UPTEC 87 05 R
JAN 1987

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UPTEC 8705 R

Date of issue
1987-02-02

Project name

Sponsoring organization

Author(s)
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Titel (swedish)

Titel (english)

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It is found that the non-circularity of the plasma cross-section always has a destabilizing effect but that the m = 1 mode can be stabilized by the external feedback if the non-circularity is small.

Keywords / indexing

Security

Language
English

Supplementary bibliographical information

Pages
35

Classification

ISSN 0346-8887

ISBN

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STABILITY PROPERTIES OF A TOROIDAL Z-PINCH IN AN EXTERNAL MAGNETIC MULTIPOLE FIELD

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Abstract

MHD stability of $m = 1$, axisymmetric, external modes of a toroidal z- pinch immersed in an external multipole field (Extrap configuration) is studied. The description includes the effects of a weak toroidicity, a non-circular plasma cross-section and the influence of induced currents in the external conductors.

It is found that the non-circularity of the plasma cross-section always has a destabilizing effect but that the $m = 1$ mode can be stabilized by the external feedback if the non-circularity is small.

1. Introduction

The equilibrium of a toroidal z-pinch situated in an external magnetic multipole field, Extrap configuration LEHNERT (1983), was studied by ERIKSSON (1986). The external multipole field is created by a number of circular conductors symmetrically arranged outside of the plasma.

The equilibrium problem was solved by expanding to leading orders in the inverse aspect ratio, $\varepsilon \ll 1$, and in a small parameter, δ, characterising the non-circularity of the plasma cross-section.

The aim of the present study is to investigate the linear MHD stability properties of these equilibria with regard to the external, axisymmetric, $m = 1$ mode and in the case of a uniform plasma current density.

The method used is the normal mode analysis and perturbation technique presented by BRYNOLF et.al. (1985). This metnod is here slightly generalized to include the effects of toroidicity, a more general shape of the plasma cross section and the feedback from induced currents in the external cunductors.

In the straight Extrap configuration, studied by BRYNOLF et.al. (1985), the equilibrium plasma is located around the central (symmetry) axis with respect to the external conductors.

In the toroidal configuration, however, the equilibria are generally displaced from the central region towards the outer conductor. In this case the non-circular plasma boundary is, to leading order, described by

$$r_p = 1 + \delta \sum_{n=2}^{N} A_n \cos n\omega \qquad (1)$$

where N is the number of external conductors. If N = 4 the cross-section is modulated by the combination of an elongation, a triangularity and a square shape. The stability properties of the toroidal plasma will depend on the combination of the different cos $n\omega$ terms.

The results indicate that the stability properties of the toroidal Extrap are essentially those of a straight pinch with the appropriate non-circular plasma cross-section. The effect of the bending of the plasma column is very small ($\varepsilon \ll 1$).

When the non-circularity is weak the axisymmetric m = 1 mode can be stabilized by the external feedback.

Experiments on the full toroidal configuration is at present in an initial phase. Some results concerning the creation of the discharge have been given by DRAKE et.al. (1986). The actual stability properties have not yet been studied.

2. Equation of Motion and Boundary Conditions.

The dynamical behaviour of the plasma is described by the linearized momentum balance equation together with the appropriate boundary conditions.

Assume a perturbation $\bar{\xi}$ with time dependence ~ exp $(i\gamma t)$. The equation of motion then reads

$$-\rho_m \gamma^2 \bar{\xi} = -\nabla p_1 + \bar{j}_1 \times \bar{B}_0 + \bar{j}_0 \times \bar{B}_1 \qquad (2)$$

Where ρ_m is the mass density and p is the pressure.

The equilibrium magnetic field $\bar{B}_0$ and current density $\bar{j}_0$ are given by

$$\bar{B}_0 = - \hat{\theta} \frac{1}{r} \times \nabla\Psi \qquad (3)$$

$$\bar{j}_0 = r \frac{d}{d\Psi} \; p_0(\Psi) \qquad (4)$$

where Ψ is the equilibrium magnetic flux function and the plasma current is in the $\hat{\theta}$ - direction, fig 1.

The corresponding perturbed quantities can be expressed in terms of $\bar{\xi}$ and $\bar{B}_0$ through

$$\bar{B}_1 = \nabla \times (\bar{\xi} \times \bar{B}_0) \qquad (5)$$

$$\bar{j}_1 = \nabla \times \bar{B} / \mu_0 \qquad (6)$$

The conditions $\nabla \cdot \bar{B} = 0$ and that the force on the plasma boundary must vanish give the boundary conditions

$$\hat{n} \cdot \left[\bar{B} \right] = 0 \qquad (7)$$

$$\left[p + \frac{B^2}{2\mu_0} \right] = 0 \tag{8}$$

$\hat{n}$ is a unit normal vector at the plasma surface and [ ] denotes the jump across the boundary.

An incompressible, i.e. $\nabla\cdot\bar{\xi} = 0$, two dimensional motion can be described by a vector potential $\Phi\hat{\theta}$ such that

$$\bar{\xi} = -\hat{\theta}\,\frac{1}{r} \times \nabla\Phi \tag{9}$$

$2\pi\Phi$ is the poloidal displacement integrated over a surface extending around the torus. By introducing Φ the equations are substantially simplified and analytical solutions can be derived for the m = 1 mode.

Inserting eq. (9) into eqs. (5) and (6) one obtains

$$\bar{B}_1 = -\hat{\theta}\,\frac{1}{r} \times \nabla\chi_p \tag{10}$$

$$\bar{j}_1 = -\hat{\theta}\,\frac{1}{\mu_0 r}\,\Delta^*\chi_p \tag{11}$$

$\chi_p \equiv \bar{B}_0\cdot\nabla\Phi$ is the perturbed magnetic flux function in the plasma region and

$$\Delta^* \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \,. \tag{12}$$

As is shown by BRYNOLF et.al. (1985) the equation of motion can be expressed in terms of the quantities $\bar{B}_0$, p_0 and Φ by first inserting eqs. (3), (4), (10) and (11) into eq. (2) and then taking the curl of this equation in order to eliminate ∇p_1.

The equations are written on a dimensionless form by letting

$B_0 \to B_{0s} \cdot B_0$

$\Phi \to Ra^2 \Phi$

$\chi \to Ra\ B_{0s}\chi$

$p \to I_p/a \cdot B_{0s}\ \chi$

$\gamma \to \omega_A \gamma$.

R = toroidal radius, a = plasma radius, I_p = total plasma current, ω_A = Alfven frequency and $B_{0s} = \mu_0\ I_p/(2\pi a)$.

