$

To avoid confusion we note that with the notations used above, the variable $\underline{k}$ is <u>not</u> affected by the ∇-operator either in (106), or in (103).

Since the stability equation (105) and the bicharacteristic system (100) and (101) do constitute a closed system of equations, they can in principle be solved. In general the rays will hit the boundaries of the plasma; thus an investigation of how the wave is reflected/transmitted there is required in order to carry the study further. Whether such a study can reveal anything of interest for the stability problem, is an open question which we shall not take up here. We should like to remark, however, that it seems possible to carry out such a study for special cases by numerical methods, while analytic results presumably are hard to obtain even in special cases such as slabs or screw pinches where the complicated expressions occurring in (100), (101), and (105) can be shown to reduce substantially.

VIII. THE SINGULAR CASE WHERE $\underline{k}\times\underline{B}_0 = \underline{0}$

As already seen in Sec.IV, the characteristic root Ω_2 coincides with Ω_4 and/or Ω_6 and the characteristic root Ω_3 coincides with Ω_5 and/or Ω_7 if $\underline{k}\times\underline{B}_0 = \underline{0}$. In this case, therefore, the multiplicity assumptions in Eckhoff (Ref.1, Sec.5) are neither satisfied for the Alfvén waves nor for at least one set of the magnetoacoustic waves. Those waves cannot therefore be studied by the approach in Eckhoff[1] in a straightforward fashion in that singular case.

In order to get hold of the nature of the singular case $\underline{k}\times\underline{B}_0 = \underline{0}$, we calculate along the rays associated with the Alfvén waves

$$\begin{aligned}\frac{d}{dt}(\underline{k}\times\underline{B}_0) &= \frac{d\underline{k}}{dt}\times\underline{B}_0 + \underline{k}\times\frac{\partial\underline{B}_0}{\partial t} + \underline{k}\times\left(\frac{d\underline{x}}{dt}\cdot\nabla\underline{B}_0\right)\\ &= \underline{B}_0\times(\nabla\underline{v}_0)\cdot\underline{k} \pm \varrho_0^{-\frac{1}{2}}\,\underline{B}_0\times(\nabla\underline{B}_0)\cdot\underline{k}\\ &+ \frac{1}{2}\varrho_0^{-\frac{3}{2}}(\underline{k}\cdot\underline{B}_0)\underline{B}_0\times\nabla\varrho_0 + \underline{k}\times\frac{\partial\underline{B}_0}{\partial t}\\ &- \underline{v}_0\cdot(\nabla\underline{B}_0)\times\underline{k} \mp \varrho_0^{-\frac{1}{2}}\,\underline{B}_0\cdot(\nabla\underline{B}_0)\times\underline{k},\end{aligned} \tag{107}$$

where the upper signs hold for the characteristic root Ω_2 and the lower signs hold for Ω_3. In the case of a homogeneous basic state (3), the right-hand side in (107) obviously vanishes. The quantity $\underline{k}\times\underline{B}_0$ is therefore conserved along the rays in that case, the Alfvén waves and the magnetoacoustic waves may therefore interact through the system of transport equations (39) in the singular case where $\underline{k}\times\underline{B}_0 = \underline{0}$. Each of the cases (54) has to be considered separately; the relevant eigenvectors are given in Sec.IV. Since the case of a homogeneous basic state is better treated by other methods, we shall not pursue that approach here.

When the basic state (3) is inhomogeneous, $\underline{k}\times\underline{B}_0 = \underline{0}$ is usually not a persistent property along the rays. This can easily be seen, for example, in the slab geometry (78) where we for simplicity consider the special case

$$\underline{v}_0 = \underline{0}, \quad \rho_0 = \rho_0(z), \quad p_0 = p_0(z), \quad \underline{B}_0 = B_0(z)\underline{e}_x. \tag{108}$$

Here $\underline{e}_x$ is the unit vector along the x-axis. Assuming that at a certain instant, $\underline{k}\times\underline{B}_0 = \underline{0}$, we have $\underline{k} = k\underline{e}_x$ and (107), (108) implies that at this instant

$$\frac{d}{dt}(\underline{k}\times\underline{B}_0) = \pm(\frac{1}{2}\rho_0^{-\frac{3}{2}} B_0^{\,2}\rho_0' - \rho_0^{-\frac{1}{2}} B_0 B_0')k\underline{e}_y, \tag{109}$$

where $\underline{e}_y$ is the unit vector along the y-axis. Since ρ_0 and B_0 are arbitrarily given functions, (109) shows that $\underline{k}\times\underline{B}_0 = \underline{0}$ usually holds only at isolated points along the rays, if at any point at all. We therefore normally do not expect the singular points where $\underline{k}\times\underline{B}_0 = \underline{0}$ to be of vital importance either for the problem of linear wave propagation or for the problem of stability. Hence we shall not study that singular case further here.

IX. THE SINGULAR CASE WHERE $\underline{k} \cdot \underline{B}_0 = 0$

As already seen in Sec. IV, all the characteristic roots except Ω_4 and Ω_5 coincide if $\underline{k} \cdot \underline{B}_0 = 0$. In this case, therefore, the multiplicity assumption in Eckhoff (Ref. 1, Sec.5) are only satisfied for the fast magnetoacoustic waves. Those waves may be studied by the approach indicated in Sec.VII, while a modification is needed in order to study all the other waves in this singular case.

In order to get hold of the nature of the singular case $\underline{k} \cdot \underline{B}_0 = 0$, we calculate

$$\frac{d}{dt}(\underline{k} \cdot \underline{B}_0) = \frac{d\underline{k}}{dt} \cdot \underline{B}_0 + \underline{k} \cdot \frac{\partial \underline{B}_0}{\partial t} + \frac{d\underline{x}}{dt} \cdot (\nabla \underline{B}_0) \cdot \underline{k} \tag{110}$$

along the rays associated with the various waves. For the mass waves we obtain from (1), (67), and (110)

$$\frac{d}{dt}(\underline{k} \cdot \underline{B}_0) = -(\underline{k} \cdot \underline{B}_0) \nabla \cdot \underline{v}_0 . \tag{111}$$

Similarly, we get for the Alfvén waves

$$\frac{d}{dt}(\underline{k} \cdot \underline{B}_0) = (\underline{k} \cdot \underline{B}_0)(\pm \frac{1}{2} \varrho_0^{-\frac{3}{2}} \underline{B}_0 \cdot \nabla \varrho_0 - \nabla \cdot \underline{v}_0), \tag{112}$$

where + holds for the characteristic root Ω_2 and - holds for Ω_3. The bicharacteristic equation (101) is seen to be singular for the slow magnetoacoustic waves, i.e., for $\kappa = 6, 7$, in the singular case $\underline{k} \cdot \underline{B}_0 = 0$ and so is (110). It is therefore not equally revealing to look at (110) for the slow magnetoacoustic waves as it is for the mass waves and for the Alfvén waves; hence we shall omit it here.

If we look at slabs (78) and screw pinches (84), we see from (111) and (112) that the quantity $\underline{k} \cdot \underline{B}_0$ is always conserved along the rays. For more general basic states (3), the quantity $\underline{k} \cdot \underline{B}_0$ is not necessarily conserved, but it easily follows from the uniqueness theorem for the initial value problem for ordinary differential equations that $\underline{k}\cdot\underline{B}_0$ must vanish everywhere along a ray if it vanishes at a point on that ray. Hence for any basic state (3), the singular case $\underline{k}\cdot\underline{B}_0 = 0$ is either totally avoided or a persistent property along the rays.

With this background we may conclude that the singular case $\underline{k}\cdot\underline{B}_0 = 0$ seems to be the "worst" possible case in a stability research, since in this case all waves except the fast magnetoacoustic waves may persistently interact along the rays, thus increasing substantially the possibility that an instability may occur. In order to treat this case we have to proceed from the transport equations (39) since the multiplicity assumptions in Eckhoff (Ref.1, Sec.5) are not satisfied.

Recapitulating, the characteristic root

$$\Omega = \underline{k}\cdot\underline{v}_0 \tag{113}$$

has multiplicity 6 in the singular case $\underline{k}\cdot\underline{B}_0 = 0$, and when $\underline{B}_0 \neq \underline{0}$ the associated eigenvectors may be chosen to be

$$\begin{aligned}
\underline{R}_1 &= \underline{r}_{11} = (\underline{0},\underline{0},1,0), \quad \underline{R}_2 = \underline{r}_{12} = k^{-1}(\underline{0},\underline{k},0,0),\\
\underline{R}_3 &= \underline{r}_2 = 2^{-\frac{1}{2}} k^{-1} B_0^{-1} (\underline{k}\times\underline{B}_0,\ -\underline{k}\times\underline{B}_0,\ 0,\ 0),\\
\underline{R}_4 &= \underline{r}_3 = 2^{-\frac{1}{2}} k^{-1} B_0^{-1} (\underline{k}\times\underline{B}_0,\ \underline{k}\times\underline{B}_0,\ 0,\ 0)\\
\underline{R}_5 &= \underline{r}_{6\perp} = 2^{-\frac{1}{2}} B_0^{-1} (\underline{B}_0,\ -c_0(2P)^{-\frac{1}{2}}\underline{B}_0,\ 0,\ (2P\rho_0)^{-\frac{1}{2}} B_0^2),\\
\underline{R}_6 &= \underline{r}_{7\perp} = 2^{-\frac{1}{2}} B_0^{-1} (\underline{B}_0,\ c_0(2P)^{-\frac{1}{2}}\underline{B}_0,\ 0,\ -(2P\rho_0)^{-\frac{1}{2}} B_0^2).
\end{aligned} \tag{114}$$

With these notations, the amplitude in the leading term of a generalized progressing wave solution of the equations (6), (7), (8), and (9) takes the following form:

$$\underline{a}_0 = \sum_{\iota=1}^{6} \sigma_\iota \underline{R}_\iota, \tag{115}$$

where the scalar functions $\sigma_1,\ldots,\sigma_6$ have to be determined by the symmetric hyperbolic system (39) with $\mu = 6$ and where we substitute

$$\underline{r}_\iota = \underline{R}_\iota \quad (\iota = 1,\ldots,6). \tag{116}$$

Furthermore, $\underline{k} = \nabla\varphi$ is determined such that $\underline{k}\cdot\underline{B}_0 = 0$ by the characteristic equation

$$\varphi_t + \underline{v}_0\cdot\nabla\varphi = 0. \tag{117}$$

By a direct calculation of the quantities $\underline{R}_\iota\cdot A^\nu\underline{R}_m$, we find that the system (39) takes the form

$$(\sigma_1)_t + \underline{v}_0\cdot\nabla\sigma_1 + \sum_{m=1}^{6}(\underline{R}_1\cdot L\underline{R}_m)\sigma_m = 0, \tag{118}$$

$$(\sigma_2)_t + \underline{v}_0\cdot\nabla\sigma_2 + \sum_{m=1}^{6}(\underline{R}_2\cdot L\underline{R}_m)\sigma_m = 0, \tag{119}$$

$$(\sigma_3)_t + (\underline{v}_0 + \rho_0^{-1/2}\underline{B}_0)\cdot\nabla\sigma_3 + \sum_{m=1}^{6}(\underline{R}_3\cdot L\underline{R}_m)\sigma_m = 0, \tag{120}$$

$$(\sigma_4)_t + (\underline{v}_0 - \rho_0^{-1/2}\underline{B}_0)\cdot\nabla\sigma_4 + \sum_{m=1}^{6}(\underline{R}_4\cdot L\underline{R}_m)\sigma_m = 0, \tag{121}$$

$$(\sigma_5)_t + (\underline{v}_0 + (2P\rho_0)^{-1/2}c_0\underline{B}_0)\cdot\nabla\sigma_5 + \sum_{m=1}^{6}(\underline{R}_5\cdot L\underline{R}_m)\sigma_m = 0, \tag{122}$$

$$(\sigma_6)_t + (\underline{v}_0 - (2P\rho_0)^{-1/2}c_0\underline{B}_0)\cdot\nabla\sigma_6 + \sum_{m=1}^{6}(\underline{R}_6\cdot L\underline{R}_m)\sigma_m = 0. \tag{123}$$

Thus we see that the transport equations in this case constitute a weakly coupled hyperbolic system. Even though the number of unknowns in (118)-(123) is only reduced by two compared to the original hyperbolic system (6)-(9), the fact that (118)-(123) is weakly coupled makes it substantially more tractable both by analytic and numerical methods. Since the number of dependent variables is only reduced by 2, it may be expected that essential information about the stability properties of the basic flow (3) is carried over to (118)-(123).

