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INTEGRABILITY AND SYMMETRIC SPACES

II. The coset spaces

L. A. Ferreira(*)
Instituto de Física Teórica - UNESP
Rua Pamplona 145
01405 - São Paulo - Brasil

ABSTRACT

It is shown that a sufficient condition for a model describing the motion of a particle on a coset space to possess a Fundamental Poisson bracket Relation, and consequently charges in involution, is that it must be a symmetric space. The conditions a hamiltonian, or any function of the canonical variables, has to satisfy in order to commute with these charges are studied. It is show that, for the case of the non compact symmetric spaces, these conditions lead to an algebraic structure which plays an important role in the construction of conserved quantities.

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Introduction

The integrability properties of physical models have been the object of a lot of study in recent years. The motivation, from the physical point of view, comes from the interesting structures appearing in some theories, like gauge theories, due to their non linear aspects. For the mathematicians, integrable systems are of great interest because of the rich algebraic structure they possess.

In this paper we study the integrability properties of models defined on coset spaces which are symmetric spaces. It is an extension of the study we have done in ref. [1] for the group manifold. We show that if one considers the motion of a particle on a coset space G/K described by a non singular lagrangean, then a sufficient condition for such model to possess a Fundamental Poisson bracket Relation (FPR)[2] and consequently charges in involution, is that the coset space must be a symmetric space[3]. This result is independent of the dynamics of the model and therefore, the existence of charges in involution in these cases is a consequence of the algebraic and geometric structures of the symmetric spaces. We show that the coset spaces which are not symmetric possess an "anomalous" FPR that does not lead to charges in involution but provides a Lax pair equation[4] for the geodesic motion on these spaces.

We study the conditions a hamiltonian, or any function of the canonical variables, has to satisfy in order to commute with the charges in involution constructed from the FPR. For the case of the non compact symmetric spaces we show that these conditions lead to an algebraic structure that plays an important role in the construction of conserved charges. This result is related to the Iwasawa decomposition of G which endows the non compact symmetric spaces with a hidden group theoretic structure[5].

The paper is organized as follows: in section 2 we discuss some algebraic properties of coset spaces related to integrability. In section 3 we show that these structures lead to the FPR when the coset space is symmetric. The conditions for the charges to be conserved are discussed in section 4. In section 5 we show how the Iwasawa decomposition leads to a hidden group theoretic structure for the non compact symmetric spaces, and the algebraic structure related to this is discussed in section 6. In section 7 and 8 we show how to construct conserved charges using representation theory of Lie algebras.

2- Some properties of coset spaces

Let G be a semisimple Lie group with Lie algebra $\underset{\sim}{g}$ and K be a closed subgroup of G with Lie algebra $\underset{\sim}{k}$. We denote by Tr the Killing form of the algebra $\underset{\sim}{g}$ and by $\underset{\sim}{p}$ the orthogonal complement of $\underset{\sim}{k}$ in $\underset{\sim}{g}$. Then we can write

$$\underset{\sim}{g} = \underset{\sim}{p} + \underset{\sim}{k} \tag{2.1}$$

and

$$\mathrm{Tr}(\underset{\sim}{p}\,\underset{\sim}{k}) = 0 \tag{2.2}$$

It follows from the invariance properties of the Killing form and the fact that $\underset{\sim}{k}$ is a subalgebra that

$$[\underset{\sim}{k},\underset{\sim}{k}] \subset \underset{\sim}{k} \ , \ [\underset{\sim}{k},\underset{\sim}{p}] \subset \underset{\sim}{p} \ , \ [\underset{\sim}{p},\underset{\sim}{p}] \subset \underset{\sim}{p} + \underset{\sim}{k} \tag{2.3}$$

we denote by $\mathcal{P}$ and $(1-\mathcal{P})$ the orthogonal projections of $\underset{\sim}{g}$ onto $\underset{\sim}{p}$ and $\underset{\sim}{k}$ respectively[6].

We are interested in the algebraic and geometrical structures of the coset space G/K which are relevant for the integrability properties of the models defined on this space. Let q^r, $r = 1,2\ldots \dim G/K$, denote a set of local coordinates on G/K, and define the quantities

$$\Pi_r = \mathcal{P}(g^{-1}\partial_r g) \quad , \; g \in G \tag{2.4a}$$

$$K_r = (1-\mathcal{P})(g^{-1}\partial_r g) \quad , \; r = 1,2\ldots \dim G/K \tag{2.4b}$$

where the derivatives are w.r.t. the coordinates of G/K. Locally we can think of G as the direct product of G/K with K and therefore a set of local coordinates on G can be taken as the coordinates q^r of G/K plus some set of local coordinates on K. Notice that Π_r and K_r are the projections of some components of the Maurer-Cartan form onto $\underset{\sim}{p}$ and $\underset{\sim}{k}$ respectively. The quantities K_r may not all be independent.

There is a kind of gauge symmetry associated to the definition (2.4) of Π_r and K_r. If we multiply g from the right by an element k of K we obtain, using the fact that $\mathcal{P}(k^{-1}\partial_r k) = \mathcal{P}(k^{-1}K_r k) = 0$, that $\Pi_r(gk) = \mathcal{P}(k^{-1}\Pi_r(g)k)$. From the commutation relations (2.3) one observes that the conjugation by elements of the subgroup K is an operation that commutes with the projection of $\underset{\sim}{g}$ onto $\underset{\sim}{p}$ or $\underset{\sim}{k}$. So we have

$$\Pi_r(gk) = k^{-1}\,\Pi_r(g)\,k \tag{2.5a}$$

Analogously

$$K_r(gk) = k^{-1} K_r(g)\, k + k^{-1}\partial_r k \qquad (2.5b)$$

It follows from the definition of Π_r and K_r that they satisfy the identity

$$\partial_r(\Pi_s + K_s) - \partial_s(\Pi_r + K_r) + [\Pi_r + K_r, \Pi_s + K_s] = 0 \quad (2.6)$$

and so projecting it onto $\underset{\sim}{p}$ and $\underset{\sim}{k}$ we get

$$\partial_r \Pi_s - \partial_s \Pi_r + [\Pi_r, K_s] - [\Pi_s, K_r] + \mathcal{P}([\Pi_r, \Pi_s]) = 0 \quad (2.7a)$$

$$\partial_r K_s - \partial_s K_r + [K_r, K_s] + (1 - \mathcal{P})([\Pi_r, \Pi_s]) = 0 \quad (2.7b)$$

These two relations are invariant under the gauge transformations (2.5).

We define the metric tensor on the coset space G/K as

$$\gamma_{rs} = \mathrm{Tr}(\Pi_r \Pi_s) \qquad (2.8)$$

It is invariant by the gauge transformations (2.5) and also by global left translations by elements of G ($g \to g'g$, $\partial_r g' = 0$). The covariant and contravariant forms of Π , as a consequence of (2.8), satisfy

$$\mathrm{Tr}(\Pi_r \Pi^s) = \delta_r{}^s \quad , \quad \Pi^r = \gamma^{rs}\,\Pi_s \tag{2.9}$$

Using (2.8), (2,7a) and the fact that (2.2) implies $\mathrm{Tr}\{\Pi_r\,\mathcal{P}([\Pi_s,\Pi_t])\} = \mathrm{Tr}(\Pi_r[\Pi_s,\Pi_t])$ one can easily check that the Christoffel symbol for G/K is given by

$$\left\{ {u \atop rs} \right\} = \frac{1}{2}\,\mathrm{Tr}\left\{\Pi^u\left(\partial_r\Pi_s + \partial_s\Pi_r - [\Pi_s,K_r] - [\Pi_r,K_s]\right)\right\} \tag{2.10a}$$

Since the set $\left\{\Pi_r\ ,\ r = 1,2,\ldots \dim G/K\right\}$ constitutes a basis for the subspace $\underset{\sim}{p}$ of $\underset{\sim}{g}$ we can write, using (2.9), for any element P of $\underset{\sim}{p}$, that

$$P = \mathrm{Tr}(\Pi^u P)\,\Pi_u \tag{2.11}$$

then we can write (2.10a) as

$$\left\{ {u \atop rs} \right\}\Pi_u = \frac{1}{2}\left(\partial_r\Pi_s + \partial_s\Pi_r - [\Pi_s,K_r] - [\Pi_r,K_s]\right) \tag{2.10b}$$

Therefore the covariant derivative of Π_r w.r.t. to the Christoffel symbol is given by

$$\mathcal{D}_r\Pi_s \equiv \partial_r\Pi_s - \left\{ {u \atop rs} \right\}\Pi_u = [\Pi_s,K_r] + \frac{1}{2}\,\mathcal{P}([\Pi_s,\Pi_r]) \tag{2.12}$$

where we have used (2.7a). The projection of $[\Pi_r,\Pi_s]$ onto

$\underset{\sim}{p}$ is not a commutator and can not be transformed into a commutator by adding to it a term like $S_{rs}^{u}\,\pi_{u}$ with $S_{rs}^{u} = S_{sr}^{u}$. So if we want to write the covariant derivation of π_r as a commutator we have to add to the connection an antisymmetric term. However, that is not what we want to do, as we explain in the next section.