The resultant form of eq (2) is

$$\gamma^2 \Delta^* \Phi = - (\bar{B}_0 \cdot \nabla - 2 \frac{a}{r} \bar{B}_0 \cdot r)\ \Delta^* \chi_p$$

$$- 2\pi \frac{d^2 p_0}{d\Psi^2} \ (\frac{r}{R})^2 (\bar{B}_0 \cdot \nabla)\ \chi_p \tag{13}$$

In the vacuum region $\nabla \times \bar{B}_1 = 0$.

Using the relation $\bar{B}_1 = -\hat{\theta}/r \times \nabla\chi_v$ one obtains

$$\Delta^* \chi_v = 0 \tag{14}$$

The boundary conditions are first linearized and it is assumed that there is no surface current in the equilibrium. Eq. (7) then gives

$$\chi_p - \chi_v = 0 \tag{15}$$

Following the procedure described by BRYNOLF el.al. (1985) the linearized form of the second boundary condition is operated on by $\bar{B}_0 \cdot \nabla$ and the momentum balance is used to eliminate the term $\bar{B}_0 \cdot \nabla p_1$. The result is

$$\gamma^2 \left(\frac{R}{r} \right)^2 \nabla\Psi \cdot \nabla\phi + 2\pi \frac{dp}{d\Psi} (\bar{B}_0 \cdot \nabla)\chi_p$$

$$+ \left[(\bar{B}_0 \cdot \nabla) \left(\frac{R}{r} \right)^2 + \left(\frac{R}{r} \right)^2 (\bar{B}_0 \cdot \nabla) \right] (\nabla\Psi \cdot \nabla) (\chi_p - \chi_v) = 0 \qquad (16)$$

3 Stability analysis

A local polar coordinate system (ϱ,ω) with the origin at the centre of the plasma is defined through (fig.1)

$$r = R + \varrho \cos \omega \qquad (17a)$$

$$z = \varrho \sin \omega \qquad (17b)$$

Using eq. (1) and specializing to a uniform current density ($dp/d\Psi = 1/\pi$) the equilibrium flux function in the plasma region is to leading orders in $\varepsilon \equiv a/R$ and δ, ERIKSSON (1986):

$$\Psi = \Psi_{00} + \delta\Psi_{01} + \varepsilon\Psi_{10} \qquad (18)$$

$$\Psi_{00} = -\frac{1}{2}(\varrho^2 - 1)$$

$$\delta\Psi_{01} = \sum_{n=2}^{N} A_n \varrho^n \cos n\omega$$

$$\Psi_{10} = -\frac{5}{8}(\varrho^2 - 1)\varrho \cos \omega$$

In Appendix A, Ψ is derived to second order in δ.

In analogy with BRYNOLF et. al. (1985) eqs. (13) - (16) are solved by expanding in ε and δ.

$$\Phi = \Phi_{00} + \delta\Phi_{01} + \delta^2\Phi_{02} + \ldots . + \varepsilon(\Phi_{10} + \delta\Phi_{11} + \ldots .) \qquad (19)$$

and correspondingly for χ, γ^2, Ψ and $\bar{B}$.

From eq (18) one obtains the operators

$$\bar{B}_{00}\cdot\nabla = -\frac{\partial}{\partial\omega} \qquad (20a)$$

$$\nabla\Psi_{00}\cdot\nabla = -\varrho\frac{\partial}{\partial\varrho} \tag{20b}$$

$$\bar{B}_{01}\cdot\nabla = \sum_{n=2}^{N} n A_n \left[\varrho^{n-1}\sin n\omega\frac{\partial}{\partial\varrho} + \varrho^{n-2}\cos n\omega\frac{\partial}{\partial\omega}\right] \tag{20c}$$

$$\nabla\Psi_{01}\cdot\nabla = \sum_{n=2}^{N} n A_n \left[\varrho^{n-1}\cos n\omega\frac{\partial}{\partial\varrho} - \varrho^{n-2}\sin n\omega\frac{\partial}{\partial\omega}\right] \tag{20d}$$

$$\bar{B}_{10}\cdot\nabla = -\frac{5}{8}\left[(\varrho^2-1)\sin\omega\frac{\partial}{\partial\varrho} + (\frac{7}{5}\varrho^2-1)\frac{\cos\omega}{\varrho}\frac{\partial}{\partial\omega}\right] \tag{20e}$$

$$\nabla\Psi_{10}\cdot\nabla = -\frac{5}{8}\left[(3\varrho^2-1)\cos\omega\frac{\partial}{\partial\varrho} - (\varrho^2-1)\frac{\sin\omega}{\varrho}\frac{\partial}{\partial\omega}\right] \tag{20f}$$

To first order in ε

$$\Delta^* = \frac{\partial}{\partial\varrho^2} + \frac{1}{\varrho}\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\frac{\partial}{\partial\omega^2} + \varepsilon\left[\frac{\sin\omega}{\varrho}\frac{\partial}{\partial\omega} - \cos\omega\frac{\partial}{\partial\varrho}\right] \tag{21}$$

$$\equiv \Delta + \varepsilon\Delta_1$$

<u>Zero order problem</u>

Assuming $\phi_{00} \sim \exp(im\omega)$ it is shown by BRYNOLF et.al. (1985) that when $m = 1$ the regular zero order solution of eqs. (13) - (16) is

$$\gamma_{00}^2 = 0 \tag{22}$$

$$\phi_{00} = -\varrho(e^{i\omega} + \sigma e^{-i\omega}) \tag{23}$$

$$\chi_{p00} = -i\varrho(e^{i\omega} - \sigma e^{-i\omega}) \tag{24}$$

$$\chi_{v00} = \chi_{p00} \cdot \frac{1}{\varrho^2} \qquad (25)$$

which is the well known result that a straight circular z-pinch is marginally stable ($\gamma_{00}^2 = 0$) with regard to a solid displacement.

Note that $\sigma = 1$ and $\sigma = -1$ represent, through eq (9), a vertical and horizontal displacement respectively.