In the case where $\underline{v}_0 \times \underline{B}_0 \equiv \underline{0}$, we note that the spatial differentiations in (118)-(123) are all along the same direction, namely along the magnetic fieldlines. This means that the number of independent variables involved in integrating (118)-(123) essentially reduces to only 2, namely the arclength along the magnetic fieldlines and the time t. The other two independent space variables will only appear as parameters during the integration of (118)-(123) when $\underline{v}_0 \times \underline{B}_0 \equiv \underline{0}$.

The integration of (118)-(123) is more complicated in the general case where $\underline{v}_0 \times \underline{B}_0 \neq \underline{0}$, since the spatial differentiations are then no longer in the same direction. In the case where the basic state (3) has magnetic surfaces such that $\underline{v}_0$ is everywhere tangent to these surfaces, however, the integration of (118)-(123) may be carried out on each of these magnetic surfaces separately. The space variable which is perpendicular to these surfaces will only appear as a parameter during the integration; hence the number of independent variables is in this case essentially reduced to 3.

For a stationary basic state (3) the streamlines never hit the boundary if they start within the plasma. In the cases most frequently studied in the literature, the magnetic fieldlines do not hit the boundary if they start within the plasma either. For these cases the boundary conditions do not cause any trouble, since we can let the initial values associated with (118)-(123) vanish in the neighborhood of the boundaries. By the properties of (118)-(123) discussed above, we then see that the solutions of (118)-(123) will vanish in the neighborhood of the boundaries also. Hence the boundary conditions will obviously be satisfied in these cases. If, on the other hand, the magnetic fieldlines do hit the boundaries, we may have to investigate how the waves are reflected/transmitted there in order to carry the study further. We shall not, however, consider such cases in this paper.

In order to study the system (118)-(123) and thus the coupling of the waves further, the quantities $\underline{R}_l \cdot L\underline{R}_m$ have to be calculated. These quantities are given in the appendix for the general non-stationary case, and are seen to consist of relatively complicated expressions involving of course the basic flow (3). We note that with the expressions in the appendix, (118)-(123) as well as (117) are all given in a coordinate-free representation; the Cartesian coordinates were only used during parts of the derivation of these equations. Furthermore we note that the expressions given in the appendix show that for any basic flow (3), Eq. (119) takes the form

$$(\sigma_2)_t + \underline{v}_0 \cdot \nabla\sigma_2 + (\underline{R}_2 \cdot L\underline{R}_2)\sigma_2 = 0. \qquad (119')$$

Thus Eq. (119) can be solved independently of the rest of the system (118)-(123). In particular we see that

$$\sigma_2 = 0 \qquad (124)$$

is always a solution. In the following sections we shall show that unless we take the solution (124), we normally get linearly growing perturbations resembling what Grad[11] has called anholonomic instabilities (see also Ref.12).

In the special cases of primary interest, namely slabs, screw pinches, general static plasma configurations, and also some other stationary plasma configurations with flow, the expressions for $\underline{R}_l \cdot L\underline{R}_m$ given in the appendix are seen to simplify considerably. We shall in the following sections look more closely at some such special cases where the simplifications are substantial.

X. THE SLAB GEOMETRY

In this section we shall look more closely at the system of transport equations (118)-(123) when the basic flow is given by (78) and (79). The phase function $\varphi(x,y,z,t)$ is in this case determined by the following two equations:

$$B_1(z)\varphi_x + B_2(z)\varphi_y = 0, \tag{125}$$

$$\varphi_t + v_1(z)\varphi_x + v_2(z)\varphi_y = 0. \tag{126}$$

The general solution of (125) and (126) is found by the method of characteristics to be

$$\varphi = \psi[B_2(z)x - B_1(z)y + (B_1(z)v_2(z) - B_2(z)v_1(z))t,\ z], \tag{127}$$

where $\psi[g,h]$ is an arbitrary function of the two variables g,h. From (127) we obtain

$$\underline{k} = \nabla\varphi = B_2 \frac{\partial\psi}{\partial g}\,\underline{e}_x - B_1 \frac{\partial\psi}{\partial g}\,\underline{e}_y + ([B_2{}'x - B_1{}'y + (B_1 v_2 - B_2 v_1)'t]\frac{\partial\psi}{\partial g} + \frac{\partial\psi}{\partial h})\underline{e}_z. \tag{128}$$

Thus in the general slab case $\underline{k}$ and hence the coefficients $\underline{R}_l \cdot LR_m$ may depend on all the variables x, y, z, t.

In order to make the system (118)-(123) more tractable by analytic methods, we shall in this paper restrict our attention to some special cases. In the first case we shall not put any restrictions on the basic flow (78) and (79), but we shall consider the following special choice of the function ψ in (127) and (128):

$$\varphi = \delta z, \qquad \underline{k} = \delta \underline{e}_z, \tag{129}$$

where δ is an arbitrary constant at our disposal. With (129) the expressions given in the appendix reduce for the slab to the following:

$$\underline{R}_3 \cdot L\underline{R}_2 = 2^{-\frac{1}{2}} B_0^{-1} (B_1 (v_2' - \rho_0^{-\frac{1}{2}} B_2') - B_2 (v_1' - \rho_0^{-\frac{1}{2}} B_1')), \tag{130}$$

$$\underline{R}_4 \cdot L\underline{R}_2 = - 2^{-\frac{1}{2}} B_0^{-1} (B_1 (v_2' + \rho_0^{-\frac{1}{2}} B_2') - B_2 (v_1' + \rho_0^{-\frac{1}{2}} B_1')), \tag{131}$$

$$\underline{R}_5 \cdot L\underline{R}_2 = - 2^{-\frac{1}{2}} \frac{\delta}{|\delta|} (\rho_0^{-\frac{1}{2}} B_0' - c_0 (2P)^{-\frac{1}{2}} B_0^{-1} (B_1 v_1' + B_2 v_2')), \tag{132}$$

$$\underline{R}_6 \cdot L\underline{R}_2 = - 2^{-\frac{1}{2}} \frac{\delta}{|\delta|} (\rho_0^{-\frac{1}{2}} B_0' + c_0 (2P)^{-\frac{1}{2}} B_0^{-1} (B_1 v_1' + B_2 v_2')), \tag{133}$$

while all the other expressions $\underline{R}_l \cdot LR_m$ vanish. Clearly, the expressions $\underline{R}_l \cdot LR_m$ depend only on the variable z in this special case, and we may therefore look for solutions of (118)-(123) of the following type:

$$\underline{\sigma} = \underline{\sigma}_0(z) \exp i(\kappa_1 x + \kappa_2 y - qt). \tag{134}$$

Substitution of (134) into (118)-(123) then shows that (134) is a solution if $q = q(z)$ and $\underline{\kappa} = (\kappa_1, \kappa_2, 0) = \underline{\kappa}(z)$ satisfy the following dispersion relation for each z:

$$(q - \underline{\kappa} \cdot \underline{v}_0)^2 (q - \underline{\kappa} \cdot (\underline{v}_0 + \rho_0^{-\frac{1}{2}} \underline{B}_0))(q - \underline{\kappa} \cdot (\underline{v}_0 - \rho_0^{-\frac{1}{2}} \underline{B}_0))$$
$$\times (q - \underline{\kappa} \cdot [\underline{v}_0 + (2P\rho_0)^{-\frac{1}{2}} c_0 \underline{B}_0])(q - \underline{\kappa} \cdot [\underline{v}_0 - (2P\rho_0)^{-\frac{1}{2}} c_0 \underline{B}_0]) = 0, \tag{135}$$

and σ_0 is of the form

$$\underline{\sigma}_0 = \chi(z) \underline{e}(z), \tag{136}$$

where $\underline{e}$ is the appropriate eigenvector corresponding to the chosen solution q of (135) and $\chi(z)$ is an arbitrary function. The relation (135) clearly shows that neither does any interaction between the mass waves, the Alfvén waves, and the magnetoacoustic waves occur, nor does any instability show up in the modes (134) in this special case.

In this special case, however, we may also consider another type of solutions than (134). In fact, we see that when φ is given by (129),

$$\underline{\alpha} = \underline{\alpha}(z,t) \tag{137}$$

is a solution of (118)-(123) if and only if $\underline{\alpha}$ satisfies the following system of ordinary differential equations for each z:

$$\frac{d\underline{\alpha}}{dt} = E\underline{\alpha}, \tag{138}$$

where $E = (\cdot R_l \cdot L R_m)$. The 6×6-matrix E is seen to be constant for each z and $\lambda = 0$ is the only eigenvalue of E. Thus it follows from the standard theory of stability for ordinary differential equations (138)[13] that $\underline{\alpha} = 0$ cannot be stable unless all the coefficients in E vanish. We have therefore shown that the trivial solution of (118)-(123) will be unstable unless the four expressions (130)-(133) all vanish. If one or more of the expressions (130)-(133) do not vanish, it is easily seen that there will be solutions (137) growing <u>linearly</u> with respect to t. These algebraic instabilities are then due to the weak coupling between the mass waves on the one hand, and the Alfvén waves and the magnetoacoustic waves on the other in the system of transport equations (118)-(123). Clearly these instabilities resemble the so-called anholonomic instabilities (see Ref.11 and 12).

