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ABSTRACT

We construct in this work, the Weil and the universal BRS algrebras of theories that can have as a gauge symmetry a free differential (Sullivan) algebra, the natural extension of Lie algebras allowing the definition of p-form gauge potentials $(p > 1)$. The finite gauge transformations of these potentials are deduced from the infinitesimal ones and the group structure is shown. The geometrical meaning of these p-form gauge potentials is given by the notion of a Quillen superconnection.

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1. Introduction

The recent study of the algebraic structure of anomalies in gauge theories using methods of differential geometry has led to a deeper understanding of the symmetries in these gauge theories at the classical and at the quantum level. In particular, the universal BRS algebra which shows how the fields and the ghosts mix together and which was known since the work of Becchi, Rouet and Stora [1] (BRS) on the renormalisation of gauge theories, has been carefully studied in ref.[2]. In this algebra two differential operators are present, the d operator which is the usual exterior derivative on a manifold and the s operator which is the BRS operator (infinitesimal gauge transformations) operating on ghosts and fields. However, only Lie algebra-valued 1-form gauge potentials can be incorporated in this formalism.

Our aim in this paper is to extand this universal BRS algebra to theories that can have also p-form gauge potentials with $p > 1$ (antisymmetric tensor fields), that is, to theories with a free differential algebra (FDA) as an underlying symmetry. This kind of gauging has been extensively studied before by d'Auria and Fré and other authors [3,4] in the group manifold approach of Ne'eman and Regge [5] and has been applied in this context to construct classical lagrangians of supergravity theories where 2- and 3-form gauge potentials appear besides the 1-forms. These FDA's appear as the natural extension of Lie algebras which allows a mathematically consistent definition of a gauge theory for these antisymmetric gauge potentials. They are also well known in the mathematical litterature from the work of Sullivan [6].

Besides its mathematical interest and the non trivial generalizations that it contains, this universal BRS algebra will be useful for the quantization of theories based on FDA's. The quantization of gauge theories of antisymmetric tensor fields has been discussed in the litterature [7,14] and the constructed BRS transformations show the same ghost for ghost mechanism which we recover here using a purely algebraic approach. This mechanism is necessary for the elimination of the redundant degrees of freedom of the fields.

We stress here from the beginning, the difference between d and s operators which is not so evident in refs.[3,4] and also the difference between the fields (which are forms over a manifold) and the ghosts (the gauge parameters). This distinction is essential when constructing this BRS algebra.

The paper is organized as follows. In section 2., we give the definition of a free differential algebra and then introduce the notion of Quillen (super)forms the natural objects defining these symmetries. In section 3., we construct the Weil algebra [8,9], associated to the preceding FDA by defining connections and curvatures. In section 4., following the same approach of Stora [10], we construct the associated BRS transformations on all fields and ghosts by defining a generalized "russian" formula. In section 5., we construct the universal BRS algebra which enables to manipulate all these objects of connections, curvatures, ghosts, ghosts of ghosts,... and then study its $d, (d+s)$ and s cohomologies in the same spirit of ref.[2]. In section 6., we show that a FDA induces a non trivial extension of the group of gauge transformations. In section 7., we consider two simple illustrative examples. Section 8. contains our conclusions

2. Definition of free differential algebras

We begin this paper by a brief review Lie (super)algebras and then introduce the concept of free differential algebras.

2.1. Lie algebras

Let $\mathcal{G}$ be a graded vector space of dimension N with a basis $(E_\alpha, \alpha = 1, ..., N)$. A Lie (super)algebra is defined by writing the graded commutator of two generators of $\mathcal{G}$ as:

$$[E_\alpha, E_\beta] = C^\gamma_{\alpha\beta} E_\gamma$$

$C^\gamma_{\alpha\beta}$ are the graded antisymmetric structure constants.

The necessary and sufficient condition to be imposed in order to have a true Lie (super)algebra is the Jacobi identity:

$$[[E_\alpha, E_\beta], E_\gamma] + permutations = 0 \tag{1}$$

or equivalently:

$$\sum_{per(\alpha,\beta,\gamma)} C^\lambda_{\alpha\mu} C^\mu_{\beta\gamma} = 0$$

But this definition is not the only one of a Lie algebra. The dual formulation is also interesting, because it can be easily extended in a natural manner to define a FDA. So, one usually defines the dual vector space of $\mathcal{G}$ as $\mathcal{G}^*$ generated by the dual basis $(\chi^\alpha, \alpha = 1, ..., N)$ which are left invariant one-forms satisfying the Maurer-Cartan equations:

$$s\chi^\alpha + \frac{1}{2} C^\alpha_{\beta\gamma} \chi^\beta \wedge \chi^\gamma = 0 \tag{2}$$

where $\wedge$ is the wedge product of forms on $\mathcal{G}^*$ which will be ommitted in all what follows and s is a differential operator acting on $\mathcal{G}^*$. Then, one we can easily see that the Jacobi identity (1) is obtained from the condition that $s^2 = 0$ on χ^α .

2.2. Free differential algebras

A dual Lie algebra $\mathcal{G}^*$ is generated in degree one. That is, all the generators (χ^α) are one-forms on $\mathcal{G}^*$. Denote this algebra by $A_1 = \mathcal{G}^*$ to show this degree. s is an action from A_1 into $A_1 \wedge A_1$ (where here $\wedge$ is the wedge product of two algebras) and such that $s^2 = 0$. As was demonstrated by Sullivan [6], one can introduce a more general algebra by adding generators of higher degree, i.e. 2-forms, 3-forms, etc. on $\mathcal{G}^*$. There is a theorem which shows how to construct the most general FDA iteratively:

i) One starts from a dual Lie algebra A_1 as before.

ii) Then introduce n generators of degree two, i.e. 2-forms $(\rho^i, i = 1, ..., n)$. These generators generate an algebra A_2 and so s will be an action from A_2 into $A_1 \wedge A_2 \oplus A_1 \wedge A_1 \wedge A_1$.

So its action on the basis (ρ^i) is:

$$s\rho^i = -D^i_{\alpha j} \chi^\alpha \rho^j + \frac{1}{6} Q^i_{\alpha\beta\gamma} \chi^\alpha \chi^\beta \chi^\gamma \tag{3}$$

where $D^i_{\alpha j}$ and $Q^i_{\alpha\beta\gamma}$ are constants such that:

$$Q^i_{\alpha\beta\gamma} = -(-)^{ab} Q^i_{\beta\alpha\gamma} = -(-)^{bc} Q^i_{\beta\gamma\alpha}$$

where the indices a, b, and c are the grades of respectively α, β and γ (0 for a bosonic index and 1 for a fermionic one).

Then the condition $s^2 = 0$ on ρ^i will impose further constraints. One finds:

$$D^i_{\beta j} D^j_{\gamma k} - D^i_{\gamma j} D^j_{\beta k} = C^\alpha_{\beta\gamma} D^i_{\alpha k} \tag{4}$$

and:

$$\sum_{per(\alpha,\beta,\gamma,\delta)} (D^i_{\alpha j} Q^j_{\beta\gamma\delta} - \frac{3}{2} \sum_\mu C^\mu_{\alpha\beta} Q^i_{\mu\gamma\delta}) = 0 \tag{5}$$

Equation (4) says that $D^{(r)}$ such that $D^{(r)}(E_\alpha)^i_j = D^i_{\alpha j}$ is a matrix representation of the Lie algebra $\mathcal{G}$ labelled by (r) and n the number of generators (ρ^i) is the dimension of this representation $D^{(r)}$, whereas eq.(5) means that Q^i (the 3-cochain of $\mathcal{G}$ with values in the representation $D^{(r)}$ such that $Q^i(E_\alpha, E_\beta, E_\gamma) = Q^i_{\alpha\beta\gamma}$) is in fact a 3-cocycle of $\mathcal{G}$. One can easily see this by writing eq.(5) in the form:

$$\sum_{per(\alpha,\beta,\gamma,\delta)} ((D^{(r)}Q)^i(E_\alpha, E_\beta, E_\gamma, E_\delta) - \frac{3}{2}Q^i([E_\alpha, E_\beta], E_\gamma, E_\delta)) = 0$$