<u>Order $\varepsilon^1 . \delta^0$</u>

To order $\varepsilon^1 \delta^0$ eqs. (13-16) become

$$(\bar{B}_{00} \cdot \nabla)\Delta\chi_{p10} = - \left[(\bar{B}_{00} \cdot \nabla)\Delta_1 + (\bar{B}_{10} \cdot \nabla)\Delta - 2\bar{B}_{00} \cdot r\Delta \right] \chi_{p00} \qquad (26)$$

$$\Delta\chi_{v10} + \Delta_1\chi_{v00} = 0 \qquad (27)$$

$$\chi_{p10} - \chi_{v10} = 0 \qquad (28)$$

$$-2 \ (\bar{B}_{00} \cdot \nabla) \ \chi_{p10} - (\bar{B}_{00} \cdot \nabla)(\nabla\Psi_{00} \cdot \nabla)(\chi_{p10} - \chi_{v10})$$

$$= \gamma_{10}^2 \nabla\Psi_{00} \cdot \nabla\phi_{00} + 2 \ (\bar{B}_{10} \cdot \nabla)\chi_{p00}$$

$$+ \left[(\bar{B}_{00} \cdot \nabla)(\nabla\Psi_{10} \cdot \nabla) + (\bar{B}_{10} \cdot \nabla)(\nabla\Psi_{00} \cdot \nabla) \right] (\chi_{p00} - \chi_{v00})$$

$$-2 \left[\varrho\cos\omega \ (\bar{B}_{00} \cdot \nabla) + \bar{B}_{00} \cdot r \right] (\nabla\Psi_{00} \cdot \nabla)(\chi_{p00} - \chi_{v00}) \qquad (29)$$

Or, with the operators and zero order quantities inserted:

$$\Delta \ \chi_{p10} = 0 \qquad , \qquad \varrho < 1 \qquad (30)$$

$$\Delta \chi_{v10} = \frac{i}{\varrho^2} (e^{i2\omega} - \sigma e^{-i2\omega}) \quad , \quad \varrho > 1 \tag{31}$$

$$\chi_{p10} - \chi_{v10} = 0 \quad , \quad \varrho = 1 \tag{32}$$

$$2 \frac{\partial \chi_{p10}}{\partial \omega} - \frac{\partial^2}{\partial \varrho \partial \omega} (\chi_{p10} - \chi_{v10}) = \gamma_{10}^2 (e^{i\omega} + \sigma e^{-i\omega})$$

$$- \frac{3}{2} (e^{i2\omega} + \sigma e^{-i2\omega}) \quad , \quad \varrho = 1 \tag{33}$$

Eqs. (30), (31) give

$$\chi_{p10} = \sum_{k=-\infty}^{\infty} a_k \varrho^{|k|} e^{ik\omega} \tag{34}$$

$$\chi_{v10} = \sum_{k=-\infty}^{\infty} b_k \varrho^{-|k|} e^{ik\omega} \tag{35}$$

Eq. (32) gives

$$a_k = b_k$$

Eq. (33) becomes

$$-2i \sum_{k=-\infty}^{\infty} k(|k| - 1) a_k e^{ik\omega} = \gamma_{10}^2 (e^{i\omega} + \sigma e^{-i\omega}) \tag{36}$$

+ terms ~ $\exp(\pm i2\omega)$.

$k = \pm 1$ then gives

$$\gamma_{10}^2 = 0 \tag{37}$$

Thus the toroidal frequency shift, associated with the leading order toroidal equilibria, $\Psi = \Psi_{00} + \varepsilon \Psi_{10}$, vanishes for both a vertical and a horizontal displacement.

Because the effect of the toroidicity is largest for m = 1 modes this suggests that the toroidal contribution to the stability properties of Extrap is in general very small.

Order ϵ^0, δ^1

Eqs. (13)-(16) become

$$\Delta\chi_{p01} = 0 \tag{38}$$

$$\Delta\chi_{v01} = 0 \tag{39}$$

$$\chi_{v01} = \chi_{p01} + f(\omega)\frac{\partial}{\partial\varrho}(\chi_{p00} - \chi_{v00}) \tag{40}$$

$$-2(\bar{B}_{00}\cdot\nabla)\chi_{p01} - (\bar{B}_{00}\cdot\nabla)(\nabla\Psi_{00}\cdot\nabla)(\chi_{p01} - \chi_{v01})$$

$$= \gamma_{01}^2\nabla\Psi_{00}\cdot\nabla\phi_{00} + 2(\bar{B}_{01}\cdot\nabla)\chi_{p00}$$

$$+\left[(\bar{B}_{00}\cdot\nabla)(\nabla\Psi_{01}\cdot\nabla) + (\bar{B}_{01}\cdot\nabla)(\nabla\Psi_{00}\cdot\nabla)\right](\chi_{p00} - \chi_{v00})$$

$$+f(\omega)\frac{\partial}{\partial\varrho}\left[2(\bar{B}_{00}\cdot\nabla)\chi_{p00} + (\bar{B}_{00}\cdot\nabla)(\nabla\Psi_{00}\cdot\nabla)\right](\chi_{p00} - \chi_{v00}) \tag{41}$$

Or, with the operators and zero order quantities inserted:

$$\Delta\chi_{p01} = 0 \tag{42}$$

$$\Delta\chi_{v01} = 0 \tag{43}$$

$$\chi_{v01} = \chi_{p01} - i\sum_{n=2} A_n\left[e^{i(n+1)\omega} - \sigma e^{-i(n+1)\omega} + e^{-i(n-1)\omega} - \sigma e^{i(n-1)\omega}\right] \tag{44}$$

$$2\frac{\partial\chi_{p01}}{\partial\omega} - \frac{\partial^2}{\partial\varrho\partial\omega}(\chi_{p01} - \chi_{v01}) = -\gamma_{01}^2(e^{i\omega} + \sigma e^{-i\omega})$$

$$- \Sigma A_n (n+1)^2 (e^{i(n+1)\omega} + \sigma e^{-i(n+1)\omega}) - \Sigma A_n (1-n^2)(e^{-i(n-1)\omega} + \sigma e^{i(n-1)\omega}) \tag{45}$$

Then

$$\chi_{p01} = \sum_{k=1}^{\infty} (a_{+k} e^{ik\omega} + a_{-k} e^{-ik\omega}) \varrho^k \tag{46}$$

$$\chi_{v01} = \sum_{k=1}^{\infty} \left[(b_{+k} \varrho^{-k} + c_{+k} \varrho^k) e^{ik\omega} + (b_{-k} \varrho^{-k} + c_{-k} \varrho^k) e^{-ik\omega} \right] \tag{47}$$

In order to incorporate the influence of induced currents in the external conductors the terms $c_k \varrho^{|k|} \exp(ik\omega)$ are retained in the perturbed vacuum flux χ_{v01}. These terms must then be identified with the induced flux from the external currents. The coefficients c_k are calculated, using Faradays law, in Appendix B.