In order that the slab shall be stable with respect to algebraically growing perturbations, it is therefore seen from (130)-(133) that (78) has to satisfy the following equations:

$$B_2(v_1' - \varrho_0^{-\frac{1}{2}}B_1') - B_1(v_2' - \varrho_0^{-\frac{1}{2}}B_2') = 0, \tag{139}$$

$$B_2(v_1' + \varrho_0^{-\frac{1}{2}}B_1') - B_1(v_2' + \varrho_0^{-\frac{1}{2}}B_2') = 0, \tag{140}$$

$$c_0(2P)^{-\frac{1}{2}}(B_1v_1' + B_2v_2') - \varrho_0^{-\frac{1}{2}}(B_1B_1' + B_2B_2') = 0, \tag{141}$$

$$c_0(2P)^{-\frac{1}{2}}(B_1v_1' + B_2v_2') + \varrho_0^{-\frac{1}{2}}(B_1B_1' + B_2B_2') = 0. \tag{142}$$

Since we assume that $\underline{B}_0 \neq 0$, it clearly follows from (139)-(142) that

$$v_1' = v_2' = B_1' = B_2' = 0. \tag{143}$$

Thus an inhomogeneous magnetic field $\underline{B}_0$ and/or a flow $\underline{v}_0$ with shear always imply that the slab (78) is unstable. From (130)-(133) and (138) we see that this instability is triggered when $\sigma_2 \neq 0$. By (115) this is equivalent to giving the perturbation of the magnetic field a nonvanishing z-component. While (138) shows that this z-component of the magnetic field and also the quantity η in (4) will be conserved for the solutions (137), usually both the other components of the magnetic field and also the quantity ζ and the x- and y-components of the velocity field will grow linearly with respect to t when (143) is not satisfied.

In the second special case we are going to consider, we shall restrict our study to slabs (78) such that

$$v_2(z) = B_2(z) = 0. \tag{144}$$

In this special case (127) is seen to be equivalent to

$$\varphi = \chi(y,z), \tag{145}$$

where $\chi(y,z)$ is an arbitrary function of the two variables y, z. For our purpose there is no essential loss of generality if we restrict our choice of phase function (145) to the following:

$$\varphi = \alpha y + \delta z, \tag{146}$$

where α, δ are arbitrary constants at our disposal. With (146) we get

$$\underline{k} = \nabla\varphi = \alpha\underline{e}_y + \delta\underline{e}_z. \tag{147}$$

With (144) and (147) the expressions given in the appendix reduce to the following for the slab:

$$\underline{R}_3\cdot L\underline{R}_1 = \underline{R}_4\cdot L\underline{R}_1 = 2^{-\frac{1}{2}}k^{-1}\alpha B_0^{-1}B_1 c_0^{-1}V' \,, \tag{148}$$

$$\underline{R}_5\cdot L\underline{R}_2 = -\,2^{-\frac{1}{2}}k^{-1}\delta B_0^{-1}B_1\{\varrho_0^{-\frac{1}{2}}B_1' - c_0(2P)^{-\frac{1}{2}}v_1'\}, \tag{149}$$

$$\underline{R}_6\cdot L\underline{R}_2 = -\,2^{-\frac{1}{2}}k^{-1}\delta B_0^{-1}B_1\{\varrho_0^{-\frac{1}{2}}B_1' + c_0(2P)^{-\frac{1}{2}}v_1'\}, \tag{150}$$

$$\underline{R}_1\cdot L\underline{R}_3 = \underline{R}_1\cdot L\underline{R}_4 = 2^{-\frac{1}{2}}k^{-1}\alpha B_0^{-1}B_1 c_0(\varrho_0^{-1}c_0^{-2}p_0' - \varrho_0^{-1}\varrho_0'), \tag{151}$$

$$\underline{R}_5\cdot L\underline{R}_3 = -\frac{1}{2}k^{-1}\alpha\{[1 - c_0(2P)^{-\frac{1}{2}}](v_1' + \varrho_0^{-\frac{1}{2}}B_1') + (\varrho_0 c_0)^{-1}(2P\varrho_0)^{-\frac{1}{2}}B_1 p_0'\}, \tag{152}$$

$$\underline{R}_6\cdot L\underline{R}_3 = -\frac{1}{2}k^{-1}\alpha\{[1 + c_0(2P)^{-\frac{1}{2}}](v_1' + \varrho_0^{-\frac{1}{2}}B_1') - (\varrho_0 c_0)^{-1}(2P\varrho_0)^{-\frac{1}{2}}B_1 p_0'\}, \tag{153}$$

$$\underline{R}_5 \cdot L\underline{R}_4 = -\frac{1}{2} k^{-1} \alpha \{[1 + c_0 (2P)^{-\frac{1}{2}}](v_1' - \rho_0^{-\frac{1}{2}} B_1')$$

$$+ (\rho_0 c_0)^{-1} (2P\rho_0)^{-\frac{1}{2}} B_1 p_0'\}, \tag{154}$$

$$\underline{R}_6 \cdot L\underline{R}_4 = -\frac{1}{2} k^{-1} \alpha \{[1 - c_0 (2P)^{-\frac{1}{2}}](v_1' - \rho_0^{-\frac{1}{2}} B_1')$$

$$- (\rho_0 c_0)^{-1} (2P\rho_0)^{-\frac{1}{2}} B_1 p_0'\}, \tag{155}$$

$$\underline{R}_3 \cdot L\underline{R}_5 = \underline{R}_4 \cdot L\underline{R}_5 = -\underline{R}_3 \cdot L\underline{R}_6 = -\underline{R}_4 \cdot L\underline{R}_6$$

$$= \frac{1}{2} k^{-1} \alpha c_0^{-1} (2P\rho_0)^{-\frac{1}{2}} B_1 V', \tag{156}$$

while all the other expressions $\underline{R}_\iota \cdot L\underline{R}_m$ vanish. Also in this case the expressions $\underline{R}_\iota \cdot L\underline{R}_m$ depend only on the variable z. Since the spatial differentiations in (118)-(123) are all along the x-axis, we may therefore in this special case look for solutions of (118)-(123) of the following type:

$$\underline{a} = \underline{a}_0 (y,z) \exp i(\kappa x - qt). \tag{157}$$

Substitution of (157) into (118)-(123) then shows that (157) is a solution if $q = q(y,z)$ and $\kappa = \kappa(y,z)$ satisfy a certain dispersion relation for each z and $\underline{a}_0$ is of the form

$$\underline{a}_0 = \psi(y,z)\underline{e}(z), \tag{158}$$

where $\underline{e}$ is the appropriate eigenvector corresponding to the chosen solution of the dispersion relation and $\psi(y,z)$ is an arbitrary function. For arbitrary κ the dispersion relation corresponding to (157) is relatively complicated. We shall therefore limit our discussion here to the case where $\kappa = 0$. In that case the dispersion relation becomes

$$q^4(q^2 - M^2) = 0, \tag{159}$$

$$M^2 = \underline{R}_3 \cdot L\underline{R}_5(\underline{R}_6 \cdot L\underline{R}_4 - \underline{R}_5 \cdot L\underline{R}_4 + \underline{R}_6 \cdot L\underline{R}_3$$

$$- \underline{R}_5 \cdot L\underline{R}_3) - 2(\underline{R}_1 \cdot L\underline{R}_3)(\underline{R}_3 \cdot L\underline{R}_1)$$

$$= \frac{\alpha^2}{k^2} V'\{\rho_0^{-1}\rho_0' - (2P)^{-1} V'\}. \tag{160}$$

From (157), (159), and (160) we clearly get the following necessary condition for stability of the special slabs (78), (79), and (144):

$$V'\{\rho_0^{-1}\rho_0' - (2P)^{-1}V'\} \geq 0. \tag{161}$$

This is the interchange stability criterion found earlier by Tserkovnikov in the special case where[14]

$$V' = g = \text{constant}. \tag{162}$$

When (161) is satisfied we see that there are modes oscillating with the local frequency M given by (160). By (147) and (160) it is seen that M has its maximum value when $\delta = 0$, and that M resembles the Brunt-Väisälä frequency in fluid mechanics.

XI. THE SCREW PINCH

In this section we shall look more closely at the system of transport equations (118)-(123) when the basic flow is given by (84) and (85). The phase function $\varphi(r,\theta,z,t)$ is in this case determined by the following two equations:

$$\frac{1}{r} B_\theta(r)\varphi_\theta + B_z(r)\varphi_z = 0, \tag{163}$$

$$\varphi_t + \frac{1}{r} v_\theta(r)\varphi_\theta + v_z(r)\varphi_z = 0. \tag{164}$$

The general solution of (163) and (164) is found by the method of characteristics to be

$$\varphi = \psi[r,\ rB_z(r)\theta - B_\theta(r)z + (B_\theta(r)v_z(r) - B_z(r)v_\theta(r))t], \tag{165}$$

where $\psi\,[g,h]$ is an arbitrary function of the two variables g,h. From (165) we get

$$\underline{k} = \nabla\varphi = ([(B_z + rB_z')\theta - B_\theta'z + (B_\theta v_z - B_z v_\theta)'t]\frac{\partial\psi}{\partial h} + \frac{\partial\psi}{\partial g})\hat{\underline{r}}$$
$$+ B_z \frac{\partial\psi}{\partial h}\hat{\underline{\theta}} - B_\theta \frac{\partial\psi}{\partial h}\hat{\underline{z}}. \tag{166}$$

Thus for the general screw pinch $\underline{k}$ and hence the coefficients $\underline{R}_\iota \cdot L\underline{R}_m$ may depend on all the variables r,θ,z,t.

In order to make the system (118)-(123) more tractable by analytic methods, we shall in this paper restrict our attention to some special cases. In the first case we shall not put any restrictions on the basic flow (84) and (85), but we shall consider the following special choice of the function ψ in (165) and (166):

$$\varphi = \alpha r, \qquad k = \alpha\hat{\underline{r}}, \tag{167}$$

where α is an arbitrary constant at our disposal. This case is analogous to the first special case considered for the slab geometry where the phase function was given by (129). With (167) the expressions given in the appendix reduce for the screw pinch to the following:

$$\underline{R}_3 \cdot L\underline{R}_2 = 2^{-\frac{1}{2}} B_0^{-1} \{ B_\theta (v_z' - \varrho_0^{-\frac{1}{2}} B_z')$$

$$- B_z [v_\theta' - \frac{1}{r} v_\theta - \varrho_0^{-\frac{1}{2}} (B_\theta' + \frac{1}{r} B_\theta)] \}, \quad (168)$$

$$\underline{R}_4 \cdot L\underline{R}_2 = 2^{-\frac{1}{2}} B_0^{-1} \{ B_z [v_\theta' - \frac{1}{r} v_\theta + \varrho_0^{-\frac{1}{2}} (B_\theta' + \frac{1}{r} B_\theta)]$$

$$- B_\theta (v_z' + \varrho_0^{-\frac{1}{2}} B_z') \}, \quad (169)$$

$$\underline{R}_5 \cdot L\underline{R}_2 = - 2^{-\frac{1}{2}} \frac{\alpha}{|\alpha|} B_0^{-1} \varrho_0^{-\frac{1}{2}} \{ \text{[illegible]} + \frac{1}{r} B_\theta) + B_z B_z' \}$$

$$+ 2^{-\frac{1}{2}} \frac{\alpha}{|\alpha|} B_0^{-1} c_0 (2P)^{-\frac{1}{2}} \{ B_\theta (v_\theta' - \frac{1}{r} v_\theta) + B_z v_z' \}, \quad (170)$$

$$\underline{R}_6 \cdot L\underline{R}_2 = - 2^{-\frac{1}{2}} \frac{\alpha}{|\alpha|} B_0^{-1} \varrho_0^{-\frac{1}{2}} \{ B_\theta (B_\theta' + \frac{1}{r} B_\theta) + B_z B_z' \}$$

$$- 2^{-\frac{1}{2}} \frac{\alpha}{|\alpha|} B_0^{-1} c_0 (2P)^{-\frac{1}{2}} \{ B_\theta (v_\theta' - \frac{1}{r} v_\theta) + B_z v_z' \}, \quad (171)$$

while all the other expressions $\underline{R}_\ell \cdot L\underline{R}_m$ vanish. Clearly, the expressions $\underline{R}_\ell \cdot L\underline{R}_m$ depend only on the variable r in this special case, we may therefore look for solutions of (118)-(123) of the following type:

$$\underline{\sigma} = \underline{\sigma}_0 (r) \exp i(\kappa_1 \theta + \kappa_2 z - qt). \quad (172)$$