3- The FPR for the symmetric spaces

In reference [1] we have shown how to obtain the Fundamental Poisson bracket Relation(FPR)[2] for any non singular lagrangian defined on a group manifold, using the fact the covariant derivative w.r.t. the Christoffel symbol of the components of the Maurer-Cartan 1-form can be written as a commutator. We see from (2.12) that a similar result does not hold in the case of the coset space G/K. However, we show in appendix I that, although the relation (2.12) does not lead to the FPR, it does lead to a Lax[4] pair equation for the geodesic motion on G/K.

The second term on the r.h.s. of (2.12) is exactly what prevents the space G/K from being a symmetric space. The coset spaces, are furnished with an automorphism σ of order two ($\sigma^2 = 1$) of G such that K is the invariant subgroup of G under σ [3]. In this case the algebra $\underset{\sim}{g}$ of G is decomposed as in (2.1) but $\underset{\sim}{p}$ and $\underset{\sim}{k}$ are now the odd and even subspaces under σ respectively, i.e.

$$\sigma(\underset{\sim}{k}) = \underset{\sim}{k} \quad , \quad \sigma(\underset{\sim}{p}) = -\underset{\sim}{p} \tag{3.1}$$

The relation (2.2) is still true, but due to the fact that σ is an automorphism it follows that the third relation in (2.3) is replaced by

$$[\underset{\sim}{p}, \underset{\sim}{p}] \subset \underset{\sim}{k} \tag{3.2}$$

Therefore the commutator $[\Pi_r, \Pi_s]$ has no component in $\underset{\sim}{p}$ and the relation (2.12) for a symmetric space G/K becomes

$$\mathcal{D}_r \Pi_s = [\Pi_s, K_r] \tag{3.3}$$

The projections of $\underset{\sim}{g}$ onto $\underset{\sim}{p}$ and $\underset{\sim}{k}$ can now be performed by the automorphism σ and the quantities Π_r and K_r are then written as

$$\Pi_r = \frac{1}{2}\left[g^{-1}\partial_r g - \sigma(g^{-1}\partial_r g)\right] \tag{3.4a}$$

$$K_r = \frac{1}{2}\left[g^{-1}\partial_r g + \sigma(g^{-1}\partial_r g)\right] \tag{3.4b}$$

Following the procedure of reference [1] we now derive the FPR for the symmetric spaces G/K. Let $\mathcal{L}(q, \dot{q}, t)$ be a non singular lagrangian describing the motion of a particle on the symmetric space G/K, and let $p_r = \frac{\partial \mathcal{L}}{\partial \dot{q}^r}$ be the canonical momentum conjugate to q^r. We define the operator

$$A = p_r \Pi^r \quad , \qquad \sigma(A) = -A \tag{3.5}$$

for each lagrangian on G/K. The Poisson bracket between the entries of this operator can be easily evaluated by noticing that due to the symmetry of the Christoffel symbol and the antisymmetry of the Poisson bracket the ordinary derivatives w.r.t. to the coordinates q^r can be replaced by covariant derivatives. Then using (3.3) one gets, in a tensor product notation, that

$$\{A \overset{\otimes}{,} A\}_{PB} = \rho s\left([\pi^s, K_r] \otimes \pi^r - \pi^r \otimes [\pi^s, K_r]\right)$$

Defining the operators

$$\mathbb{P} = K_r \otimes \pi^r - \pi^r \otimes K_r \tag{3.6a}$$

$$\mathbb{R} = K_r \otimes \pi^r + \pi^r \otimes K_r \tag{3.6b}$$

we get that

$$\{A \overset{\otimes}{,} A\}_{PB} = -\left[\frac{\mathbb{P}+\mathbb{R}}{2}, A \otimes 1\right] - \left[\frac{\mathbb{P}-\mathbb{R}}{2}, 1 \otimes A\right]$$

$$= -\frac{1}{2}\left[\mathbb{P}, A \otimes 1 + 1 \otimes A\right] - \frac{1}{2}\left[\mathbb{R}, A \otimes 1 - 1 \otimes A\right] \tag{3.7}$$

A relation like this one is called the Fundamental Poisson bracket Relation (FPR) [2,5]. From it we obtain that

$$\left\{A, \frac{\mathrm{Tr}\, A^N}{N}\right\}_{PB} = [A, B_N] \tag{3.8}$$

where the operator B_N is given by

$$B_N = \mathrm{Tr}_R\left\{\left(\frac{\mathbb{P}\cdot\mathbb{R}}{2}\right)\left(1\otimes A^{N-1}\right)\right\} = -\mathrm{Tr}_L\left\{\left(\frac{\mathbb{P}-\mathbb{R}}{2}\right)\left(A^{N-1}\otimes 1\right)\right\} \quad (3.9)$$

The subindex R(L) means we are taking the trace of the right (left) entry of the tensor product. In (3.9) we have used the fact that $\mathbb{R}$ and $\mathbb{P}$ are respectively symmetric and antisymmetric w.r.t. the interchange of the left and right entries of the tensor product. Notice that if $\mathrm{Tr}\, A^N$ is the hamiltonian of the system under consideration the eq. (3.8) is the Lax pair equation. From (3.8) we get

$$\left\{\mathrm{Tr}\, A^N, \mathrm{Tr}\, A^M\right\}_{PB} = 0 \quad (3.10)$$

So, this is analogous to the result we have obtained for the group manifolds[1], i.e., for any non singular lagrangian describing the motion of a particle on a coset space, which is a symmetric space, we can construct the FPR and consequently charges in involution. One notices that this result is independent of the dynamics of the model.

4- The conservation of the charges $\mathrm{Tr}\, A^N$

The involution property of the charges $\mathrm{Tr}\, A^N$, as we have seen in the last section, is valid for any model defined on the symmetric space G/K. However, these charges are conserved for some models only, and we now want to find what models these are. The hamiltonians of such models obviously commute with those charges, and therefore we have to find all functions of the canonical variables that have vanishing Poisson bracket with all charges $\mathrm{Tr}\, A^N$. Following the procedure of section 3 of reference [1] we get that an arbitrary function $I(p,q)$ of the canonical variables satisfies

$$\{A, I(p,q)\}_{PB} = [A, B_I] - \pi^u \Delta_u I(p,q) \tag{4.1}$$

where we have used (3.3) and (3.5), and have introduced the operators

$$B_I = \frac{\partial I}{\partial p_u} K_u \tag{4.2}$$

and

$$\Delta_u = \frac{\partial}{\partial q^u} + p_r \left\{ {r \atop u\, s} \right\} \frac{\partial}{\partial p_s} \tag{4.3}$$

Then from (4.1)

$$\{I, \frac{\mathrm{Tr}\,A^N}{N}\}_{PB} = \mathrm{Tr}(A^{N-1}\pi^u)\,\Delta_u I \tag{4.4}$$

So, there are two sufficient conditions for the function $I(p,q)$ to have vanishing Poisson bracket with all the charges $\mathrm{Tr}\,A^N$. The first one is that it has to be annihilated by all operators Δ_u

$$\Delta_u I(p,q) = 0 \tag{4.5}$$

The second one is that there must exist on operator C_I, in the algebra $\underset{\sim}{k}$ of K, such that

$$\pi^u \Delta_u I(p,q) = [A, C_I] \tag{4.6a}$$

Since the r.h.s. of (4.4) becomes the trace of a commutator and so it vanishes. Using (2.9) we can write (4.6a) as

$$\Delta_u I(p,q) = \mathrm{Tr}([A, C_I]\pi_u) \tag{4.6b}$$

These results are the same as those we have obtained for the group manifold[1], and in fact many of the results of section 3 of reference [1] still hold true in the case of the symmetric spaces G/K. In particular the theorems I, II and III of that section are still valid when we replace G by symmetric spaces G/K. The operators defined in (4.3) satisfy

$$[\Delta_u, \Delta_v] = -p_r R^r_{suv} \frac{\partial}{\partial p_s} \tag{4.7}$$

where R^r_{suv} is the Riemann-Christoffel curvature tensor for the symmetric space G/K. In appendix II of reference [1] we show that it is given by

$$R^r_{suv} \Pi_r = -[\Pi_s, [\Pi_u, \Pi_v]] \tag{4.8}$$

Using (3.3), (3.5) and (4.3) one can chek that the operator A satisfies

$$\Delta_u A = [A, K_u] \tag{4.9}$$

As a consequence of this relation, it follows that the charges $\mathrm{Tr}\, A^N$ satisfy (4.5).