Eq. (44) gives

$$b_{+k} + c_{+k} = a_{+k} - i(A_{k-1} - \sigma A_{k+1}) \tag{48a}$$

$$b_{-k} + c_{-k} = a_{-k} - i(A_{k+1} - \sigma A_{k-1}) \tag{48b}$$

The second boundary condition, eq (45), then gives, when k = 1

$$\gamma_{01}^2 = -2ic_1 + \sigma 2A_2 \tag{49a}$$

$$\sigma\gamma_{01}^2 = 2ic_{-1} + 2A_2 \tag{49b}$$

Because the coefficients A_2 and $c_{\pm 1}$ are independent one obtains, as is shown by BRYNOLF et.al. (1985), the condition that eqs. (49) are simultaneously solvable only when $\sigma = \pm 1$.

When k = 2,3,... eq. (45) gives

$$a_{+k} = \frac{k}{k-1} b_{+k} + \frac{i}{k-1} A_{k+1} \qquad (50a)$$

$$a_{-k} = \frac{k}{k-1} b_{-k} - \frac{i}{k-1} A_{k+1} \qquad (50b)$$

Using the values of $c_{\pm k}$ obtained in Appendix B (with $R_0/R_c = 20$) the first order frequency shift becomes

$$\gamma_{01}^2 = 0.0248 - 2A_2 \quad , \quad \sigma = -1 \qquad (51a)$$

$$\gamma_{01}^2 = 0.0246 + 2A_2 \quad , \quad \sigma = 1 \qquad (51b)$$

γ_{01}^2 is shown in fig. (2). The induced currents have a slightely stabilizing effect. The vertical mode becomes unstable when $|A_2| \gtrsim 0.012$ (note that $A_2 < 0$ for an elongation in the vertical direction).

Order ε^0, δ^2

Because the frequency shift γ_{02}^2 only appears in a term proportional to exp ($\pm i\omega$) in the second boundary condition it is sufficient to look at terms with this ω dependence only.

In the second order (ε^0, δ^2) stability analysis the second order equilibrium quantities are needed to form the operators ($\bar{B}_{02}\cdot\nabla$) and ($\nabla\Psi_{02}\cdot\nabla$). Calculating the terms where these operators appear it turns out that only the coefficients A_{22} and a_{22} (see Appendix A) are involved in terms proportional to exp ($\pm i\omega$).

Using the second order version of eqs. (13)-(16) and proceeding analogously as when deriving γ_{01}^2 one finds, after rather extensive but straightforward calculations

$$\gamma_{02}^2 = 2A_2^2 + \sum_{n=2} R_1(n+1)A_{n+1}^2 + \sigma\sum_{n=2} R_2(n)A_nA_{n+2} + \sigma 4a_{22} - \sigma 2A_{22}$$

$$-\sigma i6A_2c_1 - iA_{n-1}\sum_{n=2} R_3(n)c_n - iA_{n+1}\sum_{n=2} R_4(n)c_{-n} \qquad (52a)$$

$$\sigma\gamma_{02}^2 = \sigma 2A_2^2 + \sigma\sum_{n=2} R_1(n+1)A_{n+1}^2 + \sum_{n=2} R_2(n)A_nA_{n+2} + 4a_{22} - 2A_{22}$$

$$- i\,6\,A_2c_1 + iA_{n-1}\sum_{n=2} R_3(n)c_{-n} + iA_{n+1}\sum_{n=2} R_4(n)c_n \qquad (52b)$$

where

$$R_1(n) = -2\,\frac{(n-1)^3}{n-2}$$

$$R_2(n) = 2n^2 + 4n + 3$$

$$R_3(n) = 2n$$

$R_2(n) = \frac{2n^2}{n-1}$ $\qquad n = 2,3,...$

if $A_2 \neq 0$ it is known from the first order stability analysis that the polarisation of the eigenmodes is given by $\sigma = \pm 1$.

When $A_2 = 0$, $\sigma\cdot\left[\text{ eq. (52a)}\right]$ - eq. (52b) gives

$$0 = (\sigma^2-1)\{\Sigma R_2(n)A_nA_{n+2} + 4a_{22} + 2A_{22}\} \qquad (53)$$

where the relation $c_{\pm n} = -\sigma c_{\mp n}$ has been used. Eq. (53) implies $\sigma = \pm 1$ as expected.

Suppose $c_n = 0$ (no external feedback). The total frequency shift then takes the form

$$\gamma^2 = \gamma_{01}^2 + \gamma_{02}^2 + \gamma_{10}^2 = \sigma 2A_2 + 2A_2^2 + \sum_{n=2} R_1(n+1)A_{n+1}^2$$

$$+ \sigma \sum_{n=2} R_2(n)A_nA_{n+2} + \sigma\,(4a_{22} - 2A_{22}) \qquad (54)$$

If $A_2 = 0$ (no elongation) and the terms $A_nA_{n+2} = 0$ ($\rightarrow a_{22} = A_{22} = 0$) the two eigenmodes corresponding to $\sigma = \pm 1$ (horizontal and vertical) are both *unstable* ($\gamma^2 < 0$) with the same frequency shift (degeneration).

When $A_nA_{n+2} \neq 0$, i.e. when the modulation of the plasma cross-section is given by a combination of different cos nw terms, the degeneration is removed.

Suppose $A_2 \neq 0$. Eq. (54) is a second degree polynomial in A_2. If one solves for A_2 (letting $\gamma^2 = 0$) and assumes a weak non-circularity, i.e. $A_n << 1$ such that $\Sigma(R_1(n+1)A_{n+1}^2 \pm R_2(n)A_nA_{n+2}) \equiv b \pm c << 1$, one obtains the following conditions for stability:

$$A_2 > \frac{1}{2}\frac{b-c}{a^2} \qquad \sigma = 1 \tag{55a}$$

$$A_2 < -\frac{1}{2}\frac{b+c}{a^2} \qquad \sigma = -1 \tag{55b}$$

where $a = \frac{1}{2}(1+8A_n)$.

($a_{22} = A_2A_4$ and $A_{22} = \frac{7}{2}A_2A_4$ has been used.) Eqs. (55) cannot be simultaneously satisfied if a, b, c $\neq$ 0.

Hence a z-pinch of almost circular cross-section can never be m = 1 stable without external feedback.

Consider now an Extrap configuration with four external conductors i.e. $A_n \equiv 0$ when $n \geq 5$. Eq (54) becomes

$$\gamma^2 = \sigma 2A_2 + 2A_2^2 - 16A_3^2 - 27A_4^2 + \sigma 16A_2A_4 \tag{56}$$

When $A_n \ll 1$ the curves of marginal stability are shown in fig.3 for the vertical mode (upper curve) and for the horizontal mode (lower curve).

