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FREE BOSONIC STRING FIELD THEORY WITHOUT SUPPLEMENTARY FIELDS[1]

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Abstract

A covariant local action for free bosonic string fields is constructed without the use of supplementary fields. The open string case is treated in detail. Up to a mathematical conjecture which is likely to hold it is shown that the Virasoro constraints arise as a special choice of gauge. The kinetic operator turns out to be extremely simple, the gauge transformation law arising rather implicitly. The case of closed strings is briefly discussed.

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1 Introduction

Covariant bosonic string field theory interprets the first-quantized wave functionals of the free string dynamics as classical fields which have to be second-quantized along the lines of quantum field theory. Of course, the wave functionals necessary to describe the first-quantized string are not unique (for reviews of bosonic string theory see Refs. [1,2]). BRS(T) and related techniques give rise to a field spectrum including also ghost fields in addition to the pure string functional which is generated by the oscillator modes and the center of mass motion. Several formulations of bosonic string field theory have been given within this framework [3-9] or using pure geometric reasoning [10].

Here we approach the subject along the lines of the old covariant quantization, starting from the Virasoro conditions which select physical states and are usually formulated in the open or closed string Fock-space. In the appendix, the relevant notation is summarized for the case of free open strings, and some mathematical preliminaries are given.

The situation we encounter is at first sight analogous to the covariant quantization of the free massive point particle whose first-quantized wave equation,

$$(\Box - m^2)\phi(x) = 0 \tag{1.1}$$

may be interpreted as selecting the physical states contained in the total space of functions $\phi(x)$. The first step in second-quantizing the Klein-Gordon field consists of finding a classical action functional which reproduces (1.1) as its field equation. Interactions are then incorporated by adding suitable non-quadratic terms to the free Lagrangian.

In the theory of open bosonic strings, the analogue of (1.1) are the Virasoro conditions

$$L_n|\psi\rangle = 0 \tag{1.2}$$

$(n \geq 1)$, together with the mass shell equation

$$(L_0 - 1)|\psi\rangle = 0, \tag{1.3}$$

for elements $|\psi\rangle$ of the total Fock-space. Once having interpreted $|\psi\rangle$ as classical field, subject to the above equations, the next step is to look for an action functional reproducing these.

Usually, one is interested in a gauge-covariant formulation of string field theory. In most of the approaches this means that the conformal generators L_{-n} - which give rise to reparametrizations of the string world sheet - should also provide the gauge symmetries

$$\delta|\psi\rangle = L_{-n}|\Lambda\rangle \tag{1.4}$$

($|\Lambda\rangle$ unconstrained) of the action. However, this is only possible if supplementary fields are introduced and subjected to appropriate gauge transformation laws as well. The requirement then is that in a special gauge, the field equations should be identical to (1.2) and (1.3) with all supplementary fields vanishing. Moreover, the action shall be local in the space-time fields contained in $|\psi\rangle$, which in other words means that the kinetic operator must be polynomial in the derivatives ∂_μ at each mass level.

This problem has been solved by several authors in different ways [11-19], also including the interaction of string fields [20,21] and the superstring case [22]. There are a lot of arguments in favour of the symmetries (1.4) and the introduction of supplementary fields. The structure of these was also related to the ghosts in the BRST-approach [4].

However, there exists an alternative approach. Giving up the full gauge symmetry and the supplementary fields, we will construct a local action functional for free bosonic strings, formulated in terms of the Fock-space state $|\psi\rangle$ alone. It admits a gauge symmetry which is considerably *smaller* than (1.4). Upon a conjecture which is very likely to hold, it is argued in the open string case that the correct physical state conditions (1.2) and (1.3) may be obtained as a gauge-fixing by means of this symmetry. For the closed strings, the existence of an appropriate action functional without supplementary fields is not as firmly established as for the open strings, but we suggest a possible candidate which does the job up to the second mass level. While in the open string case the gauge transformation yielding the correct gauge-fixing exists as a consequence of pure algebraic and differential manipulations, one is forced to impose appropriate boundary conditions upon the space-time fields of the closed strings.

The physical significance of the actions suggested is not yet clarified. It is likely that they may be obtained by a partial gauge fixing and elimination of supplementary fields from the fully gauge-covariant actions. However, *what* has been fixed, remains unclear.