Substitution of (172) into (118)-(123) then shows that (172) is a solution if κ_1 is any given integer and if $q = q(r)$ and $\kappa_2 = \kappa_2(r)$ satisfy the following dispersion relation for each r:

$$
\begin{aligned}
&(q - \frac{1}{r}\kappa_1 v_\theta - \kappa_2 v_z)^2 (q - \frac{1}{r}\kappa_1 (v_\theta + \varrho_0^{-\frac{1}{2}} B_\theta) - \kappa_2 (v_z + \varrho_0^{-\frac{1}{2}} B_z)) \\
&\times (q - \frac{1}{r}\kappa_1 (v_\theta - \varrho_0^{-\frac{1}{2}} B_\theta) - \kappa_2 (v_z - \varrho_0^{-\frac{1}{2}} B_z)) \\
&\times (q - \frac{1}{r}\kappa_1 [v_\theta + (2P\varrho_0)^{-\frac{1}{2}} c_0 B_\theta] - \kappa_2 [v_z + (2P\varrho_0)^{-\frac{1}{2}} c_0 B_z]) \\
&\times (q - \frac{1}{r}\kappa_1 [v_\theta - (2P\varrho_0)^{-\frac{1}{2}} c_0 B_\theta] - \kappa_2 [v_z - (2P\varrho_0)^{-\frac{1}{2}} c_0 B_z]) = 0, \qquad (173)
\end{aligned}
$$

and $\underline{a}_0$ is of the form

$$\underline{a}_0 = \chi(r)\underline{e}(r), \qquad (174)$$

where $\underline{e}$ is the appropriate eigenvector corresponding to the chosen solution q of (173) and $\chi(r)$ is an arbitrary function. The dispersion relation (173), which is analogous to (135), clearly shows that no interaction between the mass waves, the Alfvén waves, and the magneto-acoustic waves occurs, nor does any instability show up in the modes (172) in this special case.

Analogous to (137) we may in this special case also consider solutions of (118)-(123) of the following type:

$$\underline{a} = \underline{a}(r,t). \qquad (175)$$

In fact, when φ is given by (167), we see that (175) is a solution of (118)-(123) if and only if for each r it satisfies the system of ordinary differential equations (138) with the expressions for $\underline{R}_l \cdot L\underline{R}_m$ found above substituted. As in the case of slab geometry, we may therefore conclude that unless the expressions (168)-(171) all vanish indentically, there will be perturbations of the basic screw pinch (84) and (85) growing linearly with respect to t. In order that the screw pinch shall be stable with respect to algebraically growing perturbations, it is therefore seen from (168) - (171) that (84) has to satisfy the following equations:

$$v_\theta' - \frac{1}{r} v_\theta = v_z' = B_\theta' + \frac{1}{r} B_\theta = B_z' = 0. \qquad (176)$$

Clearly, equations (176) can only be satisfied when

$$v_\theta = C_1 r, \quad v_z = C_2, \quad B_\theta = \frac{C_3}{r}, \quad B_z = C_4, \qquad (177)$$

where C_1, C_2, C_3, C_4 are constants. The basic flow (84) and (85) with (177) may be realizable for a tubular pinch, while a columnar screw pinch is always subject to algebraic instabilities if we let $\sigma_2 \neq 0$.

In the second special case we want to let the function $\psi[g,h]$ chosen in (165) also depend on h. In order that the phase function φ shall be independent of the variable t, we then see from (165) that we have to restrict our study to screw pinches (84) such that

$$B_\theta(r)v_z(r) - B_z(r)v_\theta(r) = 0. \qquad (178)$$

Equation (178) simply means that $\underline{v}_0 \times \underline{B}_0 = \underline{0}$, we therefore know from the discussion in Sec.IX that in this case the number of independent variables involved in integrating (118)-(123) essentially reduces to only 2, namely the arclength along the magnetic fieldlines and the time t. Since (178) implies that $\underline{k}$ is independent of t, the coefficients in (118)-(123) are also seen to be independent of t in this special case. Since $\underline{k}$ normally will depend on the arclength along the magnetic fieldlines, however, the coefficients $\underline{R}_i \cdot L\underline{R}_m$ will normally not all be constants along the magnetic fieldlines. Hence the integration of (118)-(123) is usually not trivial, but at least there exist very efficient numerical codes for integrating this system of equations along each magnetic fieldline[15].

The integration of (118)-(123) will be further considerably simplified in the special cases where the coefficients $\underline{R}_\iota \cdot L\underline{R}_m$ are constants along each magnetic fieldline. It is easily verified that this will be true for the screw pinch if and only if $\hat{\underline{r}}\cdot\underline{k}$, $\hat{\underline{\theta}}\cdot\underline{k}$, and $\hat{\underline{z}}\cdot\underline{k}$ are constants along the fieldlines. Clearly along each magnetic fieldline r and $rB_z(r)\theta - B_\theta(r)z$ are both constant; hence we see from (165) and (166) that with the assumption (174) the coefficients $\underline{R}_\iota \cdot L\underline{R}_m$ will be constants along the fieldlines if and only if $(B_z(r) + rB_z'(r))\theta - B_\theta'(r)z$ is constant along each fieldline when ψ is assumed to depend on h. This condition is easily seen to be equivalent to

$$(B_z + rB_z')\frac{1}{r} B_\theta - B_\theta' B_z = \frac{1}{r} B_\theta^2 \left(\frac{rB_z}{B_\theta}\right)' = 0, \qquad (179)$$

which is the constant pitch case where the magnetic field is given by

$$B_\theta = CrH(r), \quad B_z = DH(r). \qquad (180)$$

Here C,D are arbitrarily given constants and $H(r)$ is an arbitrarily given function.

In the special case where (84) satisfies (178) and (180), it is easily seen that (165) is equivalent to

$$\varphi = \chi(r,\ D\theta - Cz), \qquad (181)$$

where $\chi(g,h)$ is an arbitrary function of the two variables g,h. For our purposes there is no essential loss of generality if we restrict our choice of phase function (181) to the following:

$$\varphi = \alpha r + \delta(D\theta - Cz), \qquad (182)$$

where α, δ are arbitrary constants at our disposal. With (182) we get

$$\underline{k} = \nabla\varphi = \alpha\hat{\underline{r}} + \delta\frac{D}{r}\hat{\underline{\theta}} - \delta C\hat{\underline{z}}. \tag{183}$$

In this case it is seen from the expressions given in the appendix that the coefficients in (118)-(123) depend only on the variable r. Since the spatial differentiations in (118)-(123) are all along the magnetic fieldlines, we may therefore in this special case look for solutions of (118)-(123) of the following type:

$$\underline{a} = \underline{a}_0(r,\tau)\exp i(\kappa s - qt), \tag{184}$$

where s is the arclength along the magnetic fieldlines and τ is the co-ordinate perpendicular to r,s. Substitution of (184) into (118)-(123) then shows that (184) is a solution if $q = q(r,\tau)$ and $\kappa = \kappa(r,\tau)$ satisfy a certain dispersion relation for each r, and $\underline{a}_0$ is of the form

$$\underline{a}_0 = \psi(r,\tau)\underline{e}(r), \tag{185}$$

where $\underline{e}$ is the appropriate eigenvector corresponding to the chosen solution of the dispersion relation and $\psi(r,\tau)$ is an arbitrary function. Unless additional assumptions are introduced, the dispersion relation corresponding to (184) is relatively complicated. To avoid excessive algebra we shall therefore limit our discussion of that dispersion relation in this paper to the static case where the external forces are assumed to be negligible. However, we shall in the next section first show that some general simplifications are then valid for the system of transport equations (118)-(123).

XII. THE STATIC CASE WITH NO EXTERNAL FORCES

If we assume that the external forces are negligible, we may take $V = 0$ in (1). In this section we shall, in addition, restrict our attention to static basic states (3), i.e.,

$$\underline{v} \equiv \underline{0}, \quad \rho = \rho_0(\underline{x}), \quad p = p_0(\underline{x}), \quad \underline{B} = \underline{B}_0(\underline{x}). \tag{186}$$

With the assumption $V = 0$, this basic state (186) is seen to satisfy the fundamental equations (1) if and only if

$$\nabla p_0 = (\nabla \times \underline{B}_0) \times \underline{B}_0 \equiv \underline{B}_0 \cdot \nabla \underline{B}_0 - \nabla(\frac{1}{2} B_0{}^2) . \tag{187}$$

From equation (187) it follows that

$$\underline{B}_0 \cdot \nabla p_0 = 0, \tag{188}$$

$$(\underline{k} \times \underline{B}_0) \cdot \nabla p_0 = B_0^2 \ \underline{k} \cdot (\nabla \times \underline{B}_0), \tag{189}$$

$$\nabla(p_0 + \frac{1}{2} B_0{}^2) = \underline{B}_0 \cdot (\nabla \underline{B}_0). \tag{190}$$

In view of (188) and the above assumptions it follows from the expressions given in the appendix that

$$\underline{R}_\iota \cdot L\underline{R}_1 = 0; \ \iota = 1, 2, \ldots, 6. \tag{191}$$