It is seen that it is not possible to choose A_2 and A_4 (A_3 is always destabilizing) such that both modes are stabilized simultaneously. By choosing an elongation which is large enough it is always possible to make the mode which is perpendicular to the elongation stable.

If the curves in fig.3 are extended it is found that the stability regions for the two modes overlap when the elongation is large $A_2 \gtrsim 0.5$ and there is a small square shape. This stability is however likely to be a defect of the model and a different method is needed to study the case when the non-circularity is large.

Consider now the case when the effect of the induced currents in the external conductors are taken into account. Eqs. (51) and (52) give

$$\gamma^2 = -2A_2 + 2A_2^2 - 16A_3^2 - 27A_4^2 - 16A_2A_4$$

$$-0.0735A_2 + 0.00339A_3 + 0.00153A_4 + 0.0248$$

when $\sigma = -1$ and (57a)

$$\gamma^2 = 2A_2 + 2A_2^2 - 16A_3^2 - 27A_4^2 + 16A_2A_4$$

$$+ 0.0728A_2 + 0.00152A_4 + 0.0246$$

when $\sigma = 1$ (57b)

The curves of marginal stability are shown in fig.4. Stability is found within the shaded region.

When the equilibrium is square shaped i.e. only $A_4 \neq 0$ the frequency shift is shown in fig.5.

Studying, as an example, the three equilibria given by ERIKSSON (1986), fig.6, it is seen that those two with weak non-circularity are stable due to the external feedback while the third, having a large elongation, lies outside the stability region.

4 Conclusion

It has been found that the effect on the frequency shift due to the weak toroidicity ($\varepsilon \ll 1$) is in general very small. The stability properties of the toroidal Extrap can be investigated by studying the corresponding straight case with the appropriate non-circularity of the plasma cross-section.

The effect of the non-circularity is always unfavourabe to stability. The strongest destabilizing effect is caused by an elongation in which case the plasma is always unstable with respect to a displacement in the direction of the elongation.

If the non-circularity is weak the axisymmetric, $m = 1$, mode is stabilized by the feedback from the external conductors. However this effect is not large enough to explain the stability of Extrap which has been observed in experiments.

As was shown by ERIKSSON (1986) the non-elongated toroidal equilibria are extremely sensitive to a small change of the currents in the external conductors. The high precision required in these currents is hardly available in the present experiment. Therefore the equilibria which are most likely to occur are those where the plasma column is displaced towards the outer conductor and has a cross-section with a large vertical elongation. As has been shown, these equilibria are highly unstable to a vertical displacement.

BRYNOLF (1985) has studied the influence of the current density profile on the stability properties of a straight z-pinch. A "peaking" of the current profile reduces the magnitude of the second order frequency shift, γ_{02}^2 , (i.e. the growth rate of the unstable modes due to a non-circular plasma cross-section which to leading order is given by $r_p = 1+ \cos n\omega$, $n = 3, 4, \ldots$) while the frequency shift due to an elongation of the plasma cross-section, γ_{01}^2 , is independent of the current profile.

This indicates that the highly unstable modes due to the elongated cross-section found in the toroidal equilibria will not be much influenced by a change of the current profile.

Acknowledgement

The Theoretical Electrotechnics Group in Uppsala, especially Dr. Christer Wahlberg, is gratefully acknowledged for helpful discussions.

Appendix A

Equilibrium to order ε^0, δ^2

The equilibrium flux function to order ε^0 is written

$$\Psi_0 = \Psi_{00} + \delta\Psi_{01} + \delta^2\Psi_{02} + \dots \qquad (A1)$$

The first order equilibrium i.e. $\delta\Psi_{01}$ has been calculated by ERIKSSON (1986) using the pressure profile

$$\frac{dp}{d\Psi} = (\kappa^2\Psi + C)\,\frac{1}{2\pi}$$

(The factor $1/2\pi$ is due to the normalisation used in this paper.)

In order to derive $\delta^2\Psi_{02}$ the normalization is done such that I_p/(the area of the cross-section) is constant to order δ^2. Then the equilibrium equations for Ψ_{02} in the plasma and vacuum regions are

$$\Delta\Psi_{02p} + \kappa^2\Psi_{02p} = 0 \qquad (A2)$$

$$\Delta\Psi_{02v} = 0 \qquad (A3)$$

The general ansatz for Ψ_0 is written

$$\Psi_{00p} = a_0\left[\, J_0(\kappa\varrho) - J_0(\kappa)\,\right]$$

$$\Psi_{01p} = \sum_{n=2}^{N} a_n \frac{J_n(\kappa\varrho)}{J_n(\kappa)} \cos n\omega$$

$$\Psi_{02p} = \sum_{n=1}^{2N} a_{2n} \frac{J_n(\kappa\varrho)}{J_n(\kappa)} \cos n\omega$$

$$\Psi_{00v} = b_0 \ln \varrho$$

$$\Psi_{01v} = \sum_{n=2}^{N} (b_n \varrho^{-n} + c_n \varrho^{n}) \cos n\omega$$

$$\Psi_{02v} = \sum_{n=1}^{2N} (b_{2n} \varrho^{-n} + c_{2n} \varrho^{n}) \cos n\omega \qquad (A4)$$

(Note that the c_n coefficients in eq (A4) have nothing to do with the c_k coefficients in eq (47).)

The plasma radius is written

$$r_p = 1 + \delta f_1(\omega) + \delta^2 f_2(\omega) \qquad (A5)$$

where

$$f_1(\omega) = \sum_{n=2}^{N} A_n \cos n\omega$$

$$f_2(\omega) = \sum_{n=2}^{2N} A_{2n} \cos n\omega$$

N is the number of external conductors. To order δ^k terms - $\cos n\omega$ $n \leq kN$ are included in r_p.