To have a non trivial extension of A_1, Q^i must be a non trivial 3-cocycle. So, the possible extensions are in one to one correspondence with the cohomology classes of $H(\wedge^3 \mathcal{G}^*, \delta^{(r)})$, where $\delta^{(r)}$ is the covariant s operator in the $D^{(r)}$ representation: $\delta^{(r)i}_{\ \ j} = s\delta^i_j + D^i_{\alpha j}\chi^\alpha$. This can be easily proved by considering two extensions one with Q^i and the other with $Q^i + (\delta^{(r)}P)^i$ and see that they are the same modulo a redefinition of the generators from (ρ^i) to $(\rho'^i = \rho^i - P^i)$. We note that there is a theorem by Eilenberg and Chevalley [11] saying that there is no non trivial k-cocycle of $\mathcal{G}$ in the non trivial representations if $\mathcal{G}$ is semi-simple. So to have cocycles in non trivial representations, one must work with non semi-simple Lie algebras.

iii) One can continue the procedure by introducing 3-forms etc. In general, having constructed the algebra up to order $(k-1)$, we can add generators of degree k i.e. k-forms (π^A) generating an algebra A_k, then the action of s on (π^A) will be:

$$s\pi^A = -d^A_{\alpha B}\chi^\alpha\pi^B + \Omega^A[\chi^\alpha, \rho^i, ...]$$

As before, $d^{(s)}$ defined by $d^{(s)}(E_\alpha)^A_B = d^A_{\alpha B}$ forms a matrix representation of $\mathcal{G}$ and $\Omega^A[\chi^\alpha, \rho^i, ...]$ is a (k+1)-cocycle of the algebra:

$$B_{k+1} = \bigoplus_{(\sum i_q = k+1, 1 \le i_q < k)} A_{i_1} \wedge A_{i_2} \wedge \cdots \wedge A_{i_q}$$

with values in the representation $d^{(s)}$ and with respect to its covariant s operator. To have a non trivial extension, this cocycle must be non trivial.

By a theorem of Sullivan [6], this is the most general minimal extension of a dual Lie algebra $A_1 = \mathcal{G}^*$. It is free in the sense that there is no algebraic relations between the generators besides the (anti)commutativity. Any other algebra is isomorphic to the tensor product of this minimal extension with a contractible algebra of the form $\bigwedge_i(K^i, sK^i)$ where K^i are other generators.

2.3. Quillen superforms

In order to understand the action of s on $(\chi^\alpha, \rho^i, ...)$, we shall now introduce appropriate forms which have been first used by Quillen [12] in another context.

For this, consider a FDA η^*, extension of a dual Lie algebra $\mathcal{G}^*$, generated by the forms $\{\sigma^{(p)}_a\} = \{(\sigma^{(1)}_a), (\sigma^{(2)}_b), ...\}$ where the indices $a, b, ...$ label representations of $\mathcal{G}$ and $p = 1, 2, ...$ are the degrees of the forms. To each set of p-forms $(\sigma^{(p)}_a)$, we can associate a dual basis $(E^{a(p-1)})$ and then define an extension of the Lie algebra $\mathcal{G}$ as the vector space η generated by the basis $\{E^{a(p-1)}\} = \{(E^{a(0)}), (E^{b(1)}), ...\}$, $(E^{a(0)})$ are the usual generators of the Lie algebra $\mathcal{G}$ with (a) running in the adjoint representation of $\mathcal{G}$ and $E^{a(p-1)}$ commutes or anticommutes with s according to the parity of $(p-1)$: $s.E^{(p)} = (-1)^p E^{(p)}.s$.

The Quillen (super)form is defined by:

$$\Sigma = \sum_{p,a} \sigma^a_{(p)} E^{(p-1)}_a$$

It is the sum of forms of different degrees and is a generalisation of a Lie algebra valued form to an η-valued form. By definition it anticommutes with s since the product $(\sigma^a_{(p)} E^{(p-1)}_a)$ has an odd degree with respect to s.

We define also the p-cocycles:

$$C^{(q)} : \wedge^q \eta \longrightarrow \eta$$

$q = 2, 3, ...$ by their action on a basis of η, where for example $C^{(2)}$ is just the usual commutator when it acts on two generators of the Lie algebra $\mathcal{G}$. $C^{(q)}$ is completely defined by a set of coefficients $C^{(q)a}_{bc...}$ and for consistency, one must impose $s.C^{(q)} = (-)^q C^{(q)}.s$. Then the action of s on Σ is:

$$s\Sigma = -\sum_{q=2,3,...} \frac{1}{q!} C^{(q)}(\Sigma, ..., \Sigma)$$

and the cocycle conditions can be derived by definition from $s^2 = 0$:

$$\sum_{p,q} \frac{(-1)^p}{(p-1)!q!} C^{(p)}(C^{(q)}(\Sigma, ..., \Sigma), \Sigma, ..., \Sigma) = 0$$

which puts further constraints on the set of coefficients $C^{(q)a}_{bc...}$.

As an example, we shall now show that eqs.(2) and (3) can be written in a single equation using the Quillen (super)forms. So, we limit ourselves to the first extension (see section 2.2.), i.e. we introduce besides the 1-forms $\chi^\alpha, \alpha = 1, ..., N$ the 2-forms $\rho^i, i = 1, ..., n$. The extension of the Lie algebra $\mathcal{G}$ is the algebra $\eta = \mathcal{G} \oplus \Gamma$ generated by the two sets of generators $(E_\alpha, \alpha = 1, ..., N)$ and $(E_i, i = 1, ..., n)$ denoted commonly by $(E_a, a = \alpha, i)$. (E_α) are the generators of $\mathcal{G}$ defined before, whereas (E_i) are generators of Γ constructed as the dual basis of (ρ^i) and anticommuting with s.

We define a commutator c in η by its action on these generators as:

$$c : \eta \wedge \eta \longrightarrow \eta$$

$$\begin{aligned} c(E_\alpha, E_\beta) &= E_\gamma C^\gamma_{\alpha\beta} \\ c(E_\alpha, E_i) &= +E_j D^j_{\alpha i} \\ c(E_i, E_\alpha) &= -E_j D^j_{\alpha i} \\ c(E_i, E_j) &= 0 \end{aligned}$$

and a 3-cocycle q in η anticommuting with s by:

$$q : \eta \wedge \eta \wedge \eta \longrightarrow \eta$$

$$\begin{aligned} q(E_\alpha, E_\beta, E_\gamma) &= E_i Q^i_{\alpha\beta\gamma} \\ q(E_i, \cdots, \cdots) &= 0 \end{aligned}$$

The coefficients $C^\gamma_{\alpha\beta}$, $D^j_{\alpha i}$ and $Q^i_{\alpha\beta\gamma}$ are the same as before.

The Quillen (super)form is a form η-valued constructed from (χ^α) and (ρ^i):

$$\Sigma = \Sigma^a E_a = \chi^\alpha E_\alpha + \rho^i E_i$$

it is the sum of the 1-forms (χ^α) and the 2-forms (ρ^i), but have an odd total degree with respect to s.

Then, we easily check that eqs. (2) and (3) can be put in a single equation using the definitions of the commutator c and the 3-cocycle q as:

$$s\Sigma = -\frac{1}{2} c(\Sigma, \Sigma) - \frac{1}{6} q(\Sigma, \Sigma, \Sigma) \tag{6}$$

and see that $s^2\Sigma = 0$ by using the Jacobi identity, the representation and 3-cocycle conditions.

3. The Weil algebra

As said in the beginning, one has to make a difference between the s operator which defines the underlying symmetry (FDA) and acts on forms of η^* (possibly functions over a manifold) and the action of the d operator which acts on connections and curvatures.

To construct the associated Weil algebra [8,9], we must introduce for each generator of the FDA a corresponding connection and its related curvature and define the action of the exterior derivative $d = dx^\mu \partial / \partial x^\mu$ of the manifold on these new fields.

In the general case, the Weil algebra $W(\eta)$ associated to a FDA η^* is given by introducing the connection:

$$A = \sum_{p,a} A^a_{(p)} E^{(p-1)}_a$$

and the curvature:

$$F = \sum_{p,a} F^a_{(p)} E^{(p-1)}_a$$

all Quillen forms where $(E^{a(p-1)})$ were defined before. Then the the action of d on A and F is given by:

$$dA = F - \sum_p \frac{1}{p!} C^{(p)}(A, ..., A)$$

$$dF = -\sum_p \frac{(-)^p}{(p-1)!} C^{(p)}(F, A, ..., A)$$

and one can easily see that $d^2 = 0$ on A and F. We suppose as s that d also commutes or anticommutes with $C^{(p)}$ according to the parity of (p): $d.C^{(p)} = (-)^p C^{(p)}.d$. The second relation is usually known as the Bianchi identity in the Lie algebra case. This construction is a trivialization of the FDA since the d-cohomology of the Weil algebra is trivial as we shall see in section 5.2.