Equations (191) imply that in the system of transport equations (118)-(123) the equations (119)-(123) can be solved independently of (118) for $\sigma_2, \sigma_3, \ldots, \sigma_6$. Equation (118) can then afterward be solved for σ_1. In order to avoid algebraic instabilities we take the trivial solution (124) for σ_2. We are therefore left with an independent system of four transport equations (120)-(123) for $\sigma_3, \sigma_4, \sigma_5, \sigma_6$. Clearly this system will describe the interaction between the Alfvén waves and the slow magnetoacoustic waves in the singular case where $\underline{k} \cdot \underline{B}_0 = 0$ when the wavelengths are short. The coefficients in this system are found from the expressions given in the appendix to be

$$\underline{R}_3 \cdot L\underline{R}_3 = - \underline{R}_4 \cdot L\underline{R}_4 = - \frac{1}{4} \rho_0^{-\frac{3}{2}} \underline{B}_0 \cdot \nabla \rho_0 , \tag{192}$$

$$\underline{R}_4 \cdot L\underline{R}_3 = - \underline{R}_3 \cdot L\underline{R}_4 = \frac{1}{4} \rho_0^{-\frac{3}{2}} \underline{B}_0 \cdot \nabla \rho_0$$

$$+ k^{-2} \rho_0^{-\frac{1}{2}} B_0^{-2} (\underline{k} \times \underline{B}_0) \cdot (\nabla \underline{B}_0) \cdot (\underline{k} \times \underline{B}_0), \tag{193}$$

$$\underline{R}_5 \cdot L\underline{R}_3 = - \underline{R}_6 \cdot L\underline{R}_4 = - \frac{1}{2} k^{-1} \rho_0^{-\frac{1}{2}} \underline{k} \cdot (\nabla \times \underline{B}_0)$$

$$+ \frac{1}{2} (k \rho_0 c_0)^{-1} (2P\rho_0)^{-\frac{1}{2}} (\underline{k} \times \underline{B}_0) \cdot \{\nabla p_0 - \frac{\rho_0 c_0^2}{B_0^2} [\nabla (\frac{1}{2} B_0^2) + \underline{B}_0 \cdot \nabla B_0]\}, \tag{194}$$

$$\underline{R}_6 \cdot L\underline{R}_3 = - \underline{R}_5 \cdot L\underline{R}_4 = - \frac{1}{2} k^{-1} \rho_0^{-\frac{1}{2}} \underline{k} \cdot (\nabla \times \underline{B}_0)$$

$$- \frac{1}{2} (k \rho_0 c_0)^{-1} (2P\rho_0)^{-\frac{1}{2}} (\underline{k} \times \underline{B}_0) \cdot \{\nabla p_0 - \frac{\rho_0 c_0^2}{B_0^2} [\nabla (\frac{1}{2} B_0^2) + \underline{B}_0 \cdot \nabla \underline{B}_0]\}, \tag{195}$$

$$\underline{R}_3 \cdot L\underline{R}_5 = \underline{R}_4 \cdot L\underline{R}_5 = - \underline{R}_3 \cdot L\underline{R}_6 = - \underline{R}_4 \cdot L\underline{R}_6$$

$$= k^{-1} B_0^{-2} c_0 (2P\rho_0)^{-\frac{1}{2}} (\underline{k} \times \underline{B}_0) \cdot \nabla (p_0 + \frac{1}{2} \underline{B}_0^2), \tag{196}$$

$$\underline{R}_5 \cdot L\underline{R}_5 = - \underline{R}_6 \cdot L\underline{R}_6 = \frac{1}{2} \underline{B}_0 \cdot \nabla (c_0 (2P\rho_0)^{-\frac{1}{2}}), \tag{197}$$

$$\underline{R}_6 \cdot L\underline{R}_5 = - \underline{R}_5 \cdot L\underline{R}_6 = \frac{1}{2} \rho_0^{-\frac{1}{2}} c_0 B_0^{-2} \underline{B}_0 \cdot \nabla ((2P)^{-\frac{1}{2}} B_0^2). \tag{198}$$

The expressions (194)-(196) can be written in different ways by applying (189) and (190), and we see that for the static case with no external forces the simplifications are substantial in the system of transport equations (118)-(123).

In this paper we shall limit our further discussion to the static screw pinch. When external forces are neglected, it follows from the above expressions that for a static screw pinch we have

$$\underline{R}_3\cdot L\underline{R}_3 = \underline{R}_4\cdot L\underline{R}_4 = \underline{R}_5\cdot L\underline{R}_5 = 0,$$
$$\underline{R}_6\cdot L\underline{R}_6 = \underline{R}_6\cdot L\underline{R}_5 = \underline{R}_5\cdot L\underline{R}_6 = 0. \tag{199}$$

The rest of the expressions (192)-(198) will usually be nonvanishing. The system of transport equations (120)-(123) may therefore in this case be written

$$(\sigma_3)_t + A(\sigma_3)_s - M\sigma_4 + G\sigma_5 - G\sigma_6 = 0, \tag{200}$$

$$(\sigma_4)_t - A(\sigma_4)_s + M\sigma_3 + G\sigma_5 - G\sigma_6 = 0, \tag{201}$$

$$(\sigma_5)_t + \nu A(\sigma_5)_s + E\sigma_3 - F\sigma_4 = 0, \tag{202}$$

$$(\sigma_6)_t - \nu A(\sigma_6)_s + F\sigma_3 - E\sigma_4 = 0, \tag{203}$$

where s is the arclength along the magnetic fieldlines and where we have introduced the notation

$$A = \rho_0^{-\frac{1}{2}}B_0, \quad \nu = c_0(2P)^{-\frac{1}{2}}, \tag{204, 205}$$

$$M = \underline{R}_4\cdot L\underline{R}_3 = -\underline{R}_3\cdot L\underline{R}_4, \tag{206}$$

$$E = \underline{R}_5\cdot L\underline{R}_3 = -\underline{R}_6\cdot L\underline{R}_4, \tag{207}$$

$$F = \underline{R}_6\cdot L\underline{R}_3 = -\underline{R}_5\cdot L\underline{R}_4, \tag{208}$$

$$G = \underline{R}_3\cdot L\underline{R}_5 = \underline{R}_4\cdot L\underline{R}_5 = -\underline{R}_3\cdot L\underline{R}_6 = -\underline{R}_4\cdot L\underline{R}_6. \tag{209}$$

Unless either ψ in (165) is independent of h or we have the constant pitch case (180), we saw in the preceding section that the coefficients M,E,F,G will depend on s. This author does not yet know of any straightforward analytic method by which one can handle the stability problem for (200)-(203) when the coefficients depend on s. Hence we shall limit our further discussion in this paper to the constant pitch case.

In the constant pitch case (180), we assume that φ, $\underline{k}$ are given by (182) and (183), respectively. A direct calculation in (193) shows that *in this case $M = 0$, and the dispersion relation associated with a* solution of the type (184) of the system of equations (200)-(203) is easily found to be

$$q^4 - \{2G(F - E) + A^2(1 + \nu^2)\kappa^2\}q^2$$

$$+ 2G(F + E)A^2\nu\kappa^2 + A^4\nu^2\kappa^4 = 0. \tag{210}$$

A necessary and sufficient condition for exponential stability of the trivial solution of (200)-(203) in this case is seen from (210) to be that q^2 is real and nonnegative for all values of κ. For $\kappa = 0$ (210) therefore *gives the following necessary condition for stability of the* static screw pinch with constant pitch

$$G(F - E) \geq 0. \tag{211}$$

By introducing (85), (180), and (183) into (194), (195), (196), (207), (208), and (209) we obtain

$$G = -\frac{\delta}{k}c_0(2P\rho_0)^{-\frac{1}{2}}C^2H, \tag{212}$$

$$F - E = -\delta(k\rho_0c_0)^{-1}(2P\rho_0)^{-\frac{1}{2}}\left(\frac{1}{r}D^2 + rC^2\right)\left(p_0' - \frac{\rho_0c_0^{\,2}}{H}H'\right)H. \tag{213}$$

Thus we see that (211) is the well-known interchange stability criterion which usually is written *in the following way when $B_\theta \neq 0$:*

$$\frac{dp_0}{dr} - \frac{\gamma p_0}{B_z}\frac{dB_z}{dr} \geq 0. \tag{214}$$

For κ arbitrary (210) implies

$$q^2 = G(F - E) + \frac{1}{2} A^2 (1 + v^2)\, \kappa^2 \pm \{\Gamma(\kappa)\}^{\frac{1}{2}}. \tag{215}$$

$$\Gamma(\kappa) = G^2 (F - E)^2 + \frac{1}{4} A^4 (1 - v^2)^2 \kappa^4$$
$$+ G[F(1 - v)^2 - E(1 + v)^2] A^2 \kappa^2 . \tag{216}$$

Clearly q^2 is real for all values of κ if and only if

$$\Gamma(\kappa) \geq 0 \tag{217}$$

for all values of κ. Hence (217) is also a necessary condition for stability of the pinch. From (216) it is clear that (217) is always satisfied when $|\kappa|$ is sufficiently large. If

$$G[F(1 - v)^2 - E(1 + v)^2] \geq 0 \tag{218}$$

(216) shows that (217) is satisfied for all values of κ and $\Gamma(\kappa)$ has its minimum value at $\kappa = 0$. If on the other hand

$$G[F(1 - v)^2 - E(1 + v)^2] < 0, \tag{219}$$

$\Gamma(\kappa)$ is found to have its minimum value at

$$\kappa_m = \left\{\frac{2G[E(1 + v)^2 - F(1 - v)^2]}{A^2 (1 - v^2)^2}\right\}^{\frac{1}{2}}. \tag{220}$$

Thus when (219) is satisfied, (217) holds for all values of κ if and only if

$$\Gamma(\kappa_m) \geq 0. \tag{221}$$

By substituting (220) into (216), (221) is seen to be equivalent to

$$G^2 (F - E)^2 \geq \frac{G^2 [E(1 + v)^2 - F(1 - v)^2]^2}{(1 - v^2)^2} . \tag{222}$$

In view of (211) we have therefore established the following necessary condition for stability when (219) is satisfied:

$$G(F - E) \geq \frac{G[E(1 + \nu)^2 - F(1 - \nu)^2]}{1 - \nu^2} . \quad (223)$$

In view of (211) we also see that (223) is a necessary condition for stability when (219) is not satisfied. By simple manipulations (223) can be rewritten in the following way:

$$G(F - E) \geq \nu G(F + E). \quad (224)$$

Analogous to (213) we find that

$$F + E = - k^{-1} \rho_0^{-\frac{1}{2}} \underline{k} \cdot (\nabla \times \underline{B}_0)$$

$$= - k^{-1} \rho_0^{-\frac{1}{2}} \underline{B}_0^{\,2} (\underline{k} \times \underline{B}_0) \cdot \nabla p_0 = - \frac{\delta}{k} \frac{1}{r \rho_0^{\frac{1}{2}} H} \frac{dp_0}{dr} . \quad (225)$$

From (205), (212), (213), and (225) it follows that when $B_\theta \neq 0$ (224) is equivalent to

$$\frac{dp_0}{dr} - \frac{\gamma p_0}{B_z} \frac{dB_z}{dr} \geq \frac{\gamma p_0}{B_0^{\,2}} \frac{dp_0}{dr} . \quad (226)$$

We have not seen this necessary condition (226) for stability of a screw pinch with constant pitch in the literature before. Since (226) is markedly different from the interchange stability criterion (215) especially for high beta plasmas, we presume (226) is a remnant from the so-called ballooning modes which are modes with short wavelengths and which therefore can be expected to be present in our system of transport equations. As will be seen below, however, the Suydam criterion is more restrictive than (226). This is probably the reason why (226) does not appear in the conventional approaches to the stability problem for the screw pinch with constant pitch.