The flux function at the plasma boundary is, to order δ^2, given by

$$\Psi(r_p) = \Psi_{02}(1) + f_1(\omega) \left.\frac{\partial \Psi_{01}}{\partial \varrho}\right|_{\varrho=1} + \frac{f_1^{\,2}(\omega)}{2} \left.\frac{\partial^2 \Psi_{00}}{\partial \varrho^2}\right|_{\varrho=1}$$

$$+ f_2(\omega) \left.\frac{\partial \Psi_{00}}{\partial \varrho}\right|_{\varrho=1} \qquad (A6)$$

The constants a_{2n}, b_{2n}, c_{2n} A_{2n} are determined by the boundary conditions that $\Psi(r_p)$ = constant = 0 and $B_\omega(r_p)$ continous. After some calculations one obtains ($J_n \equiv J_n(\kappa)$):

$$a_{2n} = A_{2n} - \frac{1}{2} \Sigma\Sigma\, \kappa \frac{J'_j}{J_j} A_i A_j + \frac{1}{4} \kappa \frac{J'_1}{J_1} \Sigma\Sigma\, A_i A_j \qquad \text{(A7)}$$

n = 1, 2, ...

where

$$\Sigma\Sigma \equiv \left[\sum_{i=2} \sum_{j=2} (\text{such that } |i - j| = n) + \sum_{i=2} \sum_{j=2} (i + j = n) \right]$$

$$b_{2n} = A_{2n} - \frac{1}{4} \Sigma\, \Sigma\, A_i A_j - \frac{1}{2} \Sigma\, \Sigma\, A_i (b_j - c_j)\, j - c_{2n} \qquad \text{(A8)}$$

n = 1, 2, ...

$$c_{2n} = \frac{1}{2} A_{2n} \left[1 - \frac{1}{n} + \frac{\kappa}{n} \left(\frac{J'_n}{J_n} - \frac{J'_1}{J_1} \right) \right]$$

$$- \frac{1}{4n} \kappa \frac{J'_n}{J_n} \Sigma\, \Sigma\, \kappa \frac{J'_j}{J_j} A_i A_j$$

$$- \frac{1}{8} \left[1 - \frac{2}{n} + \frac{\kappa^2}{n} \left(- \frac{J'_1}{J_1} \frac{J'_n}{J_n} + \frac{J_3}{4 J_1} - \frac{3}{4} \right) \right] \Sigma\, \Sigma\, A_i A_j$$

$$- \frac{1}{4} \Sigma\, \Sigma\, A_i \left[(b_j - c_j) j + ((j - 1) b_j + (j + 1) c_j) \frac{j}{n} - \frac{1}{n} \kappa^2 \frac{J''_j}{J_j} A_j \right]$$

(A9)

n = 1, 2, ...

The coefficients $c_{2n} \equiv 0$ when $2 \leq n \leq N$ because the field from the external currents is identified with the c_n coefficients up to terms $\sim \cos N\omega$. When $N + 1 \leq n \leq 2N$ the terms including c_{2n} should be identified with the corresponding terms of the vacuum field. This determines A_{2n}.

In order to calculate the second order frequency shift a_{22} and A_{22} are needed.

When $N = 4$ and $\kappa \to 0$ eq. (A9) and (A7) give (uniform current density)

$$A_{22} = \frac{7}{2} A_2 A_4 \tag{A 10}$$

$$a_{22} = A_2 A_4 \tag{A 11}$$

Appendix B

The induced currents

The skin depth of a low frequency (γ) electromagnetic wave in a conductor (conductivity σ) is

$$\delta = (2/\mu_0 \sigma \gamma)^{1/2}$$

In Extrap the external conductors are made of coppar and the characteristic time for the development of a MHD-perturbation, the Alfven time, is $\tau \sim 10^{-6}$ s. Then $\delta \sim 10^{-4}$ m $\ll$ a (= radius of the conductor $\sim 5 \cdot 10^{-3}$ m).

Hence the external conductors behave as ideal conductors on the time scale of the perturbation.

In the case of an ideal current loop Faradays law implies that a current will be induced such that the total magnetic flux passing through the loop remains constant.

The flux through a circuit, i, can be expressed in terms of the mutual inductances M_{ij}, between circuit i and circuits j, and the current I_j as

$$\Psi_i = \sum_j I_j M_{ij} \qquad (B\ 1)$$

In the case of the Extrap configuration this gives a system of five algebraic equations.

M and I are then linearized with respect to a small displacement from the equilibrium position:

$$M_{ij} = M^0_{ij} + M^1_{ij} \qquad (B\ 2)$$

$$I_j = I^0_j + I^1_j \qquad (B\ 3)$$

With Φ_i = constant (equal to the equilibrium value) the system eqs. (B1) gives five equations for the induced currents I^1_j, $j = 1, 2, \ldots 5$, if the inductances are known.

The self inductance M_{ii} is easily calculated for a circular loop of toroidal radius R and small radius a by use of the well known formula

$$M_{ii} = \mu_0 R \left(\ln \frac{8R}{a} - \frac{7}{4} \right) \qquad (B\ 4)$$

When calculating the mutual inductances the radius of the external current rings and of the plasma are, for simplicity, taken to be zero. Then

$$M_{ij} = \frac{\mu_0}{4\pi} \int_i \int_j \frac{d\bar{s}_i \cdot d\bar{s}_j}{R_{ij}} \qquad (B\ 5)$$

M_{ij} has been calculated in the case when the plasma (= conductor 5) is placed at the centre $R = R_0$, $z = 0$ and with parameters $R_0/R_c = 20$, $a_{plasma}/R_0 = 0.01$ and $a_{conductor}/R_0 = 0.005$, fig.1.

The induced currents become $(I^1_n \rightarrow I^1_n/I_c)$ for a horizontal displacement

$$I^1_1 = -\ 0.0722 \quad \times \left(\frac{I_p}{I_c} \right)$$

$I_2^1 = I_4^1 = 0.0112$

$I_3^1 = 0.0520$

$I_5^1 = -\ 0.000827$ (B6)

and for a vertical displacement

$I_1^1 \approx I_3^1 \approx I_5^1 \approx 0$

$I_2^1 = -\ I_4^1 = -\ 0.0615 \quad \times \left(\frac{I_p}{I_c} \right)$ (B7)

The induced flux function

The magnetic flux function from four straight conductors takes the form ERIKSSON (1986)

$$\Psi = -\frac{\mu_0}{2\pi} \sum_{n=1}^{4} I_n \ln r_n \qquad \text{(B8)}$$

$$\ln r_n = \frac{1}{2} \ln R_c - \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\varrho}{R_c}\right)^k \cos k \left(\omega + \beta_n + \frac{\pi}{2}\right)$$

$$\beta_1 = -\frac{\pi}{2} \;,\quad \beta_2 = \pi \;,\quad \beta_3 = \frac{\pi}{2} \;,\quad \beta_4 = 0.$$

The origin of (ϱ, ω) is at $R = R_0$, $z = 0$.

Omitting the constant term the normalized Ψ takes the form, $\eta \equiv a/R_c$:

$$\Psi = \Sigma \frac{1}{2|n|} \eta^n \varrho^n e_n e^{in\omega} \qquad \text{(B9)}$$

$$e_{\pm 1} = I_1^1 - I_3^1 \mp i \, (I_2^1 - I_4^1)$$

$$e_{\pm 2} = I_1^1 - I_2^1 + I_3^1 - I_4^1$$

$$e_{\pm 3} = e_{\mp 1}$$

$$e_{\pm 4} = I_1^1 + I_2^1 + I_3^1 + I_4^1$$

$$e_{\pm 5} = e_{\pm 1}$$

With $\Phi_{00} = \varrho(e^{i\omega} + \sigma e^{-i\omega})$ eq. (9) gives

$\bar{\xi} = y$ if $\sigma = 1$

$\bar{\xi} = -ix$ if $\sigma = -1$.