Also, the graviton is described here - in the linearized case - by the symmetric traceless tensor contained in the closed string field at the first mass-level. Thus, upon including interactions, one expects that the gravitational sector will be described either in a not fully general covariant way or by some non-local action of the Fradkin-Vilkovisky type [23].

Nevertheless, the approach presented here has the advantage of an extremely simple kinetic operator. The price paid for this is - in addition to the remarks made above - a quite unexplicit form of the gauge transformation law. In their final form, the action functionals suggested are identical to those given in Ref. [15] for the free open and closed string fields when all supplementary fields are set equal to zero. This observation might provide a hint how to treat the superstring fields analogously.

The paper is organized as follows. Sections 2 - 8 are concerned with the free field theory of bosonic open strings. In section 2, the well-known action functionals for the mass levels 0 and 1 are reviewed. Section 3 is devoted to the construction of a Lagrangian for the fields occurring at the mass level 2. The computation is first carried out explicitly in terms of the space-time fields and then condensed in the Fock-space notation. The necessary gauge symmetry is exhibited. In section 4, the kinetic operator which should apply for all mass levels is presented. The existence of an infinite series of gauge transformations is proved in section 5. In this formulation, the gauge parameters are constrained to lie in the space defined by equation (1.2). In section 6, a mathematical conjecture concerning the kinetic operator is made. Under the assumption that it holds, it is shown that the gauge symmetries are in fact enough to admit the gauge-fixing (1.2) and (1.3). In section 7, it is outlined how the infinite series of transformation laws may be unified into a single one, involving only one unconstrained element of the Fock-space as gauge parameter. The residual gauge freedom which still exists after the Virasoro gauge has been fixed is mentioned in section 8. Section 9 presents a candidate for an appropriate action functional in the closed string case, although the matter is not so clear here as compared to the open strings. Some concluding remarks are made in section 10, discussing the properties of the kinetic operator that are responsible for the existence of a free covariant string field theory without supplementary fields, and touching the question whether there may be other formulations working equally well.

The most interesting questions of how the shortcomings of the formulation presented here might be overcome in the interacting case and what it tells us for the theory of superstrings are presently under consideration.

2 Action Functionals for the Mass Levels 0 and 1

We first consider the states of zero mass level, i.e. the tachyonic field

$$|\psi\rangle \equiv |\psi\rangle_0 = \phi(x)|0\rangle. \tag{2.1}$$

Its only field equation is the mass-shell condition (A.17)

$$(\Box + 2)\phi = 0. \tag{2.2}$$

Clearly, this field equation follows from the action

$$S = \frac{1}{2}\int d^D x \phi(\Box + 2)\phi, \tag{2.3}$$

or, when written in the language of the Fock-space,

$$S = -\frac{1}{2}(\psi|2(L_0 - 1)|\psi). \tag{2.4}$$

As well-known, the first mass level may be treated in an analogous way without complications. The states at the first mass level have the form

$$|\psi) \equiv |\psi)_1 = -iA_\mu(x)\alpha_{-1}{}^\mu|0), \tag{2.5}$$

the field equations being the mass shell equation (A.17)

$$\Box A^\mu = 0 \tag{2.6}$$

supplemented by the Virasoro condition (A.16)

$$\partial_\mu A^\mu = 0 \tag{2.7}$$

which requires $|\psi)$ to lie in $\mathbf{V}^{(0)}$ (cf. Equ. (A.18)). Considering the free Maxwell action

$$S = \frac{1}{2}\int d^D x A^\mu(\Box A_\mu - \partial_{\mu\nu}A^\nu), \tag{2.8}$$

one obtains the field equations

$$\Box A_\mu - \partial_{\mu\nu}A^\nu = 0. \tag{2.9}$$

S possesses the local gauge symmetry

$$\delta A_\mu = \partial_\mu \Lambda \tag{2.10}$$

for arbitrary $\Lambda(x)$, which implies that one can always choose (2.7) as a gauge condition and hence obtains (2.6) as the field equations. In the notation of the Fock-space, S reads (cf. Ref. [13])

$$S = -\frac{1}{2}(\psi|2(L_0 - 1) - L_{-1}L_1|\psi). \tag{2.11}$$

The gauge transformation law (2.10) has the form

$$\delta|\psi\rangle = L_{-1}|\chi\rangle \tag{2.12}$$

where

$$|\chi\rangle = \Lambda(x)|0\rangle. \tag{2.13}$$

It is important to note that checking the invariance of (2.11) under (2.12) one only has to use the fact that $|\chi\rangle$ is annihilated by L_1. No reference to the mass levels involved is needed in this computation. Moreover, the action (2.11) reduces to (2.4) if $|\psi\rangle = |\psi\rangle_0$ is inserted. Thus, S provides an action for the first two mass levels simultaneously.