From the above calculations it is clear that the necessary conditions for stability (211) and (224) imply that (217) is satisfied and therefore that q^2 is real. In order that q^2 shall also be nonnegative, it follows from (210) that we have to require

$$2G(F + E)A^2\nu\kappa^2 + A^4\nu^2\kappa^4 \geq 0 \tag{227}$$

for all values of κ. Obviously this is true if and only if

$$G(F + E) \geq 0. \tag{228}$$

We may therefore conclude that the trivial solution of (200)-(203) is exponentially stable for the constant pitch case if and only if (224) and (228) are satisfied. From (212) and (225) it follows that (228) is equivalent (when $B_\theta \neq 0$) to

$$\frac{dp_0}{dr} \geq 0, \tag{229}$$

which is the Suydam criterion for the constant pitch case[7].

Of the above necessary conditions for exponential stability of the static screw pinch with constant pitch, the Suydam criterion (229) is certainly the most restrictive one. In fact, it easily follows from (85) and (180) that

$$-\frac{1}{B_z}\frac{dB_z}{dr} \geq \frac{1}{B_0^2}\frac{dp_0}{dr} . \tag{230}$$

Hence (226) and (214) will be satisfied if (229) is satisfied. Equality in the different criteria (214), (226), and (229) are, however, the critical values where the character of the local frequency q given by (215) and (216) changes; they all therefore have significance for the modes involved. Clearly the Suydam criterion (229) cannot be satisfied for a cylindrical plasma surrounded by vacuum, and hence magnetic shear is definitely necessary in order to stabilize such screw pinches. There has been some discussion earlier in the literature about the necessity of the criterion (229) for a screw pinch with constant pitch; any doubt about this question has hopefully now been removed.

XIII. DISCUSSION

In this work we have been studying asymptotic expansions of linear wave solutions valid for short wavelengths. The waves are superimposed on an arbitrarily given solution of the ideal magnetohydrodynamic equations. Since we are concerned with waves with short wavelengths, it would have been desirable to include resistivity and possibly also other effects, but it is not clear how the asymptotic method we apply can be modified to include such effects. Thus we have been working entirely within the framework of ideal magnetohydrodynamics.

Within this framework the method applied has been shown to be powerful. The equations describing the propagation of the waves have been derived in a form which is entirely independent of the coordinate system, thus we do not have to deal with the special difficulties associated with, for instance, Hamada coordinates. It has not been necessary to introduce any assumption beyond the usual regularity assumptions on the arbitrarily given solution that represent our basic state, it may even be time dependent. Since our theory allows a gravitational potential as well as a flow in the basic state, it may be applied both in astrophysics and in problems related to thermonuclear fusion.

Plasma flow is clearly present in a rotating star. Large flows have also been observed in fusion devices after heating plasmas by neutral beams. The amount of theoretical work on waves and stability done on plasmas with flow is quite limited in comparison with static systems. This is mainly due to the increasing complexity of the problem. The energy principle of Bernstein et al.[16], which is the dominating approach to the stability problem, is, for instance, usually not applicable to problems with flow. The methods which have been applied to waves and stability problems for plasmas with flow are usually not satisfactory in one way or another, our method therefore seems very promising for such problems. In the examples discussed in this paper the effect of a basic flow has barely been touched, but we hope to take up such applications of our theory in the near future. The prospects seem very good since the method has been shown to give useful results for problems with flow in ordinary fluid dynamics by Eckhoff & Storesletten[3,4].

Our discussion of slabs and screw pinches shows that it is possible in special cases to obtain detailed analytic results for the wave solutions. It is not yet known to what extent it is possible to derive analytic results for more general cases; only future research can decide that. However, the equations we have derived for the propagation of waves seem extremely attractive for numerical methods. Since traditional numerical codes do not comprise waves with short wavelengths, a numerical code for our transport equations will therefore amend this deficiency of the traditional codes. It seems reasonable to believe that our approach will make it possible to get information about the continous spectrum (the essential spectrum) associated with the traditional normal mode approach (see Ref.5). However, since our approach is clearly different from the traditional approaches, the difficulties in applying the traditional approaches are not present for our method. Only future research can reveal the amount of difficulties involved in our approach.

Even though our method of approach does not depend on symmetries in the basic state, it must be emphasized that it is a hard task to obtain a solution of the basic ideal magnetohydrodynamic equations that is not symmetric. In fact, at least if external forces are neglected, Grad[11] claims that nonsymmetric static solutions are virtually nonexistent. However, even when the basic state is only known numerically, our method seems well suited to describe superimposed linear waves numerically. The superimposed linear waves may contain important information about the stability of the basic state, giving both the growth rates and the structure of the unstable modes. Thus it may be possible to get information about the prospects for observing the calculated solution in an actual experiment.

Due to the complicated equations involved in describing the propagation of the magnetoacoustic waves, we have not been able to decide to what extent those waves may describe possible instabilities when they do not interact with the other wave types. The mass waves and the Alfvén waves, on the other hand, have been shown to represent stable perturbations of the basic state as long as they do not interact. The exceptional case, where instabilities have been detected, is the case where the local wavenumber vector $\underline{k}$ is perpendicular to the magnetic fieldlines. In this case we have shown that the mass waves, the Alfvén waves, and the slow magnetoacoustic waves will persistently interact, and that this interaction may give rise to instabilities. In view of the asymptotic expansion we have applied, this means that waves which have short wavelengths perpendicular to the magnetic field but long wavelengths parallel to it, appear in our approach to be the most critical ones in a stability research. These results are consistent with results found earlier by other methods (see Refs. 5 and 10).

In the exceptional case where $\underline{k} \cdot \underline{B}_0 = 0$ we find that the interacting mass waves, Alfvén waves, and slow magnetoacoustic waves are described by a weakly coupled hyperbolic system. When the basic state possesses common magnetic flow-flux surfaces, the structure of that hyperbolic system shows that there will exist modes which essentially are localized to these surfaces. This is consistent with the results found by Hameiri[17]. Similarly, the structure of the hyperbolic system shows that for static basic states there will be interacting modes essentially localized to magnetic fieldlines. This is consistent with results discussed extensively in the literature[5,7-10,18].

The weakly coupled hyperbolic system found is called the system of transport equations for the interacting waves, since it is derived in essentially the same way as we derive the transport equations for the non-interacting waves in the nonsingular cases. Our calculations for this system of transport equations show that for almost all possible basic states it is possible to find perturbations which are growing linearly with respect to time. These algebraic instabilities resemble the anholonomic instabilities detected earlier by Lortz & Rebhan[12] and Grad[11]. They are excluded from the energy principle by Bernstein et al.[16] since the perturbations are restricted there by the chosen Lagrange-displacement representation. We can avoid these instabilities if we restrict the set of perturbations considered to the case where $\sigma_2 = 0$. We do not claim, however, that algebraic instabilities cannot appear when $\sigma_2 = 0$. In fact, it is very likely that algebraic instabilities can appear in marginal cases analogous to the cases discussed in ordinary fluid mechanics by Eckhoff & Storesletten[3,4] even when we have $\sigma_2 = 0$. In this paper we have not looked for such instabilities, however; we have restricted our study to exponentially growing modes when $\sigma_2 = 0$. We would like to remark that it is an open question what physical significance the detected algebraic instabilities have, since it seems possible that they may be dominated by effects we have neglected (nonlinearities, resistivity, etc.).

It is not surprising that the transport equations simplify considerably when we restrict our study to waves superimposed on a static basic state, compared to more general cases with flow. As mentioned above, the transport equations will then in the case $\underline{k}\cdot\underline{B}_0 = 0$ describe waves which essentially are localized to magnetic fieldlines. By (117) $\underline{k} = \nabla\varphi$ is in this case seen to be independent of t, and hence $\underline{k}$ may be determined by the method of characteristics applied to $\underline{k}\cdot\underline{B}_0 = 0$. Thus $\underline{k}$ is determined by ordinary differential equations along the magnetic fieldlines, and these may be solved in conjunction with the transport equations. As discussed in Sec.XII, the transport equations will be further substantially simplified if we, in addition, neglect external forces. If we then take the Fourier transform of the transport equations with respect to time, we will get a system of equations which has a structure similar to the ballooning mode equations[5,10]. It must be stressed here, however, that it is not yet clear exactly how our transport equations are related to the ballooning mode equations. Our transport equations are derived directly from the fundamental magnetohydrodynamic equations (1) by a method which is a generalization of the WKB method, while the ballooning mode equations are obtained by a method of WKB type for the variational problem for the potential energy δW[10]. It thus seems reasonable to conjecture that our system of transport equations does describe the propagation of the ballooning modes properly, and hence that our method gives the generalization of these waves superimposed on arbitrary basic states. In particular it seems beyond doubt that the Suydam[7] and the Mercier[8,9] criteria for stability must be deducible from our transport equations; we hope to be able to do that sometime in the future.

The traditional ballooning representation and the associated Fourier transform technique[10] involve as yet unresolved issues related to the convergence of the series present in such representations and their connection with the physical eigenfunctions. These difficulties are not present in our method of approach since we derive our transport equations directly from the dynamic equations governing the plasma. Furthermore, we avoid the problems associated with the spectrum since we consider the initial value problem for the transport equations along the magnetic fieldlines. Since we are studying linear waves, this does not cause any problems in toroidal geometry even though $\underline{k}$ usually will be multivalued there. We simply have to add up the waves which have the same toroidal angle modulo 2π. However, if we take the Fourier transform of the transport equations with respect to time, we will get a nonstandard eigenvalue problem for a system of ordinary differential equations along the magnetic fieldlines involving nonattractive difficulties. Neither from a numerical nor from an analytical point of view does this eigenvalue problem seem to have any advantages compared to the initial value problem as far as we can see, but only future research can clarify these points.

Even though gravity hardly is avoidable in earthbound experiments, it is customary to neglect external forces in studies related to thermonuclear fusion. Since the timescales involved in most fusion devices are very short, this may seem a reasonable approximation from a physical point of view, especially since this approximation leads to substantial simplifications in the model. Some evidence is available, however, which may call into question the validity of this approximation. First of course, this approximation affects the problem of determining basic states (preferably static ones). Second, it is known from fluid mechanics that even an arbitrarily small external force may change the stability criteria substantially[4]. In this connection it is natural to call attention to the interchange instabilities detected for the slab and the screw pinch in Secs.X and XII, respectively. At first glance these instabilities may seem completely analogous, but a closer look reveals that the structure of the unstable modes is different. The mass wave associated with σ_1 plays a fundamental role in the interchange instability for the slab where the external force is the driving force, while the fundamental role is played only by the Alfvén waves and the slow magnetoacoustic waves for the screw pinch where the magnetic field represents the driving force. Only future research can settle the question of how good the approximation is when we neglect external forces in the model. The approach we have described in this paper is applicable also without this simplifying assumption.