For example, the action of the operator d in the Weil algebra $W(\eta^*)$ of the FDA $\eta^* = \mathcal{G}^* \oplus \Gamma^*$ generated by only (χ^α) and (ρ^i) is defined by introducing the $\mathcal{G}$-valued forms, the one-form connection $A = A^\alpha_\mu(x) dx^\mu E_\alpha$ and the two-form curvature $F = \frac{1}{2}F^{\alpha}_{\mu\nu}(x)dx^{\mu}dx^{\nu}E_{\alpha}$ such that:

$$dA = F - \frac{1}{2}c(A,A) \tag{7a}$$

and:

$$dF = c(F,A) \tag{7b}$$

This corresponds to the usual Weil algebra associated to $\mathcal{G}^*$ generated by (χ^{α}).

The other part which corresponds to Γ^* is defined by Γ-valued forms, the two-form $B = \frac{1}{2}B^{i}_{\mu\nu}(x)dx^{\mu}dx^{\nu}E_i$ and its curvature $H = \frac{1}{6}H^{i}_{\mu\nu\rho}dx^{\mu}dx^{\nu}dx^{\rho}E_i$ which is a three-form. They correspond to the antisymmetric tensors $B^{i}_{\mu\nu}$ and $H^{i}_{\mu\nu\rho}$ carrying an index in a representation $D^{(r)}$ of $\mathcal{G}$. Then the action of d is:

$$dB = H - c(A,B) - \frac{1}{6}q(A,A,A) \tag{8a}$$

and:

$$dH = c(F,B) - c(A,H) - \frac{1}{2}q(F,A,A) \tag{8b}$$

we see here that d appears always in the following form $\Delta^{i}_{j} = d\delta^{i}_{j} + D^{i}_{\alpha j}A^{\alpha}$ which is the covariant exterior derivative in the $D^{(r)}$ representation.

We can easily verify that $d^2 = 0$ on the Weil algebra generated by (A^{α}), (F^{α}), (B^{i}) and (H^{i}), whereas the covariant derivative defined above obeys $\Delta^{i}_{j}\Delta^{j}_{k} = D^{i}_{\alpha k}F^{\alpha}$.

4. The BRS algebra

We have defined in the last chapters a FDA η^* as a symmetry based on an extension of a Lie algebra by introducing higher degree forms. Then the corresponding Weil algebra $W(\eta)$ has been constructed by introducing the related connections and curvatures.

Now, we want to define the action of the s operator (gauge transformations) on the fields $A^{\alpha}, B^{i}, F^{\alpha}, H^{i}$ etc. For this, we must regard now the generators of the FDA as functions (0-forms) over a manifold: $\chi^{\alpha}(x)$ and $\rho^{i}(x)$. We shall associate to them a ghost degree (which is equivalent to the form degree on $\mathcal{G}^*$ defined in chapter 2. for χ^{α} and ρ^{i}): 1 to $\chi^{\alpha}(x)$ and 2 to $\rho^{i}(x)$. We call these fields ghosts of maximal degree.

4.1. General case

The most direct way to define gauge transformations (maybe not the usual one) is by constructing the BRS algebra, i.e. by postulating the "russian" formula of Stora [10]. This formula is simply a construction of an isomorphic algebra to the Weil algebra where the d operator is replaced by $(d+s)$ and all the generators are replaced by translated ones. So all the algebraic operations which were valid in the Weil algebra remain valid in this isomorphic algebra, such as for example the fact that $d^2 = 0$ which becomes $(d+s)^2 = 0$ and hence implies that $ds + sd = 0$ since $s^2 = 0$. This algebra is in some sense an interpolation between the FDA and the Weil algebra and so defines the action of s i.e. gauge transformations on the fields.

The gauge transformations are completly defined by the ghosts $\chi^{\alpha}(x)$ and $\rho^{i}(x)$ and other ghosts which appear in the translated generators as we shall see. This kind of construction has been used in the context of anomaly calculations in gauge theories [10,13]. Here we shall follow the same steps and try to construct the BRS algebra associated to the FDA and the Weil algebra constructed in chapters 2 and 3.

We shall discuss at the beginning the case of a general FDA η^*. Its BRS algebra is constructed from the Weil algebra $W(\eta)$ by writing the following translations on the fields:

$$A^{\alpha}_{(p)} \longrightarrow A^{\alpha(transl.)} = A^{\alpha}_{(p)} + A^{\alpha(1)}_{(p-1)} + \cdots + A^{\alpha(p)}_{(0)}$$

and:

$$F^{\alpha}_{(p+1)} \longrightarrow F^{\alpha(transl.)} = F^{\alpha}_{(p+1)} + F^{\alpha(1)}_{(p)} + \cdots + F^{\alpha(p-1)}_{(2)}$$

for $p = 1,2,...$ where we introduce p ghosts for the gauge potentials $A^a_{(p)}$ and $(p-2)$ for the curvatures $F^a_{(p)}$. The superscript indicates the ghost degree and the subscript the form degree. We note that $A^a_{(p)}$ and $F^a_{(p+1)}$ are the connections and the curvatures defined in the Weil algebra whereas $A^{a(p)}_{(0)}$ are the ghosts of maximal degree of the FDA and that in order to end with a free algebra (no algebraic relations between the generators besides commutativity), one has to translate also the curvature.

The action of d and s on all the fields and the ghosts is obtained by expanding this translated algebra for each bidegree: the ghost degree and the form degree. We recover the Weil algebra at one extreme and the FDA at the other.

In this BRS algebra, one can prove the triviality of the d and $(d+s)$ cohomology algebras, whereas the s cohomology algebra is just that of the minimal algebra generated by $(A^{a(p)}_{(0)})$ and $(F^{a(p-1)}_{(2)})$ for $p = 1,2,...$ (last ghosts in the translations). We shall discuss this in the next chapter.

In order to understand this construction, we shall give two examples, the first is the usual one known as the "russian" formula of R.Stora [10] in the case of gauge theories of Lie algebras and the second one is the case of extensions to the first order.

4.2. The Russian formula

First we recall the usual construction based on a Lie algebra (usual gauge theories) [10,13,2]. The Weil algebra based on a Lie algebra $\mathcal{G}$ is generated by the $\mathcal{G}$-valued connections $A = A^\alpha E_\alpha$ and curvatures $F = F^\alpha E_\alpha$, where (E_α) are the generators of $\mathcal{G}$.

The action of d is given by:

$$dA = F - \frac{1}{2}c(A,A) = F - A^2$$

$$dF = c(F,A) = FA - AF$$

We then construct an isomorphic algebra by making the following translations:

$$d \longrightarrow d+s \qquad A \longrightarrow A+\chi \qquad F \longrightarrow F$$

and we define a bidegree for each of these quantities: one is the form degree and the other is the ghost degree. For instance, we define the bidegree $(1,0)$ for d and

A, $(2,0)$ for F and $(0,1)$ for s and χ. This isomorphic algebra is constructed by $(A+\chi)$ and F such that:

$$(d+s)(A+\chi) = F - (A+\chi)(A+\chi)$$

$$(d+s)F = F(A+\chi) - (A+\chi)F$$

Then expanding these relations for each bidegree, we find the BRS transformations on the fields:

$$(2,0) \qquad dA = F - A^2$$

$$(1,1) \qquad sA + d\chi = -A\chi - \chi A$$

$$(2,0) \qquad s\chi = -\chi^2$$

and:

$$(3,0) \qquad dF = FA - AF$$

$$(2,1) \qquad sF = F\chi - \chi F$$

Hence, we see that this defines the action of s on A and F (gauge transformations). The other equations were already known in the FDA and in the Weil algebra.

It is shown in reference [2] that in this algebra generated by

$$(A^\alpha, F^\alpha, \chi^\alpha, d\chi^\alpha, \alpha = 1,...,N)$$

or related generators , the cohomology of d and $(d+s)$ is trivial. That is, each cocycle of d or $(d+s)$ is a cobounbary. The s cohomology has also been computed and it was shown that it is connected to the cohomology of s modulo d which is the one needed for anomaly computations in gauge theories.

4.3. The extension to the first order

Similarly, we want to define gauge transformations for B^i and H^i. So, we write first the total Weil algebra given by the action of d on A^α, F^α, B^i and H^i. See eqs.(7a-b) and (8a-b).