Let T be a generator of the semisimple group G and consider the quantity $(g \in G)$

$$X_L(T) = \mathrm{Tr}(A g^{-1} T g) = \mathrm{Tr}\left\{A\left[\frac{g^{-1}Tg - \sigma(g^{-1}Tg)}{2}\right]\right\} \tag{4.10}$$

where we have used (3.5) and the fact that the automorphism σ preserves the Killing form of G. Using (4.9), (3.4) and the fact that $\partial_u(g^{-1}Tg) = [g^{-1}Tg, g^{-1}\partial_u g]$ one can check that these quantities satisfy

$$\Delta_u X_L(T) = \mathrm{Tr}\left([A, C_L(T)]\,\Pi_u\right) \tag{4.11}$$

where $C_L(T) = \frac{1}{2}[g^{-1}Tg + \sigma(g^{-1}Tg)]$. So, the quantities $X_L(T)$ satisfy (4.6), and consequently commute with all charges $\mathrm{Tr}\, A^N$. They are the generators of canonical transformations which correspond to the left translations on the symmetric space G/K by elements of G (see appendix II). The transformations induced on phase space by the left translations are not always canonical. That depends upon the hamiltonian under consideration. When the hamiltonian is one of the charges $\mathrm{Tr}\, A^N$ then the charges $X_L(T)$ are conserved and in such cases the left translations coincide with the canonical transformations generated by $X_L(T)$. See appendix I of reference [5] for more details. The Poisson bracket between any two left charges $X_L(T)$ can be easily evaluated noticing that, due to the symmetry of the Christoffel symbol, the Poisson bracket between two arbitrary functions I and I' can be written as

$$\{I, I'\}_{PB} = \Delta_u I \frac{\partial I'}{\partial p_u} - \Delta_u I' \frac{\partial I}{\partial p_u} \tag{4.12}$$

Then using (4.10), (4.11) and (3.5)

$$\{X_L(T), X_L(T')\}_{PB} = \mathrm{Tr}\left([A, C_L(T)]\,\pi_u\right)\mathrm{Tr}\left\{\pi^u\left(\frac{g^{-1}T'g - \sigma(g^{-1}T'g)}{2}\right)\right\} -$$

$$- \mathrm{Tr}\left([A, C_L(T')]\,\pi_u\right)\mathrm{Tr}\left\{\pi^u\left(\frac{g^{-1}Tg - \sigma(g^{-1}Tg)}{2}\right)\right\}$$

Using (2.11) one gets that the left charges $X_L(T)$ generate the algebra of G under the Poisson bracket

$$\{X_L(T), X_L(T')\}_{PB} = X_L([T, T']) \quad ; \quad T, T' \in \underset{\sim}{g} \qquad (4.13)$$

5- The non-compact symmetric spaces and their hidden group theoretical structure

We now restrict our study to the cases where G is a non-compact semisimple Lie group and σ is a Cartan involution, i.e., an automorphism of $\underset{\sim}{g}$ of order two satisfying[3,5]

$$\mathrm{Tr}(T\sigma(T')) \text{ is negative definite }, \quad T,T' \in \underset{\sim}{g} \qquad (5.1)$$

These cases have some interesting properties as we now explain. Since the automorphism σ preserves the Killing form of $\underset{\sim}{g}$, it follows from (5.1) that

$$\mathrm{Tr}(TT') = \begin{cases} \text{is positive definite if } T,T' \in \underset{\sim}{p} \\ \text{is negative definite if } T,T' \in \underset{\sim}{k} \\ \text{zero if } T \in \underset{\sim}{p}\ ,\ T' \in \underset{\sim}{k} \end{cases} \qquad (5.2)$$

where $\underset{\sim}{p}$ and $\underset{\sim}{k}$ are respectively the odd and even subspaces of $\underset{\sim}{g}$ under σ (see (3.1)). The subgroup K ,with algebra $\underset{\sim}{k}$, is the maximal compact subgroup of G. The metric (2.8) in these cases is positive definite. The coset space G/K is an irreducible non compact Riemannian symmetric space[3].

An important result related to these spaces is the Iwasawa decomposition of G. Let $\underset{\sim}{a}$ be a maximal abelian subspace of $\underset{\sim}{p}$ and define the subspaces

$$\underset{\sim}{g}_\alpha = \{ T \in \underset{\sim}{g} \mid [h,T] = \alpha(h)T \quad , \quad h \in \underset{\sim}{a} \} \tag{5.3}$$

where α is called a root of $(\underset{\sim}{g}, \underset{\sim}{a})$ if $\alpha \neq 0$ and $\underset{\sim}{g}_\alpha \neq 0$. According to the Iwasawa decomposition, the algebra $\underset{\sim}{g}$ can be written as (see chapter IX ref. [3])

$$\underset{\sim}{g} = \underset{\sim}{n} + \underset{\sim}{a} + \underset{\sim}{k} \tag{5.4}$$

where $\underset{\sim}{n}$ is the nilpotent subalgebra of $\underset{\sim}{g}$

$$\underset{\sim}{n} = \sum_{\alpha > 0} \underset{\sim}{g}_\alpha \tag{5.5}$$

and where the sum is over the positive roots of $(\underset{\sim}{g}, \underset{\sim}{a})$. Analogously the group G decomposes as

$$G = NAK \tag{5.6}$$

where N and A are the exponentiation of $\underset{\sim}{n}$ and $\underset{\sim}{a}$ respectively, and K is the maximal compact subgroup of G with algebra $\underset{\sim}{k}$.

As a consequence of this decomposition it follows that we can associate to each point of G/K an element of the solvable subgroup $B = NA$ and vice versa. This endows the symmetric space G/K with a hidden group theoretical structure which has no analogue for the compact symmetric spaces. We then have a natural choice of representative from each coset, namely $K=1$ and $g=b=ma$ [5]. In this "gauge" we can write Π_r and K_r, given by (3.4), as

$$\Pi_r = \frac{1}{2}\left[b^{-1}\partial_r b - \sigma(b^{-1}\partial_r b)\right] \tag{5.7a}$$

$$K_r = \frac{1}{2}\left[b^{-1}\partial_r b + \sigma(b^{-1}\partial_r b)\right] \tag{5.7b}$$

where the derivatives are w.r.t. the coordinates of G/K which now can be taken as the parameters of the subgroup B. Using this "gauge" $K=1$ we can consistenly define right translations on G/K by elements of B, since the transformation $b \to bb'$; $b, b' \in B$ induces a map of G/K onto G/K where the coset with representative b is mapped into the coset with representative bb'. The transformations on phase space induced by these right translations are not always canonical (see appendix I of ref. [5] for details). However the infinitesimal canonical transformations generated by the charges

$$X_R(t) = \mathrm{Tr}(At) = \mathrm{Tr}\left\{A\left(\frac{t-\sigma(t)}{2}\right)\right\} \tag{5.8}$$

where t is an element of the algebra $\underset{\sim}{b}$, coincide with the right translations when functions of the coordinates only are considered (see appendix II). Using (4.9) one easily gets

$$\Delta_\mu X_R(t) = \mathrm{Tr}\left\{ [A, K_\mu] \left(\frac{t - \sigma(t)}{2}\right) \right\} \tag{5.9}$$

So, the right charges $X_R(t)$, unlike $X_L(T)$, do not satisfy (4.6b) and therefore do not commute with the charges $\mathrm{Tr}\, A^N$.