As a conclusion we may say that the asymptotic expansion applied in this paper has provided us with equations describing the propagation of linear waves superimposed on an arbitrary basic state where no superfluous assumptions are made. For a static basic state we have seen that the mass waves, the Alfvén waves, and the slow magnetoacoustic waves may be localized and interacting along the magnetic field-lines. Also for general basic states we have seen that those waves may interact, but they are then usually not localized to magnetic field-lines but to magnetic flow-flux surfaces if such surfaces exist for the basic state. The roles played by the asymptotic expansion applied and the obtained transport equations are completely understood, and since they describe special perturbations (short wavelength), they may be used to obtain necessary conditions for stability. It seems likely that the obtained transport equations describe the propagation of ballooning modes without suffering from the difficulties involved in the traditional description of those waves. Whether the necessary conditions obtained by our approach are also sufficient to insure stability of a given basic state, is a problem which can only be solved by comparing the obtained necessary conditions with sufficient conditions obtained by other methods. Such results do exist for problems in ordinary fluid mechanics[3], and Hameiri[17-19] has discussed that problem in magnetohydrodynamics.

APPENDIX: THE WAVECOUPLING COEFFICIENTS

In the singular case, where $\underline{k}\cdot\underline{B}_0 = 0$, the expressions $\underline{R}_l\cdot L\underline{R}_m$, l, $m = 1, \ldots, 6$, may be calculated directly from (114) and (6)-(9). After a considerable amount af algebra for the general basic flow (3), we find

$$\underline{R}_1 \cdot L\underline{R}_1 = \frac{\gamma}{2}\,\nabla\cdot\underline{v}_0, \tag{A.1}$$

$$\underline{R}_2 \cdot L\underline{R}_1 = 0, \tag{A.2}$$

$$\underline{R}_3 \cdot L\underline{R}_1 = 2^{-\frac{1}{2}}(kB_0c_0)^{-1}(\underline{k}\times\underline{B}_0)\cdot(\underline{v}_{0t} + \underline{v}_0\cdot\nabla\underline{v}_0 - \nabla V), \tag{A.3}$$

$$\underline{R}_4 \cdot L\underline{R}_1 = \underline{R}_3 \cdot L\underline{R}_1, \tag{A.4}$$

$$\underline{R}_5 \cdot L\underline{R}_1 = -\,2^{-\frac{1}{2}}(\varrho_0B_0c_0)^{-1}\underline{B}_0\cdot\nabla p_0, \tag{A.5}$$

$$\underline{R}_6 \cdot L\underline{R}_1 = \underline{R}_5 \cdot L\underline{R}_1, \tag{A.6}$$

$$\underline{R}_1 \cdot L\underline{R}_2 = 0, \tag{A.7}$$

$$\underline{R}_2 \cdot L\underline{R}_2 = \nabla\cdot\underline{v}_0 - k^{-2}\,\underline{k}\cdot(\nabla\underline{v}_0)\cdot\underline{k}, \tag{A.8}$$

$$\begin{aligned}\underline{R}_3 \cdot L\underline{R}_2 = {} & 2^{-\frac{1}{2}}\varrho_0^{-\frac{1}{2}}B_0^{-1}\underline{B}_0\cdot(\nabla\times\underline{B}_0) + 2^{-\frac{1}{2}}k^{-2}B_0^{-1}(\underline{k}\times\underline{B}_0)\cdot(\nabla\underline{v}_0)\cdot\underline{k} \\ & + 2^{-\frac{1}{2}}k^{-2}B_0^{-1}\,\underline{k}\cdot(\nabla\underline{v}_0)\cdot(\underline{k}\times\underline{B}_0),\end{aligned} \tag{A.9}$$

$$\begin{aligned}\underline{R}_4 \cdot L\underline{R}_2 = {} & 2^{-\frac{1}{2}}\varrho_0^{-\frac{1}{2}}B_0^{-1}\underline{B}_0\cdot(\nabla\times\underline{B}_0) - 2^{-\frac{1}{2}}k^{-2}B_0^{-1}(\underline{k}\times\underline{B}_0)\cdot(\nabla\underline{v}_0)\cdot\underline{k} \\ & - 2^{-\frac{1}{2}}k^{-2}B_0^{-1}\,\underline{k}\,(\nabla\underline{v}_0)\cdot(\underline{k}\times\underline{B}_0),\end{aligned} \tag{A.10}$$

$$\begin{aligned}\underline{R}_5 \cdot L\underline{R}_2 = {} & -\,2^{-\frac{1}{2}}k^{-1}\varrho_0^{-\frac{1}{2}}B_0^{-1}(\underline{k}\times\underline{B}_0)\cdot(\nabla\times\underline{B}_0) \\ & + 2^{-\frac{1}{2}}k^{-1}B_0^{-1}c_0(2P)^{-\frac{1}{2}}\underline{B}_0\cdot(\nabla\underline{v}_0)\cdot\underline{k} \\ & + 2^{-\frac{1}{2}}k^{-1}B_0^{-1}c_0(2P)^{-\frac{1}{2}}\underline{k}\cdot(\nabla\underline{v}_0)\cdot\underline{B}_0,\end{aligned} \tag{A.11}$$

$$\underline{R}_6 \cdot L\underline{R}_2 = -\, 2^{-\frac{1}{2}} k^{-1} \rho_0^{-\frac{1}{2}} B_0^{-1} (\underline{k} \times \underline{B}_0) \cdot (\nabla \times \underline{B}_0)$$

$$- 2^{-\frac{1}{2}} k^{-1} B_0^{-1} c_0 (2P)^{-\frac{1}{2}} \underline{B}_0 \cdot (\nabla \underline{v}_0) \cdot \underline{k}$$

$$- 2^{-\frac{1}{2}} k^{-1} B_0^{-1} c_0 (2P)^{-\frac{1}{2}} \underline{k} \cdot (\nabla \underline{v}_0) \cdot \underline{k} \qquad \text{(A.12)}$$

$$\underline{R}_1 \cdot L\underline{R}_3 = 2^{-\frac{1}{2}} (k\rho_0 B_0)^{-1} (\underline{k} \times \underline{B}_0) \cdot (c_0 \nabla \rho_0 - c_0^{-1} \nabla p_0), \qquad \text{(A.13)}$$

$$\underline{R}_2 \cdot L\underline{R}_3 = 0, \qquad \text{(A.14)}$$

$$\underline{R}_3 \cdot L\underline{R}_3 = \frac{3}{4} \nabla \cdot \underline{v}_0 - \frac{1}{4} \rho_0^{-\frac{1}{2}} \underline{B}_0 \cdot \nabla \rho_0, \qquad \text{(A.15)}$$

$$\underline{R}_4 \cdot L\underline{R}_3 = k^{-2} B_0^{-2} (\underline{k} \times \underline{B}_0) \cdot (\nabla \underline{v}_0) \cdot (\underline{k} \times \underline{B}_0)$$

$$+ k^{-2} \rho_0^{-\frac{1}{2}} B_0^{-2} (\underline{k} \times \underline{B}_0) \cdot (\nabla \underline{B}_0) \cdot (\underline{k} \times \underline{B}_0)$$

$$+ \frac{1}{4} \rho_0^{-\frac{3}{2}} \underline{B}_0 \cdot \nabla \rho_0 - \frac{1}{4} \nabla \cdot \underline{v}_0, \qquad \text{(A.16)}$$

$$\underline{R}_5 \cdot L\underline{R}_3 = \frac{1}{2} k^{-1} B_0^{-2} \{1 - c_0 (2P)^{-\frac{1}{2}}\} (\underline{k} \times \underline{B}_0) \cdot (\nabla \underline{v}_0) \cdot \underline{B}_0$$

$$- \frac{1}{2} k^{-1} B_0^{-2} \{1 + c_0 (2P)^{-\frac{1}{2}}\} \underline{B}_0 \cdot (\nabla \underline{v}_0) \cdot (\underline{k} \times \underline{B}_0)$$

$$+ \frac{1}{2} (k\rho_0 c_0)^{-1} (2P\rho_0)^{-\frac{1}{2}} (\underline{k} \times \underline{B}_0) \cdot \{\nabla p_0$$

$$- \rho_0 c_0^2 B_0^{-2} [\underline{B}_0 \cdot \nabla \underline{B}_0 + \nabla (\tfrac{1}{2} B_0^2)]\}$$

$$- \frac{1}{2} k^{-1} \rho_0^{-\frac{1}{2}} \underline{k} \cdot (\nabla \times \underline{B}_0), \qquad \text{(A.17)}$$

$$\underline{R}_6 \cdot L\underline{R}_3 = \frac{1}{2} k^{-1} B_0^{-2} \{1 + c_0 (2P)^{-\frac{1}{2}}\} (\underline{k} \times \underline{B}_0) \cdot (\nabla \underline{v}_0) \cdot \underline{B}_0$$

$$- \frac{1}{2} k^{-1} B_0^{-2} \{1 - c_0 (2P)^{-\frac{1}{2}}\} \underline{B}_0 \cdot (\nabla \underline{v}_0) \cdot (\underline{k} \times \underline{B}_0)$$

$$\frac{1}{2} (k\rho_0 c_0)^{-1} (2P\rho_0)^{-\frac{1}{2}} (\underline{k} \times \underline{B}_0) \cdot \{\nabla p_0$$

$$- \rho_0 c_0^2 B_0^{-2} [\underline{B}_0 \cdot \nabla \underline{B}_0 + \nabla (\tfrac{1}{2} B_0^2)]\}$$

$$- \frac{1}{2} k^{-1} \rho_0^{-\frac{1}{2}} \underline{k} \cdot (\nabla \times \underline{B}_0), \qquad \text{(A.18)}$$

$$\underline{R}_1 \cdot L\underline{R}_4 = \underline{R}_1 \cdot L\underline{R}_3 , \tag{A.19}$$

$$\underline{R}_2 \cdot L\underline{R}_4 = 0 , \tag{A.20}$$

$$\begin{aligned}\underline{R}_3 \cdot L\underline{R}_4 = {} & k^{-2} B_0^{-2} (\underline{k} \times \underline{B}_0) \cdot (\nabla \underline{v}_0) \cdot (\underline{k} \times \underline{B}_0) \\ & - k^{-2} \rho_0^{-\frac{1}{2}} B_0^{-2} (\underline{k} \times \underline{B}_0) \cdot (\nabla \underline{B}_0) \cdot (\underline{k} \times \underline{B}_0) \\ & - \frac{1}{4} \rho_0^{-\frac{3}{2}} \underline{B}_0 \cdot \nabla \rho_0 - \frac{1}{4} \nabla \cdot \underline{v}_0 ,\end{aligned} \tag{A.21}$$

$$\underline{R}_4 \cdot L\underline{R}_4 = \frac{3}{4} \nabla \cdot \underline{v}_0 + \frac{1}{4} \rho_0^{-\frac{3}{2}} \underline{B}_0 \cdot \nabla \rho_0 , \tag{A.22}$$