To define the BRS algebra, we have to construct an isomorphic algebra to this Weil algebra where we replace d by $(d+s)$ and A,F,B and H by translated objects in such a way that all the newly introduced generators are free. The BRS algebra corresponding to the first eqs.(7a-b) was defined in section 4.2. For eqs.(8a-b), we can easily see that due to the presence of the terms $q(A,A,A)$ and $q(F,A,A)$, the translations must be the following in order to keep the algebra free:

$$B \longrightarrow B+b+\rho$$

$$H \longrightarrow H+h$$

where h, b and ρ are new Γ-valued 0,1 and 2-forms respectively. b,ρ and h can be interpreted as respectively the ghosts of B,b and H.

We associate the following bidegrees to these fields: $(2,0)$ to B, $(1,1)$ to b, $(0,2)$ to ρ, $(3,0)$ to H and $(2,1)$ to h. When we make all the translations in eqs.(8a-b) and expand them for each bidegree we get the BRS transformations:

$$\begin{aligned}
&(3,0) \qquad dB = H - c(A,B) - \frac{1}{6}q(A,A,A)\\
&(2,1) \qquad db + sB = h - c(\chi,B) - c(A,b) - \frac{1}{2}q(A,A,\chi)\\
&(1,2) \qquad d\rho + sb = -c(\chi,b) - c(A,\rho) - \frac{1}{2}q(A,\chi,\chi)\\
&(0,3) \qquad s\rho = -c(\chi,\rho) - \frac{1}{6}q(\chi,\chi,\chi)
\end{aligned}$$

and:

$$\begin{aligned}
&(4,0) \qquad dH = c(F,B) - c(A,H) - \frac{1}{2}q(F,A,A)\\
&(3,1) \qquad dh + sH = c(F,b) - c(\chi,H) - c(A,h) - q(F,\chi,A)\\
&(2,2) \qquad sh = c(F,\rho) - c(\chi,h) - \frac{1}{2}q(F,\chi,\chi)
\end{aligned}$$

So, we see that we have defined all the gauge transformations of the fields. This kind of gauge transformations is very interesting in gauge theories, it contain the well known ghost for ghost mechanism. See [7,14]. What is new here is the appearance of a ghost for H. It can be forgotten only if B and H are in the trivial representation and $q(A,A,A)$ is absent or replaced by a Chern-Simons term $tr(AF - \frac{1}{3}A^3)$, for this one must allow the appearance of terms of the form $G^i_{ab}F^aA^b$ in eq.(8a). In these cases one finds the usual gauge transformations of refs.[7,14] (ghost for ghost mechanism) or of ref.[15] (Green-Schwartz mechanism).

5.The universal BRS algebra of a free differential algebra

After having defined all the actions of s and d operators on the fields, we shall now study in more detail the algebra generated by these fields at the first extension. We shall pursue the same steps as in ref.[2] for the case of a Lie algebra.

5.1. The universal BRS algebra $A(\eta)$

Let $\eta = \mathcal{G} \oplus \Gamma$ be the extension of the finite dimensional real Lie algebra $\mathcal{G}$ with basis $(E_\alpha, \alpha = 1,..,N)$ and $(E_i, i = 1,...,n)$. We consider four copies $\mathcal{G}^*_A, \mathcal{G}^*_F, \mathcal{G}^*_\chi$ and $\mathcal{G}^*_\phi$ of the dual space $\mathcal{G}^*$ with dual basis denoted respectively by $(A^\alpha), (F^\alpha), (\chi^\alpha)$ and (ϕ^α) and also eight copies $\Gamma^*_B, \Gamma^*_H, \Gamma^*_b, \Gamma^*_\rho, \Gamma^*_h, \Gamma^*_\xi, \Gamma^*_\varsigma$ and Γ^*_θ of the dual space Γ^* with dual basis (B^i), (H^i), (b^i), (ρ^i), (h^i), (ξ^i), (ς^i) and (θ^i).

We define $A(\eta)$ as the free graded commutative algebra generated by A^α, χ^α in degree one, by H^i, h^i, ξ^i and ς^i in degree three, by $F^\alpha, \phi^\alpha, B^i, b^i, \rho^i$ in degree two and by θ^i in degree four. That is:

$$A(\eta) = A(\mathcal{G}) \otimes A(\Gamma)$$

where:

$$A(\mathcal{G}) = (\wedge \mathcal{G}^*_A) \otimes (S\mathcal{G}^*_F) \otimes (\wedge \mathcal{G}^*_\chi) \otimes (S\mathcal{G}^*_\phi)$$

is the algebra of ref.[2], and:

$$A(\Gamma) = (S\Gamma^*_B) \otimes (\wedge\Gamma^*_H) \otimes (S\Gamma^*_b) \otimes (S\Gamma^*_\rho) \otimes (S\Gamma^*_\theta) \otimes (\wedge\Gamma^*_h) \otimes (\wedge\Gamma^*_\xi) \otimes (\wedge\Gamma^*_\varsigma)$$

$\wedge\mathcal{G}^*$ is the exterior algebra $\oplus_{n\in N}\wedge^n\mathcal{G}^*$ of multilinear antisymmetric forms on $\mathcal{G}$ graded by giving the degree n to the elements of $\wedge^n\mathcal{G}^*$. $S\mathcal{G}^*$ is the algebra $\oplus_{n\in N}S^n\mathcal{G}^*$ of the polynomials on $\mathcal{G}$ evenly graded by giving the degree $(2n)$ to the elements of $S^n\mathcal{G}^*$. $\wedge\Gamma^*$ is the exterior algebra $\oplus_{n\in N}\wedge^n\Gamma^*$ of multilinear antisymmetric forms on Γ graded by giving the degree $(3n)$ to the elements of $\wedge^n\Gamma^*$. $S\Gamma^*$ is the algebra $\oplus_{n\in N}S\Gamma^*$ of the polynomials on $\mathcal{G}$ evenly graded by giving the degree $(2n)$ to the elements of $S^n\Gamma^*$ (but $(4n)$ to the elements of $S^n\Gamma^*_\theta$). $\otimes$ is the (skew)tensor product of graded algebras [16].

On the space $\eta \otimes A(\eta)$, there is a natural bilinear bracket $c(...,...)$ defined by:

$$c(X \otimes P, Y \otimes Q) = c(X,Y) \otimes (P.Q)$$

for any $X,Y \in \eta$ and $P,Q \in A(\eta)$.

There is also a trilinear bracket $q(...,...,...)$ defined by:

$$q(X\otimes P, Y\otimes Q, Z\otimes R) = q(X,Y,Z)\otimes (P.Q.R)$$

for any $X,Y,Z\in\eta$ and $P,Q,R\in A(\eta)$.

We use the same notations for the bilinear (trilinear) bracket in $\eta\otimes A(\eta)$ and the commutator (3-cocycle) in η defined in section 2.3.

The η-valued forms introduced in the last chapter are elements of $\eta\otimes A(\eta)$: $A=\sum_\alpha E_\alpha\otimes A^\alpha$ and similarly for F,χ,ϕ ,whereas $B=\sum_i E_i\otimes B^i$ and similarly for $H,h,\xi,\varsigma,b,\rho,\theta$.