Let t_r, $r = 1,2,\ldots \dim B$, denote the generators of the solvable subgroup B of G. By introducing a matrix $M_r^{\ s}$ as

$$b^{-1}\partial_r b = M_r^{\ s} t_s \tag{5.10}$$

we can write the quantities Π_r and K_r, given in (5.7), as

$$\Pi_r = M_r^{\ s} P_s \quad ; \quad K_r = M_r^{\ s} k_s \tag{5.11}$$

where we have defined

$$P_r = \frac{1}{2}\left(t_r - \sigma(t_r)\right) \quad ; \quad k_r = \frac{1}{2}\left(t_r + \sigma(t_r)\right) \tag{5.12}$$

The generators P_r constitute a basis for the subspace $\underset{\sim}{p}$ of $\underset{\sim}{g}$ (see (3.1)). The quantities k_r are elements of the subalgebra $\underset{\sim}{k}$ and may not all be independent. Using (2.8), (2.9) and (5.11) we see that for the non compact symmetric spaces we are considering

$$\Pi^r = \eta^{uv} M^{-1\ r}_{u} P_v \tag{5.13}$$

where $M^{-1\,s}_{r}$ is the inverse of the matrix defined in (5.10), and η^{rs} is the inverse of the Killing form of $\underset{\sim}{g}$ restricted to $\underset{\sim}{p}$, i.e.

$$\eta_{rs} = \mathrm{Tr}(P_r P_s) \tag{5.14}$$

According to (5.2) this is a positive definite matrix and it is non singular because $\underset{\sim}{g}$ is semisimple.

Using (3.5), (5.8) - (5.14), and (4.12) one can evaluate the Poisson bracket between any two right charges. One gets

$$\{X_R(t_r), X_R(t_s)\}_{PB} = \mathrm{Tr}\{[A, k_s]P_r - [A, k_r]P_s\} \tag{5.15}$$

and then using (5.12) again one gets that the right charges generate the algebra of the subgroup B under the Poisson bracket

$$\{X_R(t_r), X_R(t_s)\}_{PB} = - f_{rs}^{u} X_R(t_u) \tag{5.16}$$

where f_{rs}^{u} are the structure constants of B , $[t_r, t_s] = f_{rs}^{u} t_u$. Notice that a similar relation does not exist for the compact symmetric spaces.

In the gauge $K=1$, where $g = b = m a$, the left charges (4.10) corresponding to left translations by elements of the subgroup B are given by

$$X_L(t) = \mathrm{Tr}(A\, b^{-1} t\, b)\ ,\ b \in B\ ,\ t \in \underset{\sim}{b} \qquad (5.17)$$

These charges an related to the right (5.8) by

$$X_L(t_r) = a_r^{\ s}(b)\, X_R(t_s)\ ;\ r,s = 1,2 \ldots \dim B \qquad (5.18)$$

where $a_r^{\ s}(b)$ is the adjoint representation of the subgroup B

$$b^{-1} t_r\, b = a_r^{\ s}(b)\, t_s \qquad (5.19)$$

By differetiating this relation w.r.t. a parameter of the subgroup B one gets

$$\partial_u a_r^{\ s} = f_{tv}^{\ s}\, a_r^{\ t}\, M_u^{\ v} \qquad (5.20)$$

where $M_u^{\ v}$ is defined in (5.10). Since the coordinates on G/K can be taken to be the parameters of B one gets using (5.18) and (5.20) that

$$\{\quad , X_L(t_r)\}_{PB} = a_r^{\ s}\{\quad , X_R(t_s)\}_{PB} - a_r^{\ s} f_{sv}^{\ u}\, d_u^{\ v} \qquad (5.21)$$

when the operators $d_r^{\ s}$ are defined in (6.4). From (5.8), (3.5) and (5.12) - (5.14) one can easily check that

$$\chi_R(t_r) = M^{-1\,s}_{r}\, p_s \tag{5.22}$$

Then, from (6.4)

$$d_r^{\,s}\, \chi_R(t_u) = \delta_u^{\,s}\, \chi_R(t_r) \tag{5.23}$$

Therefore we conclude from (5.16) and (5.21) that the charges $\chi_R(t)$ and $\chi_L(t')$, commute

$$\{ \chi_R(t),\, \chi_L(t') \}_{PB} \doteq 0 \tag{5.24}$$

6- The algebraic structures underlying equations (4.5) and (4.6) for the case of the non compact symmetric spaces

In section 4 we introduced the operators Δ_u and have given, in terms of them, two sufficient conditions for a function of the canonical variables to commute with all charges $\mathrm{Tr}\, A^N$. We now want to show that, for the case of the non compact symmetric spaces, there is an algebraic structure underlying those conditions. This provide us with a method of construting functions satisfying those conditions based on representation theory of Lie algebras, as we explain in the next section. We believe a similar thing happens for the compact symmetric spaces.The procedure we now discuss is equivalent to the one developed, in reference [1], for the group manifold.

We introduce the operators D_r, $r = 1,2...$ dim G/K (=dim B), as some suitable linear combinations of the operators Δ_r defined in (4.3)

$$D_r = M^{-1\,s}_{r}\,\Delta_s \qquad (6.1)$$

where $M^{-1\,s}_{r}$ is the inverse of the matrix introduced in (5.10). Using (5.11) we see that, for the case of the non compact symmetric spaces, the relations (4.5) and (4.6b) take the form

$$D_r I(p,q) = 0 \qquad (6.2)$$

$$D_r I(p,q) = \mathrm{Tr}([A, C_I] P_r) \tag{6.3}$$

The operators D_r, like Δ_r, do not form a closed algebra. In order to close their algebra we need to introduce the operators

$$d_r^s = \mathrm{Tr}(A P_r) M_u^s \frac{\partial}{\partial p_u} = X_R(t_r) M_u^s \frac{\partial}{\partial p_u} \tag{6.4}$$

where $X_R(t_r)$ are the right charges introduced in (5.8). The algebra generated by the operators D_r and d_r^s, which we denote by $\underset{\sim}{L}$, have the following commutation relations

$$[D_r, D_s] = f_{rs}^u D_u - R_{vrs}^u(0) d_u^v \tag{6.5a}$$

$$[d_r^s, D_u] = F_{ru}^v d_v^s - F_{vu}^s d_r^v \tag{6.5b}$$

$$[d_r^s, d_u^v] = \delta_u^s d_r^v - \delta_r^v d_u^s \tag{6.5c}$$

f_{rs}^u are the structure constants of the solvable subgroup B of G, $R_{vrs}^u(0)$ is the Riemann-Christoffel symbol for the non compact symmetric space G/K evaluated at the origin, i.e., the coset which correspond to the subgroup K. The structure constants F_{rs}^u are defined by

$$[P_r , k_s] = F_{rs}^{u} \, P_u \tag{6.6}$$

where P_r and k_r were defined in (5.12).

The commutation relations (6.5) of the algebra $\underset{\sim}{L}$, are similar to those obtained in reference [1] for the case of the group manifold, although the group playing a role here, namely B, is not semisimple. The operators d_r^s generate an invariant subalgebra of $\underset{\sim}{L}$ which is isomorphic to the algebra $gl(m)$, where m is the dimension of G/K (or B). The proof of relations (6.5) is very similar to that of relations (4.6) in reference [1]. Using (6.1), (4.7) and the relation (I.4) of appendix I of reference [1] one can easily check that

$$[D_r , D_s] = f_{rs}^{u} D_u - M_r^{-1u} M_s^{-1v} p_t R_{wuv}^{t} \frac{\partial}{\partial p_w} \tag{6.7}$$

At the origin of G/K, Π_r and P_r coincide and therefore from (4.8) we have

$$R_{suv}^{r}(0) P_r = -[P_s, [P_u , P_v]] \tag{6.8}$$

Therefore using (4.8), (2.9), (5.11) and (3.5)

$$\begin{aligned} M_r^{-1u} M_s^{-1v} p_t R_{wuv}^{t} &= - M_w^{u} p_t \mathrm{Tr}(\pi^t [P_u , [P_r , P_s]]) = \\ &= M_w^{u} \mathrm{Tr}(A P_v) R_{urs}^{v}(0) \end{aligned}$$

Substituting this in (6.7) and using (6.4) one gets (6.5a).

From (5.11) and (2.9) we have

$$\mathrm{Tr}(\Pi^r P_s) = M^{-1}{}_s{}^u \,\mathrm{Tr}(\Pi^r \Pi_u) = M^{-1}{}_s{}^r \qquad (6.9)$$

and then using (6.4) and (3.5) one can easily obtain (6.5c).

Using (6.1), (6.4), (4.3), (4.9) and (5.11) one gets

$$[d_r{}^s, D_u] = -\mathrm{Tr}(AP_r)\, M^{-1}{}_u{}^v \left[\partial_v M_w{}^s - \left\{{t \atop wv}\right\} M_t{}^s\right] \frac{\partial}{\partial p_w} -$$

$$-\mathrm{Tr}([A, k_u]P_r)\, M_w{}^s \frac{\partial}{\partial p_w}$$

From (3.3) and (5.11)

$$M^{-1}{}_u{}^v \mathcal{D}_v \Pi_w = M^{-1}{}_u{}^v \left(\partial_v M_w{}^s - \left\{{t \atop wv}\right\} M_t{}^s\right) P_s = M^{-1}{}_u{}^v [\Pi_w, K_v] =$$

$$= M_w{}^t [P_t, k_u]$$

Therefore using (6.6) and the fact that the generators P_s are linearly independent one obtains the relation (6.5b). So, we have proved the commutation relations (6.5).