3 Action Functionals for the Mass Level 2

Trying to extend the action (2.11) to include also the fields at the second mass level, one faces a well-known problem. Generalizing (2.12), one usually requires the action to be invariant under the gauge symmetry (cf. Equ. (1.4))

$$\delta|\psi\rangle_2 = L_{-1}|\chi\rangle_1 + L_{-2}|\chi\rangle_0 \tag{3.1}$$

where the $|\chi\rangle_n$ are arbitrary states at the mass levels n. However, it is well-known that the field content of the second mass level may not be described by a local action admitting the gauge symmetry (3.1). The action considered by Banks and Peskin [13]

$$S = \frac{\text{[illegible]}}{2}\text{[illegible]}(L_0 - 1)P|\psi\rangle \tag{3.2}$$

where P is the projector onto $\mathbf{V}^{(0)}$, turns out to become nonlocal at the second mass level due to the existence of zeros in the Kac-determinant. (In other words, P contains rational functions of L_0.) Usually, this problem is overcome by the introduction of supplementary fields which merely serve to maintain the symmetry (3.1) *and* locality.

However, there is another possibility which has apparently not been pursued before. Insisting on an action functional which involves only $|\psi\rangle$ and which is local, containing only first and second space-time derivatives, it is clear that one must sacrifice at least part of the full symmetry (3.1). In the following we will show how this works, beginning with the second mass level.

The Fock-states in question are given by

$$|\psi\rangle \equiv |\psi\rangle_2 = (-iv_\mu(x)\alpha_{-2}{}^\mu - \frac{1}{2}h_{\mu\nu}(x)\alpha_{-1}{}^\mu\alpha_{-1}{}^\nu)|0\rangle \tag{3.3}$$

the field equations again being the mass-shell condition (A.17)

$$(\Box - 2)v_\mu = (\Box - 2)h_{\mu\nu} = 0 \tag{3.4}$$

together with the constraints (A.16) (cf. Equs. (A.19))

$$C_\mu \equiv v_\mu - \frac{1}{2}\,\partial^\rho h_{\rho\mu} = 0 \tag{3.5}$$

$$B \equiv \partial_\mu v^\mu + \frac{1}{4}\,h^\mu_\mu = 0. \tag{3.6}$$

The question now is whether there is an action for the fields v_μ, $h_{\mu\nu}$ admitting enough gauge freedom to choose (3.5) and (3.6) as gauge conditions.

Let us make an ansatz for the Lagrangian

$$\begin{aligned} c\mathcal{L} \;=\; & \tfrac{1}{2}\,v^\mu(\Box - 2)v_\mu + \tfrac{\lambda}{2}\,h^{\mu\nu}(\Box - 2)h_{\mu\nu} \\ & + 2a(v^\mu v_\mu - v^\mu\partial^\rho h_{\rho\mu} - \tfrac{1}{4}\,h^{\mu\nu}\partial^\rho_\mu h_{\rho\nu}) \\ & + 2b(v^\mu\partial_{\mu\rho}v^\rho + \tfrac{1}{2}\,v^\mu\partial_\mu h^\rho_\rho - \tfrac{1}{16}\,h^\mu_\mu h^\rho_\rho). \end{aligned} \tag{3.7}$$

The numerical factors within the brackets are chosen such that the field equations arising by variation are of the form

$$\mathcal{V}_\mu \equiv (\Box - 2)v_\mu + 4aC_\mu + 4b\partial_\mu B = 0, \tag{3.8}$$

$$\mathcal{H}_{\mu\nu} \equiv \lambda(\Box - 2)h_{\mu\nu} + a(\partial_\mu C_\nu + \partial_\nu C_\mu) - b\eta_{\mu\nu}B = 0. \tag{3.9}$$