With these notations, we get:

$$dA = F-\frac{1}{2}c(A,A) \qquad dF = c(F,A)$$
$$d\chi=\phi \qquad d\phi = 0$$

and:

$$sA=-\phi-c(A,\chi) \qquad sF=c(F,\chi)$$
$$s\chi=-\frac{1}{2}c(\chi,\chi) \qquad s\phi=c(\phi,\chi)$$

for the Lie algebra part, together with:

$$dB = H-c(A,B)-\frac{1}{6}q(A,A,A)$$
$$dH = c(F,B)-c(A,H)-\frac{1}{2}q(F,A,A)$$
$$db=\xi \qquad d\xi=0$$
$$d\rho=\varsigma \qquad d\varsigma=0$$
$$dh=\theta \qquad d\theta=0$$

and:

$$sB = h-\xi-c(\chi,B)-\frac{1}{2}q(A,A,\chi)-c(A,b)$$
$$sH=-\theta+c(F,b)-c(\chi,H)-q(F,\chi,A)-c(A,h)$$
$$sb=-\varsigma-c(\chi,b)-\frac{1}{2}q(A,\chi,\chi)-c(A,\rho)$$
$$s\rho=-c(\chi,\rho)-\frac{1}{6}q(\chi,\chi,\chi)$$

$$sh=c(F,\rho)-c(\chi,h)-\frac{1}{2}q(F,\chi,\chi)$$
$$s\xi=-c(\chi,\xi)+c(\phi,b)+c(F,\rho)-c(A,\varsigma)-\frac{1}{2}q(F,\chi,\chi)+q(A,\phi,\chi)$$
$$-\frac{1}{2}c(c(A,A),\rho)+\frac{1}{4}q(c(A,A),\chi,\chi)$$
$$s\varsigma=c(\phi,\rho)-c(\chi,\varsigma)-\frac{1}{2}q(\phi,\chi,\chi)$$
$$s\theta=c(\phi,h)-c(F,\varsigma)-c(\chi,\theta)-q(F,\phi,\chi)$$
$$-c(c(F,A),\rho)-\frac{1}{2}q(c(F,A),\chi,\chi)$$

We have defined $dA^\alpha,...,dB^i,...,sA^\alpha,...,sB^i,...$ in $A(\eta)$ by:

$$dA=\sum_\alpha E_\alpha\otimes dA^\alpha,...,dB=\sum_i E_i\otimes dB^i$$

and:

$$sA=\sum_\alpha E_\alpha\otimes sA^\alpha,...,sB=\sum_i E_i\otimes sB^i$$

Then,

$$(A^\alpha,...,B^i,...)\longrightarrow(dA^\alpha,...,dB^i,...)$$

and:

$$(A^\alpha,...,B^i,...)\longrightarrow(sA^\alpha,...,sB^i,...)$$

extend respectively uniquely as antiderivations d and s of $A(\eta)$. Then one can check that d and s are of degree one and satisfy:

$$d^2=0 \qquad ds+sd=0 \qquad and \qquad s^2=0$$

Thus d,s and $(d+s)$ are three differentials on the graded algebra $A(\eta)$, we denote the corresponding graded algebras of cohomology by $H(d),H(s)$ and $H(d+s)$. We shall study these algebras in the next section.

We introduce now the underlying bigraduation on $A(\eta)$ by giving the following bidegrees:

$$A^\alpha\longrightarrow(1,0) \qquad F^\alpha\longrightarrow(2,0) \qquad \chi^\alpha\longrightarrow(0,1) \qquad \phi^\alpha\longrightarrow(1,1)$$
$$B^i\longrightarrow(2,0) \qquad b^i\longrightarrow(1,1) \qquad \rho^i\longrightarrow(0,2) \qquad H^i\longrightarrow(3,0)$$
$$h^i\longrightarrow(2,1) \qquad \xi^i\longrightarrow(2,1) \qquad \varsigma^i\longrightarrow(1,2) \qquad \theta^i\longrightarrow(3,1)$$

So, we have:

$$A(\eta) = \bigoplus_{(r,s)\in N^2} A^{r,s}(\eta)$$

with:

$$A^{r,s}(\eta).A^{k,l}(\eta) \subset A^{r+k,s+l}(\eta)$$

and the total degree of an homogeneous element of bidegree (r,s) is $(r+s)$. The differentials d and s are respectively of bidegrees $(1,0)$ and $(0,1)$ so the cohomology algebras $H(d)$ and $H(s)$ are bigraded algebras,

$$H(d) = \oplus_{(r,s)\in N^2} H^{r,s}(d)$$

and:

$$H(s) = \oplus_{(r,s)\in N^2} H^{r,s}(s)$$

whereas $H(d+s)$ is simply a graded algebra.

We shall refer to the above structure as *the universal BRS algebra of the FDA* η^*.

We remark that the subalgebra generated by only A^α, F^α, B^i and H^i is stable by d and that equipped with d, it is just the Weil algebra of the FDA η : $W(\eta)$. See chapter 3.

5.2. The d and $(d+s)$ cohomologies of $A(\eta)$

We have the following theorem concerning $H(d)$ and $H(d+s)$ similar to a theorem of reference [2].

Theorem:

(a) The d cohomology of $A(\eta)$ is trivial, i.e. $H^{k,l}(d) = 0$ for any positive integers k,l such that $k+l \geq 1$ and $H^{0,0}(d) = R$.

(b) The $(d+s)$ cohomology of $A(\eta)$ is trivial, i.e. $H^n(d+s) = 0$ for any integer $n \geq 1$ and $H^0(d+s) = R$.

Proof:

(a) $A(\eta)$ is freely generated by the homogeneous elements

$$A^\alpha, F^\alpha, B^i, b^i, \rho^i, h^i$$

and

$$dA^\alpha, dF^\alpha, dB^i, db^i, d\rho^i, dh^i$$

Thus, $A(\eta) = (\otimes_\alpha\{A^\alpha, dA^\alpha\}) \otimes \cdots \otimes (\otimes_i\{B^i, dB^i\}) \otimes \cdots$ a tensor product of graded differential algebras. By the Kunneth theorem [17], we have:

$$H(A(\eta),d) = \otimes H(\otimes_\alpha\{A^\alpha, dA^\alpha\}) \otimes \cdots \otimes H(\otimes_i\{B^i, dB^i\}) \otimes \cdots$$

But an algebra of the form $\{a, da\}$ is a contractible algebra and so has a trivial cohomology by Sullivan theorem [6]. Then:

$$H(d) = H^{0,0}(d) = R$$

(b) From the definition of $A(\eta)$, it follows that $A(\eta)$ is also generated by

$$A^\alpha, F^\alpha, B^i, b^i, \rho^i, h^i$$

and by $(d+s)A^\alpha, ..., (d+s)h^i$. Thus again $(A(\eta), d+s)$ is contractible so that:

$$H(d+s) = H^0(d+s) = R$$

5.3. The s cohomology of $A(\eta)$

To study $H(s)$, we first write the free graded commutative differential algebra $(A(\eta), s)$ as a tensor product of a contractible algebra and a minimal algebra [6]. This can be done by noticing that $A(\eta)$ is freely generated by the homogeneous elements:

$$A^\alpha, sA^\alpha, \chi^\alpha, F^\alpha, B^i, sB^i, b^i, sb^i, \rho^i, H^i, h^i, sH^i$$

now the algebra generated by:

$$A^\alpha, sA^\alpha, B^i, sB^i, b^i, sb^i, h^i, sh^i$$

is contractible, whereas the algebra generated by $\chi^\alpha, F^\alpha, \rho^i$ and h^i is isomorphic to:

$$S\mathcal{G}_F^* \otimes \wedge\mathcal{G}_\chi^* \otimes \wedge\Gamma_h^* \otimes S\Gamma_\rho^*$$

as graded algebra and is minimal for s in view of:

$$\begin{aligned} s\chi &= -\frac{1}{2}c(\chi,\chi) \\ sF &= c(F,\chi) \\ s\rho &= -c(\chi,\rho) - \frac{1}{2}q(\chi,\chi,\chi) \\ sh &= c(F,\rho) - c(\chi,h) - \frac{1}{2}q(F,\chi,\chi) \end{aligned}$$

We see that:

i) $s\chi$ and $s\rho$ is the differential over $\wedge\mathcal{G}^* \otimes S\Gamma^*$ which enables to define the cohomology of $\eta = \mathcal{G} \oplus \Gamma$ with real values.

ii) sF and sh define the action of η on the space: $S\mathcal{G}^* \otimes \wedge\Gamma^*$.

Then we can say that the s cohomology of $A(\eta)$ coincides with the s cohomology of the subalgebra generated by the elements $\chi^\alpha, F^\alpha, \rho^i$ and h^i. However, we can not calculate the cohomology $H(s)$ exactly as in the reductive Lie algebra case where $H(s)$ was the tensor product of the algebra of invariants polynomials $I_S(\mathcal{G})$ on $\mathcal{G}$ with the algebra of invariant forms $I_\wedge(\mathcal{G})$ on $\mathcal{G}$: $H(s) = I_S(\mathcal{G}) \otimes I_\wedge(\mathcal{G})$. See [2]. Here a phenomenon of creating and killing cohomology appears as we shall see in the following example.