The second term on the r.h.s. of (6.5a) constitute a non central extention of the algebra $\underset{\sim}{b}$ of the solvable subgroup B . It can, in fact, be removed by a redefinition of the operators D_r . For that purpose we introduce the operators

$$\bar{D}_r = D_r + C_{rv}^{u}\, d_u^{v} \tag{6.10}$$

where the coefficients are determined by imposing that the operators $\bar{D}_r$ generate the algebra $\underset{\sim}{b}$. Using the commutation relations (6.5) one can check that

$$[\,\bar{D}_r, \bar{D}_s] = f_{rs}^{t}\, \bar{D}_t + Y_{rsv}^{u}\, d_u^{v} \tag{6.11}$$

where

$$\begin{aligned} Y_{rsv}^{u} = {} & f_{rs}^{t}(F_{vt}^{u} - C_{tv}^{u}) + (F_{vs}^{t} - C_{sv}^{t})(F_{tr}^{u} - C_{rt}^{u}) - \\ & - (F_{vr}^{t} - C_{rv}^{t})(F_{ts}^{u} - C_{st}^{u}) \end{aligned} \tag{6.12}$$

and where we have used the fact that

$$X_{rsv}^{u} \equiv F_{vr}^{t} F_{ts}^{u} - F_{vs}^{t} F_{tr}^{u} - f_{rs}^{t} F_{vt}^{u} - R_{vrs}^{u}(0) = 0 \tag{6.13}$$

To prove this relation we use (6.5) and the Jacobi identity for the operators D_r , D_s and d_u^{v} . We obtain

$$X_{rsu}^{t}\, d_t^{v} - X_{rst}^{v}\, d_u^{t} = 0 \tag{6.14}$$

Collecting the coefficients of each operator we get that :

$X^{t}_{rsu} = 0$ if $u \neq t$, and $X^{u}_{rsu} = X^{v}_{rsv}$ for any u and v. Therefore we have that $X^{u}_{rsu} = \sum_t X^{t}_{rst}/\dim B$, for any u. Using the symmetry properties of the Riemann-Christoffel curvature tensor we get from (6.13) that $\sum_u X^{u}_{rsu} = -f^{t}_{rs}\sum_u F^{u}_{ut}$. In appendix III we show that

$$F^{u}_{rs} = \frac{1}{2}\left(f^{u}_{rs} - I^{u}_{rs}\right) \tag{6.15}$$

with

$$I^{u}_{rs} = \eta^{ut}\left(\eta_{rv} f^{v}_{ts} + \eta_{sv} f^{v}_{tr}\right) \tag{6.16}$$

where η_{rs} is the Killing form of $\underset{\sim}{g}$ restricted to $\underset{\sim}{p}$ (see (5.14)). Then we get that $\sum_u F^{u}_{ut} = 0$ and therefore all the X^{v}_{rsu} vanish. Notice that (6.13) provides an expression for the curvature tensor of G/K at the origin in terms of the structure constants of the subgroup B.

Therefore, from (6.12) and (6.15) we see there are two possible ways of making the constants Y^{u}_{rsv} to vanish. In the first one we take $C^{u}_{rs} = F^{u}_{sr}$ and in the second we take $C^{u}_{rs} = F^{u}_{rs}$ and use the Jacobi identity for the structure constants of B. Then, from (6.10) and (6.11) we get that the operators

$$D^{+}_{r} = D_{r} + F^{u}_{rv}\, d_{u}^{\ v} \tag{6.17a}$$

$$D_r^- = D_r + F_{vr}^u d_u^v \tag{6.17b}$$

satisfy

$$[D_r^+, D_s^+] = f_{rs}^t D_t^+ \quad ; \quad [D_r^-, D_s^-] = f_{rs}^t D_t^- \tag{6.18}$$

Using the commutation relations (6.5) and (6.13) one gets

$$[D_r^+, D_s^-] = f_{rs}^t D_t^- \tag{6.19}$$

So, the operators D_r^+ and D_r^- generate an algebra which is the semidirect product of two copies of the algebra $\underset{\sim}{b}$. The subalgebra generated by D_r^- is invariant. Again, using (6.5), one obtains

$$[d_r^s, D_u^-] = 0 \tag{6.20}$$

$$[d_r^s, D_u^+] = f_{ru}^t d_t^s - f_{tu}^s d_r^t \tag{6.21}$$

We now evaluate the explicit from of the operators D_r^+ and D_r^- and show they are related to the canonical transformations generated by the charges $X_R(t)$ and $X_L(t)$. From (6.1), (4.3) and (2.10a) we have

$$D_r = M^{-1}{}_r{}^s \frac{\partial}{\partial q^s} - p_u \operatorname{Tr}\left\{ P_r \left(\frac{\partial \pi^u}{\partial q^v} - [\pi^u, K_v] \right) \right\} \frac{\partial}{\partial p_v}$$

where we have used (2.9), (5.11) and (2.7a) for the case of G/K being a symmetric space, i.e. with $\mathcal{P}([\pi_r, \pi_s])$.
From (5.22), (5.11) - (5.14), (6.4) and (6.6) we get

$$D_r = \{ \quad , X_R(t_r)\}_{PB} - F_{rv}^{u}\, d_u^{v} \tag{6.22}$$

Therefore from (6.17), (6.15) and (5.21)

$$D_r^+ = \{ \quad , X_R(t_r)\}_{PB} \tag{6.23a}$$

$$D_r^- = a^{-1}{}_r{}^s(b) \{ \quad , X_L(t_s)\}_{PB} \tag{6.23b}$$

where $a_r^{\ s}(b)$ is the adjoint representation of the subgroup B (see (5.19)). Therefore the operator D_r^+ generate the same infinitensimal canonical transformation as $X_R(t_r)$ does. However these transformations coincide with the induced transformations on phase space by the right translations on G/K in some cases only. The case where the charge $\operatorname{Tr} A^N$ is the hamiltonian is not one of those (see appendix I of ref. [3] for details). The operator D_r^- is related to the infinitensimal canonical transformation generated by $X_L(t_r)$. Notice that, when acting on functions of the coordinates only, both operators, D_r^+ and D_r^-, generate right translations on G/K.

7- The base states for the representation of the algebra $\underset{\sim}{L}$

In this section we construct the base states of the representation of the algebra $\underset{\sim}{L}$ defined by the linear differential operators D_r and d_r^s. The solutions of equations (6.2) and (6.3) are states in this representation and therefore they can be written as linear combination of these base states. This reduces the problem of solving the differential equations (6.2) and (6.3) to an algebraic problem in representation theory of Lie algebras.

Our discussion applies for the non compact symmetric spaces G/K introduced in section 5. We shall denote by h_i , $i = 1,2 \ldots \dim \underset{\sim}{a}$, the generators of the maximal abelian subspace $\underset{\sim}{a}$ of $\underset{\sim}{p}$ (see (5.4)). They satisfy

$$[h_i, h_j] = 0 \qquad ; \qquad \sigma(h_i) = -h_i \tag{7.1}$$

The generators of the nilpotent subalgebra $\underset{\sim}{n}$ of $\underset{\sim}{g}$, defined in (5.5), will be denoted by e_α^a , $a = 1,2 \ldots m_\alpha$, where α is a positive root of $(\underset{\sim}{g},\underset{\sim}{a})$ and m_α its multiplicity. So we have $\dim \underset{\sim}{n} = \sum_{\alpha>0} m_\alpha$. According to (5.3) the step operators e_α^a satisfy

$$[h_i, e_\alpha^a] = \alpha_i \, e_\alpha^a \tag{7.2}$$

Since the generators h_i are odd under the automorphism

σ we see that $\sigma(e_\alpha^a)$ is a step operator for a negative root of $(\underset{\sim}{g},\underset{\sim}{a})$

$$[h_i, \sigma(e_\alpha^a)] = -\alpha_i\, \sigma(e_\alpha^a) \qquad (7.3)$$

Therefore the intersection of the subalgebra $\underset{\sim}{m}$ and its image under σ is empty. From (5.12) we get that the generators of the odd subspace $\underset{\sim}{p}$ of $\underset{\sim}{g}$ are given by

$$P_i = h_i \quad ; \quad i = 1,2 \ldots \dim \underset{\sim}{a} \qquad (7.4a)$$

$$P_\alpha^a = \frac{1}{2}\left(e_\alpha^a - \sigma(e_\alpha^a)\right) \qquad \alpha \equiv \text{positive root of } (\underset{\sim}{g},\underset{\sim}{a}) \qquad (7.4b)$$

$$a = 1,2 \ldots m_\alpha$$

and the quantities k_r are

$$k_i = 0 \quad ; \quad k_\alpha^a = \frac{1}{2}\left(e_\alpha^a + \sigma(e_\alpha^a)\right) \qquad (7.5)$$