The game consists now of playing around with the expressions $\mathcal{H}^\mu_\mu$, $\partial^\mu\mathcal{V}_\mu$, $\partial^\nu\mathcal{H}_{\mu\nu}$ and $\partial^{\mu\nu}\mathcal{H}_{\mu\nu}$ in orer to guess the required gauge transformation law. In the case $D = 26$ (D appears as δ^μ_μ when taking the trace of (3.9)), one finds the solution

$$a = \frac{1}{2}, \qquad b = -\frac{1}{4}, \qquad \lambda = \frac{1}{4}, \tag{3.10}$$

and the two gauge symmetries

$$\delta_1(v_\mu, h_{\mu\nu}) = (\chi_\mu, \partial_\mu\chi_\nu + \partial_\nu\chi_\mu) \tag{3.11}$$

$$\delta_2(v_\mu, h_{\mu\nu}) = (-\frac{1}{2}(\Box + 3)\partial_\mu\Lambda, -3\partial_{\mu\nu}\Lambda + \frac{1}{2}\,\eta_{\mu\nu}\Lambda) \tag{3.12}$$

where the gauge parameter $\chi_\mu(x)$ has to satisfy

$$\partial_\mu\chi^\mu = 0. \tag{3.13}$$

Instead of χ^μ, one could also use an unconstrained gauge parameter $\xi^\mu(x)$ and insert

$$\chi^\mu = \partial^{\mu\nu}\xi_\nu - \Box\xi^\mu \tag{3.14}$$

into (3.11). The constraint (3.13) is then automatically satisfied.

In order to see how the choice of gauge works, we consider the following combination of the field equations

$$\partial^\mu \mathcal{V}_\mu + \mathcal{H}^\mu_\mu \equiv 3\partial_\mu C^\mu + \frac{9}{2} B = 0. \tag{3.15}$$

This situation is reminiscent of the case of a massive vector field, where the Lorentz gauge condition follows from the Proca equation without the use of any gauge symmetry. Next we use (3.12) which implies

$$\delta_2 B = -\frac{1}{2}\Box(\Box - 2)\Lambda \tag{3.16}$$

to attain the gauge condition

$$B = 0 \tag{3.17}$$

from which also follows

$$\partial_\mu C^\mu = 0. \tag{3.18}$$

The remaining gauge freedom (3.11)

$$\begin{aligned} \delta_1 C_\mu &= -\tfrac{1}{2}(\Box - 2)\chi_\mu \\ \delta_1 B &= 0 \end{aligned} \tag{3.19}$$

does not affect (3.17). By virtue of (3.13) and (3.18) one may choose χ_μ such that $\delta_1 C_\mu = -C_\mu$, thus arriving at the final gauge condition

$$C_\mu = B = 0 \tag{3.20}$$

and the field equations (3.4).

Thus we have shown that the Lagrangian

$$c\mathcal{L} = \frac{1}{2} v^\mu(\Box - 2)v_\mu + \frac{1}{8} h^{\mu\nu}(\Box - 2)h_{\mu\nu} + C^\mu C_\mu + \frac{1}{2} B^2 \tag{3.21}$$

satisfies all requirements. Using (A.19) and setting $c = 1/2$, the action reads in the Fock-notation

$$S = \int d^D x \mathcal{L} = -\frac{1}{2}(\psi|2(L_0 - 1) - L_{-1}L_1 - \frac{1}{2} L_{-2}L_2|\psi). \tag{3.22}$$

Clearly, this form of the action also applies for the mass levels 0 and 1 (cf. Equs. (2.4) and (2.11)). The gauge transformation laws (3.11) and (3.12) read

$$\delta_1|\psi\rangle = L_{-1}|\chi\rangle \tag{3.23}$$

$$\delta_2|\psi\rangle = (L_{-2}L_0 - \frac{3}{2}L_{-1}^2)|\Lambda\rangle \tag{3.24}$$

where

$$|\chi\rangle = -i\chi_\mu(x)\alpha_{-1}{}^\mu|0\rangle \tag{3.25}$$

and

$$|\Lambda\rangle = \Lambda(x)|0\rangle, \tag{3.26}$$

the constraint (3.13) just expressing the requirement that $|\chi\rangle$ is an element of $\mathbf{V}^{(0)}$,

$$L_1|\chi\rangle = 0. \tag{3.27}$$

Again, the invariance of (3.22) under (3.23) and (3.24) is checked without reference to any mass level number but merely follows from the assumption that $|\chi\rangle$ and $|\Lambda\rangle$ lie in $\mathbf{V}^{(0)}$ (see appendix for the definitions of $\mathbf{V}^{(n)}$ and $\mathbf{W}^{(n)}$). The verification of (3.24) makes use of $D = 26$. Off the critical dimension, there is no gauge symmetry of the type (3.24).