5.4. Example

Consider the simple example of $SU(2)$ Lie algebra:

$$s\chi^\alpha = -\frac{1}{2}\varepsilon^\alpha_{\beta\gamma}\chi^\beta\chi^\gamma$$

where $\varepsilon^\alpha_{\beta\gamma}$ are the structure constants of $SU(2)$ Lie algebra. The only invariant form on $SU(2)$ is $tr\chi^3$ since $SU(2)$ is of rank 1. See for example [2]. So we can extand our algebra by introducing a scalar 2-form ρ such as:

$$s\rho = -\frac{1}{3}tr\chi^3$$

Then by redoing the same steps as before: constructing the Weil algebra and the BRS algebra. We find that the minimal algebra for calculating the s cohomology is generated by $\chi^\alpha, F^\alpha, \rho$ and h such that:

$$\begin{aligned} s\chi^\alpha &= -\frac{1}{2}\varepsilon^\alpha_{\beta\gamma}\chi^\alpha\chi^\gamma \\ sF^\alpha &= \varepsilon^\alpha_{\beta\gamma}F^\beta\chi^\gamma \\ s\rho &= -\frac{1}{3}tr\chi^3 \\ sh &= trF\chi^2 \end{aligned}$$

So, we see that $tr\chi^3$ is no more a non trivial s-cocycle since it can be expressed as $s\rho$. However a new non trivial s-cocycle appears namely : $h + trF\chi$ as can be seen easily from the last relations. So now the generators of the cohomology algebra $H(s)$ must contain trF^2 and $(h + trF\chi)$ for this extension of the $SU(2)$ Lie algebra.

6. Finite gauge transformations

The BRS algebra of the last chapter has been constructed by cohomological methods in the dual algebra where all the transformations were linear. We shall now show that by eliminating the ghost degrees, we can find the finite gauge transformations that generalize the well known ones: $A^g = g^{-1}Ag + g^{-1}dg$ and $F^g = g^{-1}Fg$ where g is a group valued function over a manifold to the case where 2-form gauge potentials are also present. We shall show that they form a group, i.e. the product of two successive gauge transformations is again a gauge transformation. This is reflected in the BRS algebra by the fact that $s^2 = 0$ on all the fields.

From the BRS algebra, one can easily write the infinitesimal gauge transformations on all the fields just by eliminating the ghost degrees. For example, one finds for the Lie algebra part:

$$\delta(v)A(x) = dv(x) + A(x)v(x) - v(x)A(x)$$
$$\delta(v)F(x) = F(x)v(x) - v(x)F(x)$$
$$\delta(v)u(x) = u(x)v(x) - v(x)u(x)$$

$v(x)$ is the infinitesimal parameter of the gauge transformation and it is a commuting 0-form Lie algebra valued. It replaces the ghost $\chi(x)$. $u(x)$ is also a 0-form Lie algebra valued. s is replaced by δ which depends on $v(x)$ and it carries no ghost degree. Then, by doing some trivial calculations and using the Jacobi identity, we obtain the following finite transformations:

$$exp(\delta(v))A = e^{-v}Ae^{v} + e^{-v}de^{v}$$
$$exp(\delta(v))F = e^{-v}Fe^{v}$$
$$exp(\delta(v))u = e^{-v}ue^{v}$$

which are the usual gauge transformations when g is connected to the identity gauge transformation $g = exp(v^{\alpha}E_{\alpha})$, exp is the usual exponential map $exp : \mathcal{G} \longrightarrow G$. So one recovers the usual gauge transformations using this simple trick.

Similarly, let us apply this for the Γ-part of the algebra. From the BRS algebra, we can write the following infinitesimal transformations:

$$\delta(v,\tilde{b},\tilde{h},\tilde{\rho})B = d\tilde{b} + c(A,\tilde{b}) + \tilde{h} - c(v,B) + \frac{1}{2}q(A,A,v) \qquad (9a)$$
$$\delta(v,\tilde{b},\tilde{h},\tilde{\rho})\tilde{a} = -d\tilde{\rho} - c(A,\tilde{\rho}) - c(v,\tilde{a}) \qquad (9b)$$
$$\delta(v,\tilde{b},\tilde{h},\tilde{\rho})\tilde{\sigma} = -c(v,\tilde{\sigma}) \qquad (9c)$$
$$\delta(v,\tilde{b},\tilde{h},\tilde{\rho})H = d\tilde{h} + c(A,\tilde{h}) - c(v,H) + c(F,\tilde{b}) + q(F,v,A) \qquad (9d)$$
$$\delta(v,\tilde{b},\tilde{h},\tilde{\rho})\tilde{g} = +c(F,\tilde{\rho}) - c(v,\tilde{g}) \qquad (9e)$$

where δ now depends on $v(x), \tilde{b}(x), \tilde{h}(x)$ and $\tilde{\rho}(x)$. $\tilde{h}, \tilde{b}$ and $\tilde{\rho}$ are respectively 2, 1 and 0-forms (without ghost degrees) which correspond to h, b and ρ used in the BRS algebra and similarly for $\tilde{g}, \tilde{a}$ and $\tilde{\sigma}$.

Exponentiating as before these infinitesimal variations and using the representation and cocycle conditions, we find the following finite gauge transformations with $g = e^{v}$ and $\delta = \delta(v,\tilde{b},\tilde{h},\tilde{\rho})$:

$$exp(\delta)\tilde{\sigma} = g^{-1}\tilde{\sigma}g$$
$$exp(\delta)\tilde{a} = g^{-1}(\tilde{a} - d\tilde{\rho} - c(A,\tilde{\rho}))g$$
$$exp(\delta)\tilde{g} = g^{-1}(\tilde{g} + c(F,\tilde{\rho}))g$$
$$exp(\delta)B = g^{-1}(B + \tilde{h} + d\tilde{b} + c(A,\tilde{b}))g + \Phi(A,g)$$
$$exp(\delta)H = g^{-1}(H + d\tilde{h} + c(A,\tilde{h}) + c(F,\tilde{b}))g - \Psi(A,F,g)$$

where:

$$\Phi(A,g) = \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sum_{m=0}^{\infty}(-)^{m}c(\underbrace{v,c(v,...,c(v,}_{m \text{ times}}Q^{(n-m)})...)$$

$$\Psi(A,F,g) = \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sum_{m=0}^{\infty}(-)^{m}c(\underbrace{v,c(v,...,c(v,}_{m \text{ times}}P^{(n-m)})...)$$

and:

$$Q^{(p)} = \delta^{p}q(v,A,A)$$
$$P^{(p)} = \delta^{p}q(v,F,A)$$

where δ is the gauge transformation operator. We note that $\Phi(A,g)$ and $\Psi(A,F,g)$ are a sort of non-abelian extension of this symmetry which vanishes if the 3-cocycle is zero. Despite the fact that these finite transformations are very complicated, they form a group as we shall show.

We remark that when B is Γ-algebra valued $B = B^{i}E_{i}$, then we can easily show by using:

$$c(E_i,T_a) = -D^{j}_{ai}E_j$$

that:

$$e^{-v}Be^{v} = E_iD^{i}_{j}(e^{v})B^{j}$$

or simply:

$$g^{-1}Bg = D(g)B$$

which is the usual action of the representation $D(g)$ of g on B.

Another interesting property of the infinitesimal transformations (9a-e) is the fact that they form a closed algebra under the commutator. One can easily prove by using the Jacobi identity, the representation and the 3-cocycle conditions the following commutator identity which applies to all the fields:

$$[\delta(v,\tilde{b},\tilde{h},\tilde{\rho}),\delta(u,\tilde{a},\tilde{g},\tilde{\sigma})] = \delta(-c(v,u),q(v,u,A),q(v,u,F),0)$$

We note that this formula contains the usual result $[\delta_v,\delta_u] = \delta_{[u,v]}$ which applies to A and F.

So now, by the Haussdorf's formula:

$$exp(\delta_1).exp(\delta_2) = exp(\delta_1+\delta_2+\frac{1}{2}[\delta_1,\delta_2]+\frac{1}{12}[\delta_1,[\delta_1,\delta_2]]+\frac{1}{12}[[\delta_1,\delta_2],\delta_2]+\cdots)$$

where $\delta_1 = \delta(v_1,\tilde{b}_1,\tilde{h}_1,\tilde{\rho}_1)$ and $\delta_2 = \delta(v_2,\tilde{b}_2,\tilde{h}_2,\tilde{\rho}_2)$ are two successive gauge transformations and since the algebra of gauge transformations is closed under the commutator by the last result, we have:

$$exp(\delta_1).exp(\delta_2) = exp(\delta_3)$$

for the finite gauge transformations, where:

$$\delta_3 = \delta(v_3,\tilde{b}_3,\tilde{h}_3,\tilde{\rho}_3)$$

with:

$$v_3 = v_1+v_2+\frac{1}{2}[v_1,v_2]+\frac{1}{12}[v_1,[v_1,v_2]]+\cdots$$

which corresponds to the usual group multiplication:

$$(g_1.g_2) = (g_1g_2) \Longleftrightarrow e^{v_1}.e^{v_2} = e^{v_3}$$

and with:

$$\tilde{b}_3 = \tilde{b}_1+\tilde{b}_2+q(v_1,v_2,A)+\frac{1}{12}q(v_1,c(v_1,v_2),A)+\cdots$$
$$\tilde{h}_3 = \tilde{h}_1+\tilde{h}_2+q(v_1,v_2,F)+\frac{1}{12}q(v_1,c(v_1,v_2),F)+\cdots$$
$$\tilde{\rho}_3 = \tilde{\rho}_1+\tilde{\rho}_2$$

the transformations on $\tilde{b}$ and $\tilde{h}$ are A and F dependent.