So, from (6.6) we see that the structure constants F_{rs}^u are zero whenever the index s correspond to a generator h_i of $\underset{\sim}{a}$. Then, from (6.5b) we see that

$$[D_i, d_r^s] = 0 \qquad (7.6)$$

The Cartan subalgebra of the algebra $\underset{\sim}{L}$ defined in (6.5) is generated by the operators D_i, $i=1,2\ldots\dim \underset{\sim}{a}$, and d_r^r , r = 1,2... dim G/K, since from (6.8), (7.4a) and (7.1) we have

$$R_{r\,i\,j}^{\;s}(0) = 0 \qquad \begin{array}{l} r,s = 1,2 \ldots\ldots \dim G/K \\ i,j = 1,2 \ldots \dim \underset{\sim}{a} \end{array} \tag{7.7}$$

and then, from (6.5) and (7.1)

$$[D_i, D_j] = [d_r^r, d_s^s] = 0 \tag{7.8}$$

Therefore the dimension of the Cartan subalgebra of $\underset{\sim}{L}$ is the sum of the dimensions of the Cartan subalgebras of $\underset{\sim}{b}$ and $gl(m)$, i.e., dim $\underset{\sim}{a}$ + dim G/K.

We take the base states of the representation of $\underset{\sim}{L}$ as the eigenfunctions of the generators of its Cartan subalgebra

$$D_i\, J_{\mu,\nu}(p,q) = \nu_i\, J_{\mu,\nu}(p,q) \tag{7.9a}$$

$$d_r^r\, J_{\mu,\nu}(p,q) = \mu_r\, J_{\mu,\nu}(p,q) \tag{7.9b}$$

where the eigenvalues μ_r and ν_i are the weights of the representation. The analysis that now follows is very similar to that of section 5 of reference [1] for the case of the group manifold. So, in analogy to that case, one get from (6.4) that the solutions of (7.9b) are given by

$$J_{\mu,\nu}(p,q) = \Theta_\nu(q)\; J_\mu(p,q) \tag{7.10}$$

where

$$J_\mu(p,q) = \prod_{r=1}^{\dim G/K} \left[X_R(t_r) \right]^{\mu_r} \tag{7.11}$$

and $\Theta_\nu(q)$ are arbitrary functions of the coordinates only. We show below that these functions, unlike the case of the group manifold[1], do not depend upon μ.

From (6.23), (5.24) and (5.16) one gets

$$D_r^- \, J_\mu(p,q) = 0 \tag{7.12a}$$

$$D_r^+ \, J_\mu(p,q) = \sum_{s,u=1}^{\dim B} \mu_s \, f_{rs}^{u} \, \frac{X_R(t_u)}{X_R(t_s)} \, J_\mu(p,q) \tag{7.12b}$$

From (7.1) and (7.2) we see that $J_\mu(p,q)$ is an eigen-function of the operators D_i^+

$$D_i^+ \, J_\mu(p,q) = \sum_{\alpha>0} \alpha_i \sum_{a=1}^{m_\alpha} \mu_{(\alpha,a)} \, J_\mu(p,q) \tag{7.13}$$

where we have denoted by $\mu_{(\alpha,a)}$ the eigenvalues of the operators $d_r^{\;r}$ corresponding to the generators e_α^a.

of $\underset{\sim}{b}$. The sums in (7.13) are over the positive roots of ($\underset{\sim}{g}$, $\underset{\sim}{a}$) and their degeneracy.

Since the structure constants F_{rs}^{u} vanish when the index s correspond to a generator h_i of $\underset{\sim}{a}$ (see (6.6) and (7.5)), we see from (6.17a) the operators D_i^- and D_i coincide. Then from (7.12a) we get

$$D_i \, J_\mu(p,q) = 0 \tag{7.14}$$

and then the functions $\Theta_\nu(q)$, introduced in (7.10), have to be eigenfunctions of D_i in order to $J_{\mu,\nu}$ to satisfy (7.9a). In addition, since $\Theta_\nu(q)$ does not depend upon the momenta, we see from (6.17a) that it is also eigenfunction of D_i^+ . So we have

$$D_i \Theta_\nu(q) = D_i^- \Theta_\nu(q) = D_i^+ \Theta_\nu(q) = \nu_i \, \Theta_\nu(q) \tag{7.15}$$

From (6.1) and (4.3) we see that $\Theta_\nu(q)$ is, in fact, an eigenfunction of the operators $\nabla_i \equiv M_i^{-1\,s} \partial_s$. According to appendix I of reference [1] the operators

$$\nabla_r = M_r^{-1\,s} \, \partial_s \tag{7.16}$$

satisfy

$$[\nabla_r, \nabla_s] = f_{rs}^{u} \, \nabla_u \tag{7.17}$$

They are the generators of right translations on B (or G/K, see appendix II) by elements of B , and constitute a representation of the algebra $\underset{\sim}{b}$. Then the eigenfunctions $\Theta_\nu(q)$ are the base states of this representation. Since the functions $J_\mu\ (p,q)$, defined in (7.11), are the base states of the representation of the algebra $gl(m)$ defined by the d_r^s , we see that the representation of the algebra $\underset{\sim}{L}$, with base states (7.10), decomposes as the product of the representations of the algebras $\underset{\sim}{b}$ and $gl(m)$, given respectively by the operators ∇_r and d_r^s.

The functions $J_{\mu,\nu}(p,q)$, defined in (7.10), are also eigenfunctions of the operators D_i^+ and D_i^- ,which are the generators of the Cartan subalgebra of the algebra generated by the operators D_r^+ and D_r^- (see (6.18) and (6.19)). From (7.12a), (7.13) and (7.15) we get

$$D_i^+ J_{\mu,\nu}(p,q) = \left[\nu_i + \sum_{\alpha>0} \alpha_i \sum_{a=1}^{m_\alpha} \mu_{(\alpha,a)}\right] J_{\mu,\nu}(p,q) \tag{7.18a}$$

$$D_i^- J_{\mu,\nu}(p,q) = \nu_i \ J_{\mu,\nu}(p,q) \tag{7.18b}$$

We can easily calculate the eigenfunctions $\Theta_\nu(q)$ The coordinates q^r on G/K can be taken to be the parameters of the subgroup $B = NA$. If we parametrize the elements of the abelian subgroup A as $a = \exp(\phi^i h_i)$, we get that

$$b^{-1}\frac{\partial}{\partial\phi^i}b = a^{-1}\frac{\partial a}{\partial\phi^i} = h_i \tag{7.19}$$

Therefore from (5.10) we see that $M_i^{\ j} = \delta_i^{\ j}$ and $M_i^{\ (\alpha,a)} = 0$. On the other hand, when the index r in (5.10) correspond to a generator of the nilpotent subgroup N, it is easy to see from the commutation relation (7.2) that the r.h.s. of (5.10) with not have any term propotional to h_i. Then we conclude that $M_{(\alpha,a)}^{\ i} = 0$ and the matriz $M_r^{\ s}$ has a block diagonal form. As a consequence of this we get that $M^{-1}{}_i^{\ j} = \delta_i^{\ j}$ and the operator ∇_i, introduced in (7.16), is just the ordinary derivative w.r.t. to ϕ^i, the parameters of A. Then we have

$$\nabla_i\,\Theta_\nu(q) = \frac{\partial\,\Theta_\nu(q)}{\partial\phi^i} = \nu_i\,\Theta_\nu(q) \tag{7.20}$$

and so

$$\Theta_\nu(\phi,\rho) = f_\nu(\rho)\,e^{\nu.\phi} \tag{7.21}$$

where $f_\nu(\rho)$ is an arbitrary function of the parameters, ρ, of the nilpotent subgroup N. It accounts for the degeneracy of the eigenvalues ν_i.