Hence the horisontal displacement is imaginary and in the - x direction so that the corresponding $e_{\pm n}$ coefficients should be multiplied by - i. $c_{\pm n}$ in eq. (47) becomes using eq. (B6) and (B7):

For the horizontal mode (σ=-1)

$$c_{\pm 1} = i\ 0.0621 \ .\ \eta$$

$$c_{\pm 2} = i\ 0.0106\ \eta^2$$

$$c_{\pm 3} = i\ 0.0207\ .\ \eta^3$$

$$c_{\pm 4} = -i\ 0.00027\ .\ \eta^4$$

$$c_{\pm 5} = i\ 0.0124\ .\ \eta^5 \qquad \text{(B10)}$$

for the vertical mode (σ = 1)

$$c_{\pm} = \pm i\ 0.0615\ \eta$$

$$c_{\pm 2} = c_{\pm 4} = 0$$

$$c_{\pm 3} = \mp i\ 0.0205\ \eta^3$$

$$c_{\pm 5} = \pm i\ 0.0123\ \eta^5 \qquad \text{(B11)}$$

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Figure Captions

Fig. 1

The toroidal Extrap configuration. The external conductors (N = 4, with currents I_n) are symmetrically arranged around $r = R_0$, $z = 0$ in cylindrical coordinates (r, θ, z). The local polar coordinates (ρ, ω) and the plasma are centered around $r = R = R_0 + \Delta$, $z = 0$.

Fig. 2

The first order $(\sim \delta^1)$ frequency shift due to the elongation A_2 and the external feedback.

Fig. 3

The second order $(\sim \delta^2)$ stability regions without external feedback.

Fig. 4

Same as fig.3 including the external feedback. Stability is found within the shaded region.

Fig. 5

The frequency shift for a square shaped equilibrium with the external feedback.

Fig. 6

Equilibrium field patterns

a) $A_2 = -0.091$, $A_3 = -0.042$, $A_4 = -0.0087$

b) $A_2 \approx 0$, $A_3 = -0.0090$, $A_4 = -0.0040$

c) $A_2 \approx A_3 \approx 0$, $A_4 = -0.0085$

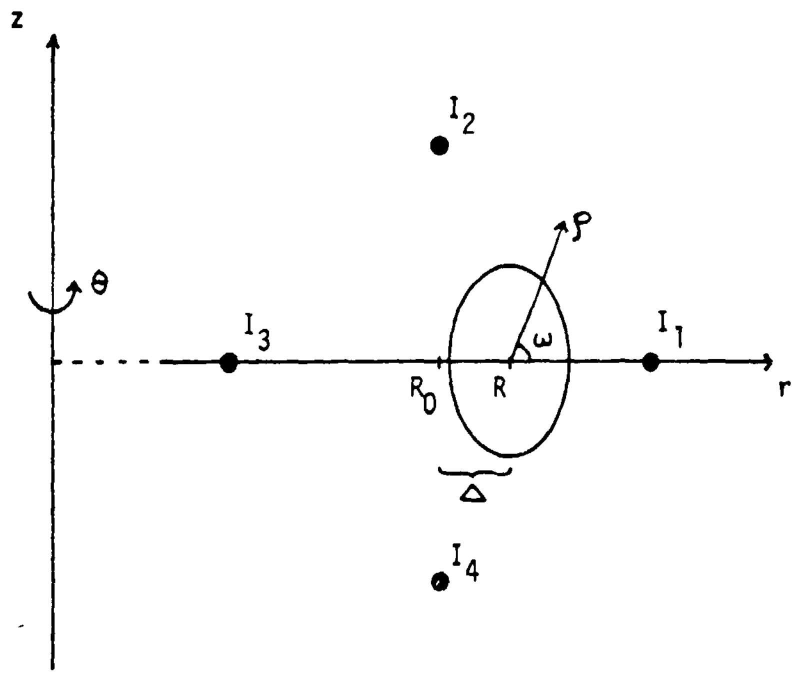
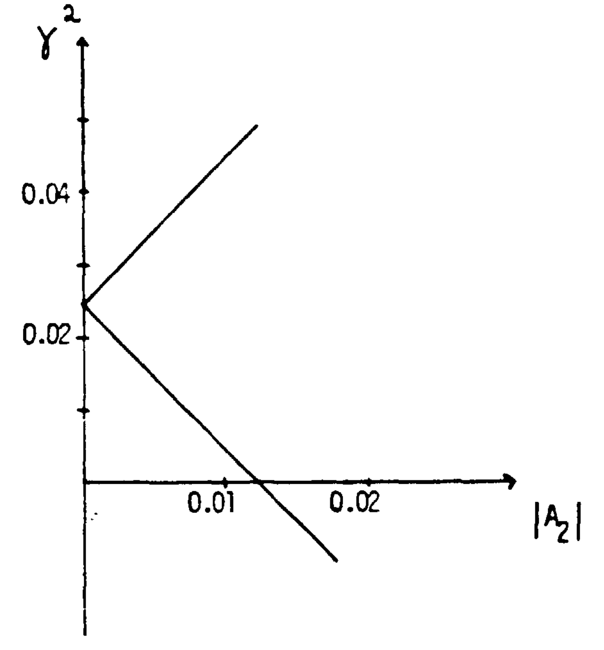
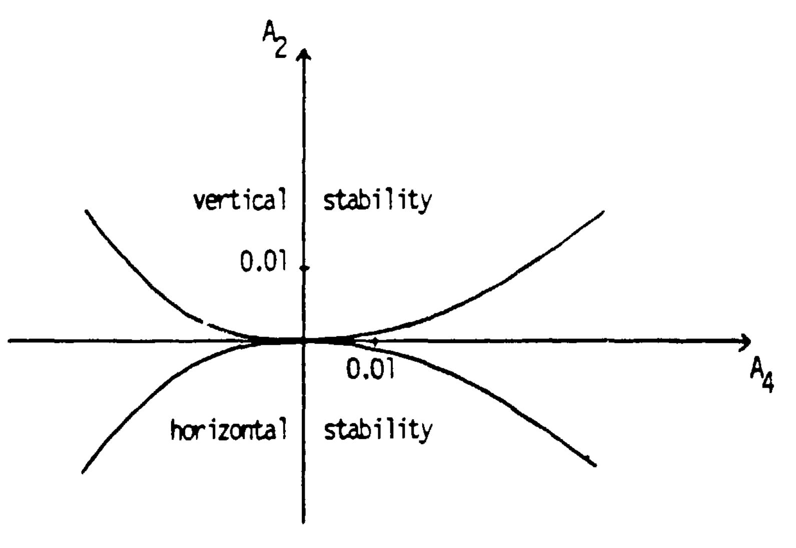