So in view of this, the set of finite gauge transformations forms a group under multiplication. We remark that the finite gauge transformations show the same phenomenon of ghost for ghost mechanism at the group level.

7. Examples

Now, we give two simple examples to illustrate all this formalism.

7.1. Example 1

Take the following dual Lie algebra $\mathcal{G}^*$ defined by $(\epsilon^{ab},e^a,\eta^\alpha)$ such as:

$$s\epsilon^{ab} = \epsilon^{ac}\epsilon^{cb}$$
$$s\eta^\alpha = \frac{1}{4}\epsilon^{ab}(\Gamma^{ab}\eta)^\alpha$$
$$se^a = \epsilon^{ab}e^b$$

where $\epsilon^{ab},\eta^\alpha$ and e^a are 1-forms respectively in the adjoint, in the spinorial and in the vectorial representations of $SO(3,1)$. η^α is of Majorana type $\bar{\eta}_\alpha = \eta^\beta C_{\beta\alpha}$, where $C_{\beta\alpha}$ is the conjugation charge matrix.

We can easily show that $\Omega^{ab} = \bar{\eta}\Gamma^a\eta e^b - \bar{\eta}\Gamma^b\eta e^a$ is a non trivial 3-cocycle in the adjoint representation. So, we can have an extension of $\mathcal{G}^*$ by introducing a 2-form β^{ab} in this representation such that:

$$s\beta^{ab} = \epsilon^{ac}\beta^{cb} - \epsilon^{bc}\beta^{ca} + \Omega^{ab}$$

The Lie algebra $\mathcal{G}$ is generated by $M_{ab},\bar{Q}_\alpha,P_a$ ($\bar{Q}_\alpha$ is a graded generator carrying a fermionic degree). Their commutation relations are the following:

$$c(M_{ab},M_{cd}) = +\eta_{ac}M_{bd}-\eta_{ad}M_{bc}-\eta_{bc}M_{ad}+\eta_{bd}M_{ac}$$
$$c(M_{ab},\bar{Q}_\alpha) = -\frac{1}{2}\bar{Q}_\beta\Gamma_{ab}{}^\beta_\alpha$$
$$c(M_{ab},P_c) = +\eta_{ac}P_b-\eta_{cb}P_a$$

and all the other commutators are zero. This is not the superPoincaré algebra since: $c(\bar{Q}_\alpha,\bar{Q}_\beta) = 0$.

For the Γ algebra part, we introduce the generators N_{ab} anticommuting with s. The commutator with M_{ab} is defined by:

$$c(M_{ab},N_{cd}) = +2\eta_{ac}N_{bd}-2\eta_{ad}N_{bc}-2\eta_{bc}N_{ad}+2\eta_{bd}N_{ac}$$

and the 3-cocycle by:

$$q(Q^\alpha,\bar{Q}_\beta,P_a) = -N_{ab}\Gamma^{b\alpha}_{\ \beta}$$

and all other commutators and 3-cocycles are zero.

So, using the Quillen form:

$$\Sigma = \frac{1}{2}\epsilon^{ab}M_{ab} + \bar{Q}_\alpha\eta^\alpha + e^a P_a + \frac{1}{2}\beta^{ab}N_{ab}$$

$(\mathcal{G} \oplus \Gamma)$-algebra valued, we recover the form of eq.(6).

One can easily check by applying the s operator that one has the following s-invariants: $tr\epsilon^3$ and: $\beta^{ab}\epsilon^{bc}\epsilon^{ca} - 2\bar{\eta}\Gamma^a\eta e^b\epsilon^{ba}$. The first invariant is the usual one known in the dual Lie algebra $\mathcal{G}^*$ and the second exists because β^{ab} is in the same representation as ϵ^{ab}.

We shall now show that one can find the d-invariants and Chern-Simons terms [18] associated to these s-invariants. For this, we construct the related Weil algebra by introducing connections $\omega^{ab}, \psi^\alpha, V^a$ (1-forms on a manifold) and B^{ab} (2-forms) and the corresponding curvatures R^{ab}, ρ^α, T^a (2-forms) and C^{ab} (3-forms) such that:

$$\begin{aligned}
d\omega^{ab} &= R^{ab} + \omega^{ac}\omega^{cb} \\
d\psi^\alpha &= \rho^\alpha + \frac{1}{4}\omega^{ab}(\Gamma_{ab}\psi)^\alpha \\
dV^a &= T^a + \omega^{ab}V^b \\
dB^{ab} &= C^{ab} + \omega^{ac}B^{cb} - B^{ac}\omega^{cb} + \bar{\psi}\Gamma^a\psi V^b - V^a\bar{\psi}\Gamma^b\psi
\end{aligned}$$

and:

$$\begin{aligned}
dR^{ab} &= -R^{ac}\omega^{cb} + \omega^{ac}R^{cb} \\
d\rho^\alpha &= -\frac{1}{4}R^{ab}(\Gamma_{ab}\psi)^\alpha + \frac{1}{4}\omega^{ab}(\Gamma_{ab}\rho)^\alpha \\
dT^a &= -R^{ab}V^b + \omega^{ab}T^b \\
dC^{ab} &= -R^{ac}B^{cb} + \omega^{ac}C^{cb} + 2\bar{\psi}\Gamma^a\rho V^b - \bar{\psi}\Gamma^a\psi T^b - (a \longleftrightarrow b)
\end{aligned}$$

Then we can easily verify by direct computation that:

i) $tr\epsilon^3$ is associated to trR^2 and to the usual Chern-Simons term $tr(\omega R - \frac{1}{3}\omega^3)$.

ii) $(\beta^{ab}\epsilon^{bc}\epsilon^{ca} - 2\bar{\eta}\Gamma^a\eta e^b\epsilon^{ba})$ is associated to $I = (C^{ab}R^{ba} + 4\bar{\psi}\Gamma^a\rho T^a)$ and its Chern-Simons term is:

$$J = (\frac{1}{2}B^{ab}R^{ba} - \frac{1}{2}C^{ab}\omega^{ba} - \bar{\psi}\Gamma^a\psi T^a - 2\bar{\psi}\Gamma^a\rho V^a - \bar{\psi}\Gamma^a\psi V^b\omega^{ba} + \frac{1}{2}B^{ab}\omega^{bc}\omega^{ca})$$

and one has $dJ = I$. We remark that I is an odd form on space-time and that J is an even form. This is the contrary of what is obtained in usual Lie algebras.

7.2. Example 2

The FDA η^* of the Sohnius-West model [19] defined in ref.[3] is generated by ϵ^{ab}, η^α, e^a, K 1-forms respectively in the adjoint, in the spinorial, in the vectorial and in the scalar representations of $SO(3,1)$ and by S 2-form in the scalar representation, can be written in the form (6) provided we define the η-valued form:

$$\Sigma = \frac{1}{2}\epsilon^{ab}M_{ab} + e^a P_a + \bar{Q}_\alpha\eta^\alpha + K\sigma + S\tau$$

η is generated by $(M_{ab}, P_a, \bar{Q}_\alpha)$ the superPoincaré generators and σ the chiral generator in degree zero and by τ a generator of degree one which anticommutes with s.