From the commutation relations (6.5c) we see that the operators $d_r^{\ s}$, for $r \neq s$, are the step operators of the algebra $gl(m)$ since

$$[d_u^u, d_r^s] = (\lambda_r^s)_u d_r^s \tag{7.22}$$

where λ_r^s are the roots of $gl(m)$ with components

$$(\lambda_r^s)_u = \delta_r^u - \delta_u^s \tag{7.23}$$

When acting with the step operator $d_r^s\,(r \neq s)$ on the function $J_{\mu,\nu}(p,q)$, defined in (7.11), we get, using (5.23) and (7.23), that

$$d_r^s J_\mu(p,q) = \mu_s J_{\mu+\lambda_r^s}(p,q) \tag{7.24}$$

and since $\Theta_\nu(q)$ does not depend upon the momenta we get that the base states (7.10) of the representation satisfy

$$d_r^s J_{\mu,\nu}(p,q) = \mu_s J_{\mu+\lambda_r^s,\nu}(p,q) \tag{7.25}$$

So the step operators $d_r^s\,(r \neq s)$ change the eigeivalue μ by the root λ_r^s of $gl(m)$ and leaves the eigenvalue ν unchanged.

Using (7.2), (7.17) and (7.20) we get

$$\begin{aligned} \nabla_i \nabla_{(\alpha,a)} \Theta_\nu(q) &= \{\nabla_{(\alpha,a)}\nabla_i + [\nabla_i, \nabla_{(\alpha,a)}]\}\, \Theta_\nu(q) = \\ &= (\nu_i + \alpha_i)\, \nabla_{(\alpha,a)}\, \Theta_\nu(q) \end{aligned} \tag{7.26}$$

So the action of the step operator $\nabla_{(\alpha,a)}$ on $\Theta_\nu(q)$ produces an eigenfunction with eigenvalue $\nu+\alpha$. Then we write

$$\nabla_{(\alpha,a)}\,\Theta_\nu(q) = N_{(\alpha,a),\nu}\;\Theta_{\nu+\alpha}(q) \tag{7.27}$$

where $N_{(\alpha,a),\nu}$ are some constants to be determined. From (4.3), (6.1), (7.16), (6.17), (6.4) and the fact that the functions $\Theta_\nu(q)$ do not depend upon the momenta we see that $D_{(\alpha,a)}\,\Theta_\nu = D^{\pm}_{(\alpha,a)}\,\Theta_\nu = \nabla_{(\alpha,a)}\,\Theta_\nu$. Therefore using (7.12a) and (7.27) we get

$$D^{-}_{(\alpha,a)}\;J_{\mu,\nu}(p,q) = N_{(\alpha,a),\nu}\;J_{\mu,\,\nu+\alpha}(p,q) \tag{7.28}$$

And from (6.17b), (7.25) and (7.28)

$$\begin{aligned} D_{(\alpha,a)}\;J_{\mu,\nu}(p,q) = {}& N_{(\alpha,a),\nu}\;J_{\mu,\,\nu+\alpha}(p,q) - \\ & - \sum_{t,u=1}^{\dim B} \mu_t\,F^{u}_{t(\alpha,a)}\;J_{\mu+\lambda^t_u,\,\nu}(p,q) \end{aligned} \tag{7.29}$$

Using (6.17a), (7.29), (7.25) and (6.15) one gets

$$\begin{aligned} D^{+}_{(\alpha,a)}\;J_{\mu,\nu}(p,q) = {}& N_{(\alpha,a),\nu}\;J_{\mu,\nu+\alpha}(p,q) + \\ & + \sum_{t,u=1}^{\dim B} \mu_t\,f^{u}_{(\alpha,a)t}\;J_{\mu+\lambda^t_u,\,\nu}(p,q) \end{aligned} \tag{7.30}$$

Therefore the operators $D_{(\alpha,a)}$ and $D^{+}_{(\alpha,a)}$ shift the eigenvalues μ and ν of the base states by roots of $gl(m)$ and $(\underset{\sim}{g}, \underset{\sim}{a})$ respectively whilst the operator $D^{-}_{(\alpha,a)}$ shifts the eigenvalue ν only by roots of $(\underset{\sim}{g}, \underset{\sim}{a})$. This is a consequence of the fact that $D^{-}_{(\alpha,a)}$ commute with all operators d_r^s , whilst the operators $D_{(\alpha,a)}$ and $D^+_{(\alpha,a)}$ do not.

8- The construction of conserved charges

In the last section we have seen that the structures constants F^{u}_{rs}, defined in (6.6), vanish whenever the index s correspond to a generator of the abelian subspace $\underset{\sim}{a}$ of $\underset{\sim}{p}$, since, from (7.5), we have that $k_i = 0$ (i = 1,2.... dim). Therefore the operators D^-_i, introduced in (6.17b), coincide with D_i and consequently the solutions of (6.2) are annihilated by them. Using (6.5a), (6.13), (6.17b) and (7.2) we get

$$[D_i , D_{(\alpha,a)}] = \alpha_i D^-_{(\alpha,a)} \tag{8.9}$$

and so the solutions of (6.2) are also annihilated by $D^-_{(\alpha,a)}$. Then we have obtained that the solutions of (6.2) satisfy

$$D^-_r I(p,q) = 0 \quad , \quad r = 1,2 \ldots \dim G/K \tag{8.2}$$

and therefore, from (6.23b), we see they have to be invariant by the infinitesimal canonical transformations generated by the charges $X_L(t)$. However they do not have to be invariant by the canonical transformations generated by the charges $X_R(t)$ since from (6.17), (6.15) and (8.2) we get

$$(D^+_r - f^{u}_{rv} d^{v}_{u}) I(p,q) = 0 \tag{8.3a}$$

and

$$\left(D_r^+ + I_{rv}^u d_u^v\right) I(p,q) = 0 \qquad (8.3b)$$

The solutions of equations (6.2) and (6.3) are states in the representation of the algebra $\underset{\sim}{L}$ defined by the linear operators D_r and d_r^s. Consequently they can be written as linear combinations of the base states of this representation constructed in the last section.The base states appearing in the expansion of the solutions of (6.2) should correspond to zero eigenvalue ν_i only.From (7.10), (7.12a) and the fact that $D_r^- \Theta_\nu(q) = \nabla_r \Theta_\nu(q)$ we get that such states satisfy

$$D^-_{(\alpha,a)} J_{\mu,0}(p,q) = J_\mu(p,q)\, \nabla_{(\alpha,a)} \Theta_0(q) \qquad (8.4)$$

Therefore, in order to (8.2) to be satisfied, the function $\Theta_0(q)$ has to be annihilated by the operators $\nabla_{(\alpha,a)}$. Since it is a zero mode of ∇_i already, it follows that it must be annihilated by all operators ∇_r and therefore it must be a constant. So the solutions of (6.2) are linear combinations of the functions $J_\mu(p,q)$ only. From (7.11), (7.12), (7.23) and (7.24) we see that $J_\mu(p,q)$ satisfies (8.2) and (8.3a) but not (8.3b) since

$$\left(D_r^+ + I_{rv}^u d_u^v\right) J_\mu = -2 \sum_{t,u=1}^{\dim G/K} \mu_t\, F_{t\,r}^{\,u}\, J_{\mu+\lambda_u^t} \qquad (8.5)$$

where we have used (6.15). Therefore the coefficients of the expansion of the solutions of (6.2) in terms of $J_\mu(p,q)$ are determined by (8.3b) using (8.5). The solutions of (6.3) would be expanded in terms of the functions $J_{\mu,\nu}(p,q)$, and not only $J_\mu(p,q)$, and could be constructed using a similar procedure. However we have not completed understood the structures underlying this method of constructing solutions of (6.2) and (6.3), and we could not check if there exist more solutions besides those we have discussed.

9- Conclusions

The work presented in this paper and in the first[1] of this series establishes that the Fundamental Poisson bracket Relation (FPR) is intrinsically related to the algebraic and geometric properties of the symmetric spaces. It does not depend upon the dynamics of the model and it can be constructed in a quite simple and general way for any non singular lagrangian describing the motion of a particle on these spaces. We believe that this result can help understanding the origin of the FPR in other models that apparently are not related to symmetric spaces. The reason is that it has been conjectured[7] that integrable systems can be obtained, by using a reduction procedure[5,7], from models defined on symmetric spaces. In addition the Poisson bracket algebra satisfied by any set of quantities is respected by this reduction and therefore a FPR can be obtained for the reduced model under this procedure.

The construction of conserved charges using the representation theory of the algebra $\underset{\sim}{L}$ is not completely understood yet. Its importance relies much more on the fact that it sheds some light on the structures responsible for the existence of conserved charges than on the effective power it has in their pratical construction.