$$\begin{aligned}\underline{R}_5 \cdot L\underline{R}_4 = {} & \frac{1}{2} k^{-1} B_0^{-2} \{1 + c_0 (2P)^{-\frac{1}{2}}\} (\underline{k} \times \underline{B}_0) \cdot (\nabla \underline{v}_0) \cdot \underline{B}_0 \\ & - \frac{1}{2} k^{-1} B_0^{-2} (1 - c_0 (2P)^{-\frac{1}{2}}) \; \underline{B}_0 \cdot (\nabla \underline{v}_0) \cdot (\underline{k} \times \underline{B}_0) \\ & + \frac{1}{2} (k \rho_0 c_0)^{-1} (2P\rho_0)^{-\frac{1}{2}} (\underline{k} \times \underline{B}_0) \cdot \{\nabla p_0 \\ & - \rho_0 c_0^{2} B_0^{-2} [\underline{B}_0 \cdot \nabla \underline{B}_0 + \nabla (\frac{1}{2} B_0^{2})]\} \\ & + \frac{1}{2} k^{-1} \rho_0^{-\frac{1}{2}} \underline{k} \cdot (\nabla \times \underline{B}_0) ,\end{aligned} \tag{A.23}$$

$$\begin{aligned}\underline{R}_6 \cdot L\underline{R}_4 = {} & \frac{1}{2} k^{-1} B_0^{-2} (1 - c_0 (2P)^{-\frac{1}{2}}) (\underline{k} \times \underline{B}_0) \cdot (\nabla \underline{v}_0) \cdot \underline{B}_0 \\ & - \frac{1}{2} k^{-1} B_0^{-2} (1 + c_0 (2P)^{-\frac{1}{2}}) \; \underline{B}_0 \cdot (\nabla \underline{v}_0) \cdot (\underline{k} \times \underline{B}_0) \\ & - \frac{1}{2} (k \rho_0 c_0)^{-1} (2P\rho_0)^{-\frac{1}{2}} (\underline{k} \times \underline{B}_0) \cdot \{\nabla p_0 \\ & - \rho_0 c_0^{2} B_0^{-2} [\underline{B}_0 \cdot \nabla \underline{B}_0 + \nabla (\frac{1}{2} B_0^{2})]\} \\ & + \frac{1}{2} k^{-1} \rho_0^{-\frac{1}{2}} \underline{k} \cdot (\nabla \times \underline{B}_0) ,\end{aligned} \tag{A.24}$$

$$\underline{R}_1 \cdot L\underline{R}_5 = 2^{-\frac{1}{2}} \rho_0^{-1} B_0^{-1} \underline{B}_0 \cdot (c_0 \nabla \rho_0 - c_0^{-1} \nabla p_0) , \tag{A.25}$$

$$\underline{R}_2 \cdot L\underline{R}_5 = 0 , \tag{A.26}$$

$$\underline{R}_3 \cdot L\underline{R}_5 = k^{-1}B_0^{-2}\underline{B}_0\cdot(\nabla\underline{v}_0)\cdot(\underline{k}\times\underline{B}_0)$$

$$+ k^{-1}(2P\rho_0)^{-\frac{1}{2}}(c_0 + \frac{1}{2}\rho_0^{-1}B_0^{2}c_0^{-1})\underline{k}\cdot(\nabla\times\underline{B}_0)$$

$$- \frac{1}{2}(k\rho_0 c_0)^{-1}(2P\rho_0)^{-\frac{1}{2}}(\underline{k}\times\underline{B}_0)\cdot\nabla p_0$$

$$+ k^{-1}B_0^{-2}c_0(2P\rho_0)^{-\frac{1}{2}}(\underline{k}\times\underline{B}_0)\cdot(\nabla\underline{B}_0)\cdot\underline{B}_0, \qquad \text{(A.27)}$$

$$\underline{R}_4 \cdot L\underline{R}_5 = \underline{R}_3 \cdot L\underline{R}_5 \qquad \text{(A.28)}$$

$$\underline{R}_5 \cdot L\underline{R}_5 = \frac{1}{2}B_0^{-2}\{1 - c_0^{2}(2P)^{-1}\}\underline{B}_0\cdot(\nabla\underline{v}_0)\cdot\underline{B}_0$$

$$+ \{\frac{1}{4} + (\frac{1}{2}c_0^{2} + \frac{\gamma}{4}\rho_0^{-1}B_0^{2})(2P)^{-1}\}\nabla\cdot\underline{v}_0$$

$$+ \frac{1}{2}\underline{B}_0\cdot\nabla\{c_0(2P\rho_0)^{-\frac{1}{2}}\}, \qquad \text{(A.29)}$$

$$\underline{R}_6 \cdot L\underline{R}_5 = \frac{1}{2}B_0^{-2}\{1 + c_0^{2}(2P)^{-1}\}\underline{B}_0\cdot(\nabla\underline{v}_0)\cdot\underline{B}_0$$

$$+ \{\frac{1}{4} - (\frac{1}{2}c_0^{2} + \frac{\gamma}{4}\rho_0^{-1}B_0^{2})(2P)^{-1}\}\nabla\cdot\underline{v}_0$$

$$+ (\frac{\gamma}{4} - 1)(\rho_0 c_0)^{-1}(2P\rho_0)^{-\frac{1}{2}}\underline{B}_0\cdot\nabla p_0$$

$$+ \frac{1}{2}\rho_0^{-\frac{1}{2}}c_0 B_0^{-2}\underline{B}_0\cdot\nabla\{(2P)^{-\frac{1}{2}}B_0^{2}\}, \qquad \text{(A.30)}$$

$$\underline{R}_1 \cdot L\underline{R}_6 = \underline{R}_1 \cdot L\underline{R}_5, \qquad \text{(A.31)}$$

$$\underline{R}_2 \cdot L\underline{R}_6 = 0, \qquad \text{(A.32)}$$

$$\underline{R}_3 \cdot L\underline{R}_6 = k^{-1}B_0^{-2}\underline{B}_0\cdot(\nabla\underline{v}_0)\cdot(\underline{k}\times\underline{B}_0)$$

$$- k^{-1}(2P\rho_0)^{-\frac{1}{2}}(c_0 + \frac{1}{2}\rho_0^{-1}B_0^{2}c_0^{-1})\,\underline{k}\cdot(\nabla\times\underline{B}_0)$$

$$+ \frac{1}{2}(k\rho_0 c_0)^{-1}(2P\rho_0)^{-\frac{1}{2}}(\underline{k}\times\underline{B}_0)\cdot\nabla p_0$$

$$- k^{-1}B_0^{-2}c_0(2P\rho_0)^{-\frac{1}{2}}(\underline{k}\times\underline{B}_0)\cdot(\nabla\underline{B}_0)\cdot\underline{B}_0, \qquad \text{(A.33)}$$

$$\underline{R}_4 \cdot L\underline{R}_6 = \underline{R}_3 \cdot L\underline{R}_6, \qquad \text{(A.34)}$$

$$\underline{R}_5 \cdot L\underline{R}_6 = \frac{1}{2} B_0^{-2} \{1 + c_0^2 (2P)^{-1}\} \underline{B}_0 \cdot (\nabla \underline{v}_0) \cdot \underline{B}_0$$

$$+ \{ \frac{1}{4} - (\frac{1}{2} c_0^2 + \frac{\gamma}{4} \rho_0^{-1} B_0^2)(2P)^{-1} \} \nabla \cdot \underline{v}_0$$

$$+ (1 - \frac{\gamma}{4})(\rho_0 c_0)^{-1} (2P\rho_0)^{-\frac{1}{2}} \underline{B}_0 \cdot \nabla p_0$$

$$- \frac{1}{2} \rho_0^{-\frac{1}{2}} c_0 B_0^{-2} \underline{B}_0 \cdot \nabla \{(2P)^{-\frac{1}{2}} B_0^2\}, \qquad (A.35)$$

$$\underline{R}_6 \cdot L\underline{R}_6 = \frac{1}{2} B_0^{-2} \{1 - c_0^2 (2P)^{-1}\} \underline{B}_0 \cdot (\nabla \underline{v}_0) \cdot \underline{B}_0$$

$$+ \{ \frac{1}{4} + (\frac{1}{2} c_0^2 + \frac{\gamma}{4} \rho_0^{-1} B_0^2)(2P)^{-1} \} \nabla \cdot \underline{v}_0$$

$$- \frac{1}{2} \underline{B}_0 \cdot \nabla \{c_0 (2P\rho_0)^{-\frac{1}{2}}\}. \qquad (A.36)$$

In the manipulations necessary in order to obtain the above expressions, we have applied the assumption that $\underline{k} \cdot \underline{B}_0 = 0$ and that (3) satisfies (1). Furthermore, we have used that $\underline{k} = \nabla \varphi$ satisfies the equation

$$\underline{k}_t + \underline{v}_0 \cdot \nabla \underline{k} + (\nabla \underline{v}_0) \cdot \underline{k} = 0, \qquad (A.37)$$

which follows from (117) in view of the fact that $\nabla \underline{k}$ is a symmetric tensor.

REFERENCES

1. K.S. Eckhoff, J. Diff. Eq. 40, 94 (1981).

2. K.S. Eckhoff, J. Diff. Eq. 43, 281 (1982).

3. K.S. Eckhoff and L. Storesletten, J. Fluid Mech. 89, 401 (1978).

4. K.S. Eckhoff and L. Storesletten, J. Fluid Mech. 99, 433 (1980).

5. E. Hameiri, Comm. Pure Appl. Math. 38, 43 (1985).

6. E. Hameiri and P. Laurence, J. Math. Phys. 25, 396 (1984).

7. B.R. Suydam, in Proc. Second International Conference on the Peaceful Uses of Atomic Energy (United Nations, Geneva, 1958) 31, 157.

8. C. Mercier, Nucl. Fusion 1, 47 (1960).

9. C. Mercier, Nucl. Fusion Suppl. 2, 801 (1962).

10. J.W. Connor, R.J. Hastie and J.B. Taylor, Proc. R. Soc. Lond. A. 365, 1 (1979).

11. H. Grad, in Proc. of the Workshop on Mathematical Aspects of Fluid and Plasma Dynamics, Trieste, Italy, 1984, edited by C. Cercignani, S. Rionero, and M. Tessarotto (Università degli Studi di Trieste, Facoltà di Scienze, Istituto di Meccanica, Trieste, 1985), p.253.

12. D. Lortz and E. Rebhan, Phys. Lett., 35A, 236 (1971).

13. L. Cesari, Asymptotic Behavior and Stability Problems in Ordinary Differential Equations (Springer, Berlin, 1958).

14. W.A. Newcomb, Phys. Fluids 4, 391 (1961).

15. G.E. Forsythe and W.R. Wasow, Finite-Difference Methods for Partial Differential Equations (Wiley, New York, 1960).

16. I.B. Bernstein, E.A. Frieman, M.D. Kruskal, and R.M. Kulsrud, Proc. R. Soc. Lond. A 244, 17 (1958).

17. E. Hameiri, Phys. Fluids 26, 230 (1983).

18. E. Hameiri, Phys. Fluids 23, 889 (1980).

19. E. Hameiri, J. Math. Phys. 22, 2080 (1981).

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