Fig. 1



γ²
0.04
0.02
0.01
0.02
|A₂|

Fig. 2



A₂
vertical stability
0.01
0.01
A₄
horizontal stability

Fig. 3

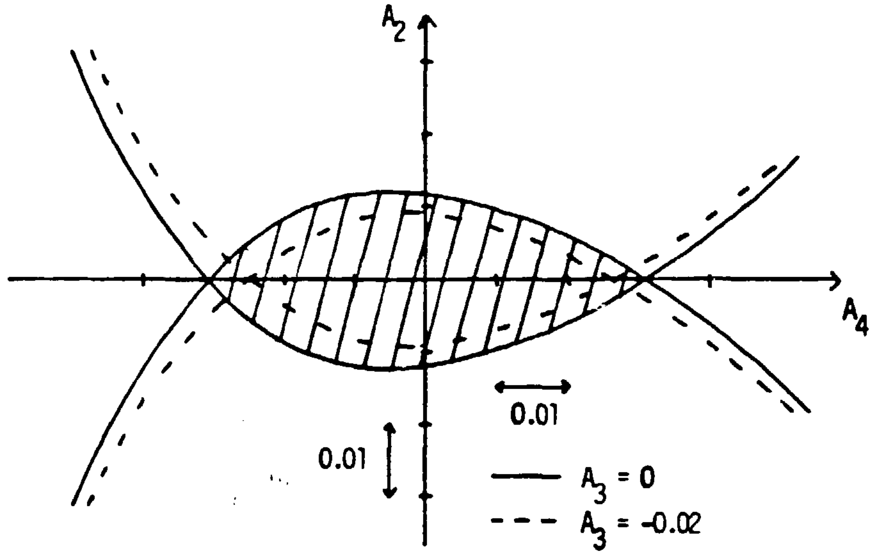


Fig. 4

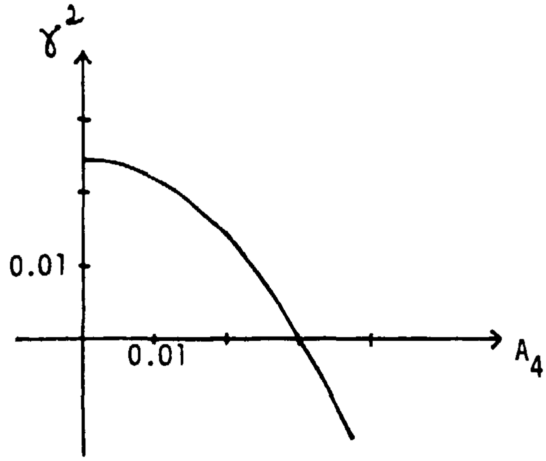


Fig. 5

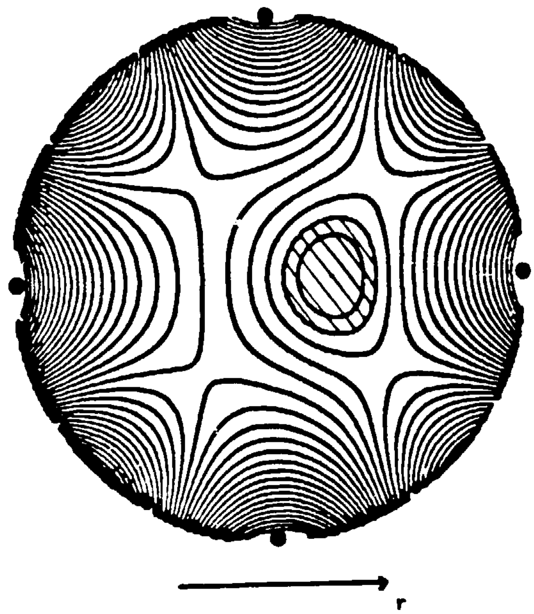


Fig. 6a

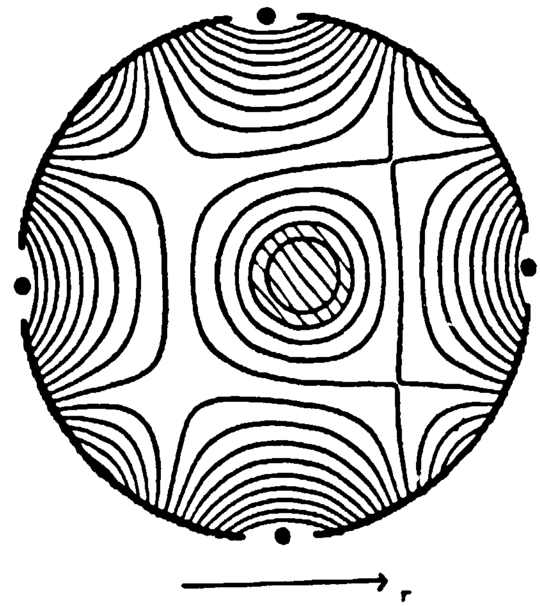


Fig. 6b

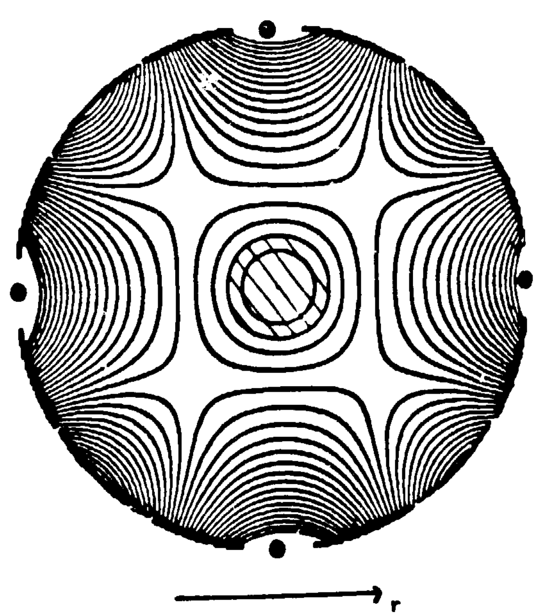


Fig. 6c