The kinetic operator appears in the literature as a *part* of string field actions but never in the form presented here (cf. Refs. [13,15]).

The question that will concern us for the next three sections is whether this scheme carries over to all higher mass levels. In the following section, the problem is posed in a more convenient language.

4 Action Functionals for the Higher Mass Levels

From now on we will abandon the notation using component fields and rather work instead with the elements of the Fock-space directly. The major ingredient for the following computations is the Virasoro algebra (A.22). Following the scheme of the last section, the kinetic operator should have the form

$$\mathcal{K} = 2(L_0 - 1) - \sum_{n=1}^{\infty} a_n L_{-n} L_n, \tag{4.1}$$

the action functional

$$S = -\frac{1}{2}(\psi|\mathcal{K}|\psi) \tag{4.2}$$

being local and containing only first and second space-time derivatives. We know already $a_1 = 1$, $a_2 = 1/2$. If it is possible to determine the a_n such that the Virasoro constraints (A.16) arise as possible gauge-fixing conditions, (4.2) constitutes an action for *all* field components contained in a general element $|\psi\rangle$ of the Fock-space.

The field equations read

$$\mathcal{K}|\psi\rangle = 0. \tag{4.3}$$

Since

$$\mathcal{K}^\dagger = \mathcal{K}, \tag{4.4}$$

the action is invariant under some gauge symmetry if and only if the field equations (4.3) are, i.e.

$$\mathcal{K}\delta|\psi\rangle = 0. \tag{4.5}$$

Extending the transformation laws (3.23) and (3.24), we ask whether there exist operators

$$\mathcal{S}_{-n} = \mathcal{L}^{(n)}_{-i} a^i_{(n)}(L_0) \tag{4.6}$$

for every $n \geq 1$ such that

$$\delta|\psi\rangle = \mathcal{S}_{-n}|\chi\rangle \tag{4.7}$$

satisfies (4.5) for *all* $|\chi\rangle \in \mathbf{V}^{(0)}$ (see appendix for the definition of $\mathcal{L}^{(n)}_{-1}$). Thereby the $a^i_{(n)}$ shall be polynomials in L_0. We already know that such operators exist for $n = 1$ and 2 if $D = 26$, namely

$$\mathcal{S}_{-1} = L_{-1} \tag{4.8}$$

$$\mathcal{S}_{-2} = L_{-2}L_0 - \frac{3}{2} L^2_{-1}. \tag{4.9}$$

In order to define operators of the type (4.6) unambiguously, we always place the L_0's to the right of the $\mathcal{L}^{(n)}_{-i}$.

Inserting (4.7) into the invariance condition (4.5), one obtains

$$\mathcal{K}\mathcal{S}_{-n}|\chi\rangle = 0 \tag{4.10}$$

for all $|\chi\rangle \in \mathbf{V}^{(0)}$ which immediately turns out to be a system of ℓ_n linear equations for the ℓ_n unknown $a^i_{(n)}$ (ℓ_n being the number of different $\mathcal{L}^{(n)}_i$'s). Using the ansatz (4.6) and commuting the operators L_p occurring in $\mathcal{K}$ to the right, one finds that

$$\mathcal{K}\mathcal{L}^{(n)}_{-i}|\xi\rangle \equiv \mathcal{L}^{(n)}_{-j}\mathcal{K}^{(n)j}{}_i(L_0)|\xi\rangle \tag{4.11}$$

for any $|\xi\rangle \in \mathbf{V}^{(0)}$. The $\mathcal{K}^{(n)j}{}_i$ are polynomials (of degree one) in L_0. Inserting $a^i_{(n)}(L_0)|\chi\rangle$ for $|\xi\rangle$ and using the requirement that the resulting expression vanishes for all $|\chi\rangle \in \mathbf{V}^{(0)}$, one obtains the system of equations

$$\mathcal{K}^{(n)j}{}_i(h) a^i_{(n)}(h) = 0 \tag{4.12}$$

which as to be valid for all real numbers h. To be more precise, one could insert a state of type (A.26) into (4.10) and deduce (4.12) for almost all values of h (and hence for all since we are dealing with polynomials) as indicated in the appendix. Clearly, if (4.12) holds, then also (4.10) is valid for any $|\chi\rangle \in \mathbf{V}^{(0)}$, irrespectively of $|\chi\rangle$ being an eigenvector of L_0 or not.