We believe our analysis could be further explored from a geometrical point of view. It would be interesting to see how these structures are carried to the quantum theory and if they are also present in two dimensional models defined on symmetric spaces.

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Appendix I: The "anomalous" FPR

In this appendix we derive an "anomalous" Fundamental Poisson bracket Relation (FPR) for the coset spaces which are not symmetric spaces and then show that it leads to a Lax pair equation for the geodesic motion on these spaces. Let $\mathcal{L}\ (q, \dot{q}, t)$ be a non singular lagrangian describing the motion of a particle on a coset space G/K and let $p_r = \frac{\partial \mathcal{L}}{\partial \dot{q}^r}$ be the canonical momenta conjugated to the coordinates q^r. We introduce an operator A in the same way as we did in (3.5), but in this case A does not have to be odd under any automorphism of the algebra of G. Using eq. (2.12) and playing with the symmetry of the Christoffel symbol and the antisymmetry of the Poisson bracket we get

$$\{A \overset{\otimes}{,} A\}_{PB} = p_r \Big\{ [\pi^r, K_u] \otimes \pi^u - \pi^u \otimes [\pi^r, K_u] + \\ + \frac{1}{2}\, \mathcal{P}([\pi^r, \pi_u]) \otimes \pi^u - \frac{1}{2}\, \pi^u \otimes \mathcal{P}([\pi^r, \pi_u]) \Big\} \qquad \text{(I.1)}$$

Defining the operators $\mathbb{P}$ and $\mathbb{R}$ as we did in (3.6) and an operator $\mathbb{S}$ as

$$\mathbb{S} = \frac{1}{2}\, \pi_r \otimes \pi^r \qquad \text{(I.2)}$$

we get

$$\{A \overset{\otimes}{,} A\}_{PB} = -\left[\frac{\mathbb{P}+\mathbb{R}}{2}, A\otimes 1\right] - \left[\frac{\mathbb{P}-\mathbb{R}}{2}, 1\otimes A\right] +$$

$$+ \mathcal{P}_R([\mathcal{S}, 1\otimes A]) - \mathcal{P}_L([\mathcal{S}, A\otimes 1]) \qquad (I.3)$$

where $\mathcal{P}_R$ $(\mathcal{P}_L)$ means we are acting with the projection $\mathcal{P}$ of $\underset{\sim}{g}$ onto $\underset{\sim}{p}$ (see section 2) on the right (left) entry of the tensor product. The relation above is the "anomalous" FPR for the coset space G/K. Using it we get

$$\{A, \tfrac{1}{2}\mathrm{Tr}\, A^2\}_{PB} = [A, B_2] + \frac{1}{2}\pi^u\, \mathrm{Tr}\{\mathcal{P}([\pi_u, A])A\} -$$

$$- \frac{1}{2}\mathcal{P}([\pi_u, A])\, \mathrm{Tr}(A\pi^u) \qquad (I.4)$$

where we have defined the operator

$$B_2 = \mathrm{Tr}_R\left\{\left(\frac{\mathbb{P}+\mathbb{R}}{2}\right)(1\otimes A)\right\} = -\mathrm{Tr}_L\left\{\left(\frac{\mathbb{P}-\mathbb{R}}{2}\right)(A\otimes 1)\right\} \qquad (I.5)$$

and where the subindex R (L) means we are taking the trace of the right (left) entry of the tensor product. Since the operator A is in the subspace $\underset{\sim}{p}$ we see from (2.2) that

$$\mathrm{Tr}\{\mathcal{P}([\pi_u, A])A\} = \mathrm{Tr}([\pi_u, A]\, A) = 0 \qquad (I.6)$$

Using (2.11) we get that the last term on the r.h.s. of (I.4) vanishes and therefore we have

$$\left\{ A , \frac{\mathrm{Tr}\, A^2}{2} \right\}_{PB} = [A , B_2] \quad . \tag{I.7}$$

Notice that the same result is not necessarily true for $\mathrm{Tr}\, A^N$ with $N > 2$ since (I.4) would contain traces of more than two generators and the calculations above would be representation dependent. The hamiltionian for the geodesic motion on G/K is $\mathrm{Tr}\, A^2$ and therefore eq. (I.7) becomes the Lax pair equation for such model

$$\frac{dA}{dt} = [A, B_2] \tag{I.8}$$

It then follows that the charges $\mathrm{Tr}\, A^N$ are conserved but are not necessarily in involution.

Appendix II

There exists a matrix realization of the points of a symmetric space G/K in terms of a matrix representation of the group G. This is called the principal variable and it is given by [7,8]

$$x(g) = g\,\sigma(g)^{-1} \qquad \text{(II.1)}$$

To each coset there corresponds a matrix x, since $x(gk) = x(g)$. Under left translations on G/K by elements of G ($\delta_L g = \varepsilon T g$, $T \in \underset{\sim}{g}$, $g \in G$), the principal variable transforms as

$$\delta_L x(g) = \varepsilon \left[T x(g) - x(g)\,\sigma(T) \right] \qquad \text{(II.2)}$$

The canonical transformations generated by the charges $X_L(T)$, defined in (4.10), correspond to these left translations when acting on functions of the coordinates only since

$$\varepsilon \{ x(g), X_L(T) \}_{PB} = \varepsilon\, g\, \Pi_u\, \sigma(g)^{-1}\, \mathrm{Tr} \left\{ \pi^u \left(g^{-1} T g - \sigma(g^{-1} T g) \right) \right\}$$

$$= \delta_L x(g) \qquad \text{(II.3)}$$

where we have used (2.11) in the last equality. For the non compact symmetric spaces, where the Iwasawa decomposition holds, we can consistenly define right translations on G/K by elements of B, ($\delta_R b = \varepsilon b t$, $t \in \underset{\sim}{b}$, $b \in B$). Under these translations the principal variable transforms as

$$\delta_R x(b) = \varepsilon \left[b t b^{-1} x(b) - x(b)\, \sigma(b t b^{-1}) \right] \tag{II.4}$$

The canonical transformations generated by the charges $X_R(t)$, defined in (5.8), correspond to these right translations when acting on functions of the coordinates only since, using (2.11) again, we get

$$\begin{aligned} \varepsilon \{ x(b), X_R(t) \}_{TB} &= \varepsilon\, b\, \Pi_\mu\, \sigma(b)^{-1}\, T_\lambda \left\{ \Pi^\mu (t - \sigma(t)) \right\} = \\ &= \delta_R x(b) \end{aligned} \tag{II.5}$$

When extended to phase space the left and right translations, (II.2) and (II.4), are canonical transformations in some cases only. Therefore the canonical transformations generated by $X_L(T)$ and $X_R(t)$, when acting on functions of the velocities, do not always coincide with the left and right translations. See appendix I of ref. [5] for more details.

Appendix III

In this appendix we prove relations (6.15) and (6.16). Using the definition (5.12) of P_r and k_r we get

$$[P_r, k_s] = \frac{1}{2}\left(f_{rs}^{u} - I_{rs}^{u} \right) P_u \tag{III.1}$$

where f_{rs}^{u} are the structures constants of $\underset{\sim}{b}$, $[t_r, t_s] = f_{rs}^{u} t_u$, and I_{rs}^{u} is defined by $(I_{rs}^{u} = I_{sr}^{u})$

$$I_{rs}^{u} P_u = \frac{1}{2}[\sigma(t_r), t_s] - \frac{1}{2}[t_r, \sigma(t_s)] \tag{III.2}$$

Using (5.14) and the fact that the Killing form of $\underset{\sim}{g}$ is invariant under the involutive automophism σ we get

$$\eta_{vu} I_{rs}^{u} = -\frac{1}{2}\left[f_{vs}^{t}\, \mathrm{Tr}\{t_t \sigma(t_r)\} + f_{vr}^{t}\, \mathrm{Tr}\{t_t \sigma(t_s)\} \right] \tag{III.3}$$

Again from (5.14), we have that

$$\eta_{rs} = \frac{1}{2}\mathrm{Tr}(t_r t_s) - \frac{1}{2}\mathrm{Tr}(t_r \sigma(t_s)) \tag{III.4}$$

But since $f_{vs}^{t}\,\mathrm{Tr}(t_t t_r) + f_{vr}^{t}\,\mathrm{Tr}(t_t t_s) = 0$, we can add it to (III.3) and use (III.4) to get

$$I_{rs}^{u} = \eta^{uv}\left[\eta_{rt} f_{vs}^{t} + \eta_{st} f_{vr}^{t}\right] \qquad \text{(III.5)}$$

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