The commutators and cocycles of the generators are given by:

$$\begin{aligned}
c(M_{ab}, M_{cd}) &= +\eta_{ac}M_{bd} - \eta_{ad}M_{bc} - \eta_{bc}M_{ad} + \eta_{bd}M_{ac} \\
c(M_{ab}, P_c) &= +\eta_{ac}P_b - \eta_{cb}P_a \\
c(M_{ab}, \bar{Q}_\alpha) &= -\frac{1}{2}\bar{Q}_\beta\Gamma_{ab}{}^\beta_\alpha \\
c(Q^\alpha, \bar{Q}_\beta) &= -\frac{i}{2}P_a\Gamma^{a\alpha}_\beta \\
c(\bar{Q}_\alpha, \sigma) &= -\frac{i}{2}\bar{Q}_\beta\Gamma_5{}^\beta_\alpha
\end{aligned}$$

and:

$$q(Q^\alpha, \bar{Q}_\beta, P_a) = \frac{i}{2}\tau\Gamma_a{}^\alpha_\beta$$

and all others are zero.

If one expands eq.(6) for each degree, one recovers the explicit algebra given in ref.[3] which was used to construct the action of the Sohnius-West model in the group manifold approach:

$$\begin{aligned}
s\epsilon^{ab} &= \epsilon^{ac}\epsilon^{cb} \\
s\eta^\alpha &= \frac{1}{4}\epsilon^{ab}(\Gamma_{ab}\eta)^\alpha + \frac{i}{2}(\Gamma_5\eta)^\alpha K \\
se^a &= \epsilon^{ab}e^b + \frac{i}{2}\bar{\eta}\Gamma^a\eta \\
sK &= 0 \\
sS &= \frac{i}{2}\bar{\eta}\Gamma^a\eta e_a
\end{aligned}$$

The associated Weil algebra is generated by the connections $\omega^{ab}, \psi^{\alpha}, V^{a}, A$ all 1-forms together with the 2-form T and their curvatures $R^{ab}, \rho^{\alpha}, R^{a}, \bar{R}$ and R^{*} such that:

$$
\begin{aligned}
d\omega^{ab} &= R^{ab} + \omega^{ac}\omega^{cb} \\
d\psi^{\alpha} &= \rho^{\alpha} + \frac{1}{4}\omega^{ab}(\Gamma_{ab}\psi)^{\alpha} + \frac{i}{2}(\Gamma_5\psi)^{\alpha}A \\
dV^{a} &= R^{a} + \omega^{ab}V^{b} + \frac{i}{2}\bar{\psi}\Gamma^{a}\psi \\
dA &= \bar{R} \\
dT &= R^{*} + \frac{i}{2}\bar{\psi}\Gamma^{a}\psi V^{a}
\end{aligned}
$$

The Bianchi identities are:

$$
\begin{aligned}
dR^{ab} &= \omega^{ac}R^{cb} - R^{ac}\omega^{cb} \\
d\rho^{\alpha} &= \frac{1}{4}\omega^{ab}(\Gamma_{ab}\rho)^{\alpha} - \frac{1}{4}R^{ab}(\Gamma_{ab}\psi)^{\alpha} - \frac{i}{2}(\Gamma_5\rho)^{\alpha}A + \frac{i}{2}(\Gamma_5\psi)^{\alpha}\bar{R} \\
dR^{a} &= \omega^{ab}R^{b} - R^{ab}V^{b} - i\bar{\rho}\Gamma^{a}\psi \\
d\bar{R} &= 0 \\
dR^{*} &= -i\bar{\rho}\Gamma^{a}\psi V_{a} - \frac{i}{2}\bar{\psi}\Gamma^{a}\psi R_{a}
\end{aligned}
$$

Then, from the BRS algebra, it is easy to deduce the following infinitesimal transformations:

$$
\begin{aligned}
\delta(1)\omega^{ab} &= d\epsilon_1^{ab} + \epsilon_1^{ac}\omega^{cb} - \omega^{ac}\epsilon_1^{cb} \\
\delta(1)V^{a} &= de_1^{a} + \epsilon_1^{ab}V^{b} - \omega^{ab}e_1^{b} - i\bar{\psi}\Gamma^{a}\eta_1 \\
\delta(1)\psi &= d\eta_1 + \frac{1}{4}\epsilon_1^{ab}(\Gamma_{ab}\psi) - \frac{1}{4}\omega^{ab}(\Gamma_{ab}\eta_1) + \frac{i}{2}(\Gamma_5\eta_1)A - \frac{i}{2}(\Gamma_5\psi)k_1 \\
\delta(1)A &= dk_1 \\
\delta(1)T &= dt_1 - i\bar{\psi}\Gamma^{a}\eta_1 V^{a} + \frac{i}{2}\bar{\psi}\Gamma^{a}\psi e_1^{a} + r_1 \\
\delta(1)t_1 &= ds_1 \\
\delta(1)s_1 &= 0
\end{aligned}
$$

$$
\begin{aligned}
\delta(1)R^{ab} &= -R^{ac}\epsilon_1^{cb} + \epsilon_1^{ac}R^{cb} \\
\delta(1)R^{a} &= -R^{ab}e_1^{b} + \epsilon_1^{ab}R^{b} - i\bar{\eta}_1\Gamma^{a}\rho \\
\delta(1)\rho &= -\frac{1}{4}R^{ab}(\Gamma_{ab}\eta_1) + \frac{1}{4}\epsilon_1^{ab}(\Gamma_{ab}\rho) - \frac{i}{2}(\Gamma_5\rho)k_1 + \frac{i}{2}(\Gamma_5\eta_1)R^{*} \\
\delta(1)R^{*} &= 0 \\
\delta(1)\bar{R} &= dr_1 + i\bar{\eta}_1\Gamma^{a}\rho V^{a} - i\bar{\psi}\Gamma^{a}\rho e_1^{a} + i\bar{\psi}\Gamma^{a}\eta_1 R^{a} \\
\delta(1)r_1 &= 0
\end{aligned}
$$

$\delta(1)$ is defined by the set of infinitesimal parameters $\epsilon_1^{ab}, e_1^{a}, \eta_1^{\alpha}, k_1$ and s_1 0-forms and functions on space-time, t_1 a 1-form and r_1 a 2-form.

One can show after some trivial calculations that:

$$[\delta(1), \delta(2)] = \delta(12)$$

such as:

$$
\begin{aligned}
\epsilon_{(12)}^{ab} &= -\epsilon_1^{ac}\epsilon_2^{cb} + \epsilon_2^{ac}\epsilon_1^{cb} \\
e_{(12)}^{a} &= +i\bar{\eta}_2\Gamma^{a}\eta_1 \\
\eta_{(12)}^{\alpha} &= \frac{i}{2}(\Gamma_5\eta_2)^{\alpha}k_1 - \frac{i}{2}(\Gamma_5\eta_1)^{\alpha}k_2 \\
t_{(12)} &= i\bar{\eta}_2\Gamma^{a}\eta_1 V^{a} - i\bar{\psi}\Gamma^{a}\eta_1 e_2^{a} + i\bar{\psi}\Gamma^{a}\eta_2 e_1^{a} \\
r_{(12)} &= -i\bar{\eta}_2\Gamma^{a}\eta_1 R^{a} + i\bar{\rho}\Gamma^{a}\eta_1 e_2^{a} - i\bar{\rho}\Gamma^{a}\eta_2 e_1^{a}
\end{aligned}
$$

which shows that these small variations form a closed algebra by the commutator. The first parameter is just the product of two successive rotations and the second is the translation induced by two supersymmetry transformations. We note that the variations of t and r depend on the connections and the curvatures of the 1-forms.

7. Conclusion

A FDA has been known in the context of the group manifold approach to be the underlying symmetry of some supergravity theories and in general of theories containing p-form gauge potentials ($p > 1$). We have shown in this work how to construct consistent Weil and BRS algebras of a general FDA by stressing the difference between the d and s operators. The construction was done geometrically by postulating some few cohomological principles as the fact that $d^2 = s^2 = ds + sd = 0$ and introducing the minimal number of ghosts one needs in order to have free algebras. This BRS algebra which shows a ghost for ghost mechanism will be useful in the quantization of gauge theories of antisymmetric gauge fields. Together with the corresponding antighosts which we have not considered here it will also allow us to construct invariant lagrangians. We have shown also that the infinitesimal gauge transformations in the BRS algebra induce finite gauge transformations on p-forms generalizing the known ones on 1-form potentials. The natural object defining a FDA, the Quillen (super)connection, which is the formal sum of forms of different degrees has been introduced. It may help to undestand the geometrical meaning of gauge theories of p-forms on a manifold. Finally, by considering two simple illustrative examples, we have shown that this construction contains many interesting features as for example a new form for the Chern-Simons terms and also a new kind of gauge transformations.

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