The first 1×1 matrix $\mathcal{K}^{(1)}$ turns out to be identically zero. This agrees with the fact that - due to (4.8) - $\mathcal{K}$ annihilates every element of $\mathbf{V}^{(1)}$, for any value of the space-time dimension D. Counting the $\mathcal{L}^{(2)}_i$ as

$$\mathcal{L}^{(2)}_1 = L_2, \qquad \mathcal{L}^{(2)}_2 = L^2_1, \tag{4.13}$$

one finds

$$\mathcal{K}^{(2)i}{}_j(h) = \begin{pmatrix} 2 - \frac{1}{4}D & -3h \\ -3 & -2h \end{pmatrix} \tag{4.14}$$

for general D. Its determinant is given by

$$\det \mathcal{K}^{(2)}(h) = \frac{1}{2}\, h(D - 26). \tag{4.15}$$

This illustrates that off the critical dimension, there is no solution $a^i_{(2)}(h)$. If $D = 26$, one reads off from (4.9)

$$a^i_{(2)}(h) = \begin{pmatrix} h \\ -3/2 \end{pmatrix}. \tag{4.16}$$

We adopt the convention that the $a^i_{(n)}(h)$ have no common polynomial factor, i.e. they never vanish simultaneously. The rank of the matrix $\mathcal{K}^{(2)}$ turns out to be 1 for *all* values of h. If $h = -9/4$, the eigenvalue 0 appears twice in $\mathcal{K}^{(2)}$, but

$$\mathcal{K}^{(2)}(-\frac{9}{4}) = \begin{pmatrix} -9/2 & 27/4 \\ -3 & 9/2 \end{pmatrix} \tag{4.17}$$

is non-zero and hence non-diagonalizable (i.e. there is no basis of eigenvectors). As a consequence, the polynomials (4.16), found as the *generic* solution of (4.12), in fact constitue the general solution for any h. This is responsible for the fact that the symmetries (3.23), (3.24) admit the required gauge condition for *any* $|\psi\rangle_2$ which is subject to the field equation (4.3). We will return to this point later.

For the moment, we consider the system (4.12) for higher n. Counting the $\mathcal{L}_i^{(3)}$ as given in (A.25), one may compute $\mathcal{K}^{(3)}$ for arbitrary a_3 and D. One finds that the equation (4.12) admits non-trivial polynomials $a^i_{(3)}(h)$ if and only if $a_3 = 1/3$ and $D = 26$. Leaving D arbitrary, one again arrives at the result that $\det \mathcal{K}^{(3)}$ is a polynomial in h and D, multiplied by $D - 26$. The $a^i_{(3)}(h)$ are unique up to common numerical factors, i.e. if $D = 26$, the generic rank of $\mathcal{K}^{(3)}$ is 2.

It has been checked on the computer - using the algebraic program REDUCE - that this scheme holds up to $n = 7$. Nontrivial solutions of (4.12) arise if and only if $a_n = 1/n$ and $D = 26$. For illustration, we quote here the results for the cases $n = 3$ and 4:

$$S_{-3} = L_{-3}a^1_{(3)}(L_0) + L_{-2}L_{-1}a^2_{(3)}(L_0) + L^3_{-1}a^3_{(3)}(L_0), \tag{4.18}$$

where

$$\begin{aligned} a^1_{(3)}(h) &= \frac{1}{13}(5 - 8h)h \\ a^2_{(3)}(h) &= 2h + 1 \\ a^3_{(3)}(h) &= -3/2 \end{aligned} \tag{4.19}$$

and

$$S_{-4} = L_{-4}a^1_{(4)}(L_0) + L_{-3}L_{-1}a^2_{(4)}(L_0) + L^2_{-2}a^3_{(4)}(L_0) + L_{-2}L^2_{-1}a^4_{(4)}(L_0) + L^4_{-1}a^5_{(4)}(L_0) \tag{4.20}$$

with

$$\begin{aligned} a^1_{(4)}(h) &= -48h^4 - 174h^3 - 36h^2 - 774h \\ a^2_{(4)}(h) &= 176h^3 + 710h^2 - 200h + 130 \\ a^3_{(4)}(h) &= 72h^3 + 456h^2 + 672h \\ a^4_{(4)}(h) &= -394h^2 - 2084h - 1690 \\ a^5_{(4)}(h) &= 197h + 845. \end{aligned} \tag{4.21}$$

Motivated by these results, we fix the kinetic operator as

$$\mathcal{K} = 2(L_0 - 1) - \sum_{n=1}^{\infty} \frac{1}{n} L_{-n}L_n \tag{4.22}$$

and conjecture that for any $n \geq 1$ there is a set of polynomials $a^i_{(n)}(h)$ such that (4.10) (or equivalently (4.12)) holds, provided $D = 26$. The $a^i_{(n)}$ should be unique up to common factors. Moreover, we expect the sequence of gauge symmetries obtained in this way being large enough to bring any solution of the field equation (4.3) into the Virasoro gauge (A.16). This last statement may equivalently be expressed as follows. Let $|\psi\rangle$ be any element of the Fock-space satisfying (4.3). Then $|\psi\rangle$ is of the form

$$|\psi\rangle = |\chi_0\rangle + \sum_{p=1}^{\infty} \mathcal{S}_{-p}|\chi_p\rangle \tag{4.23}$$

where all $|\chi\rangle$'s lie in $\mathbf{V}^{(0)}$. $|\chi_0\rangle$ is the "physical" part of $|\psi\rangle$, satisfying the mass-shell equation (A.17). According to the considerations given in the appendix, we expect the decomposition (4.23) not to be unique. The arbitrariness we are left with constitutes a residual gauge invariance of the type (4.7) which leaves the Virasoro conditions invariant. For illustration, consider a state of the form

$$|\psi\rangle = \mathcal{S}_{-1}|\chi\rangle \equiv L_{-1}|\chi\rangle \tag{4.24}$$

with $|\chi\rangle \in \mathbf{V}^{(0)}$. This state obeys the field equation (4.3) and is an element of $\mathbf{V}^{(1)}$. According to (4.23), it is a pure gauge. If, however, $|\chi\rangle$ additionally satisfies

$$L_0|\chi\rangle = 0, \tag{4.25}$$

it follows (multiplying (4.24) by L_1) that $|\psi\rangle$ is also in $\mathbf{V}^{(0)}$. Thus, $|\psi\rangle$ may equally well be interpreted as the physical contribution $|\chi_0\rangle$ in (4.23). At the first mass level this just means that the Lorentz condition (2.7) still admits the gauge freedom (2.10) with

$$\Box\Lambda = 0. \tag{4.26}$$

As indicated in the appendix, the non-uniqueness of the decomposition (4.23) is connected with the zeros in the Kac-determinants ($h = 0$ being the unique zero of $\mathcal{M}^{(1)}(h) = 2h$, in agreement with (4.25)). However, trying to fix the gauge completely means to give up the covariant formalism. In quantized string field theory, this is not even necessary, because the covariant gauge - together with appropriate Feynman boundary conditions - suffices to guarantee the existence of the propagator.

The fact that the gauge parameters $|\chi\rangle$ are constrained to be elements of $\mathbf{V}^{(0)}$ is not really a shortcoming of the present formalism. One can always express them in terms of unconstrained parameters along the lines of equation (3.14). The complicated form of the gauge transformation laws is the price we have to pay when using the simple action (4.2) and (4.22). We will return to the residual gauge freedom and the question of unconstrained gauge parameters in sections 7 and 8.

In the following two sections we will partly prove the conjectures made here. During the course of the investigations, the problem has been attacked by several different approaches, and it seems extremely unlikely that the assumptions which remain unproven are incorrect.

5 Existence Proof

In this section we prove the existence of the operators $\mathcal{S}_{-n}$ for all $n \geq 1$. The kinetic operator is taken over from (4.22), the space-time dimension is left unspecified for the moment. Since the general form of the matrices $\mathcal{K}^{(n)i}{}_j$ arises rather implicitly, we proceed along another line of reasoning.

Making extensive use of the Virasoro algebra (A.22), one may show

$$L_1\mathcal{K} = -\sum_{n=1}^{\infty} \frac{1}{n} L_{-n}\mathcal{A}_{n+1} \tag{5.1}$$

$$L_2\mathcal{K} = \frac{1}{4}(26-D)L_2 - 3\mathcal{A}_2 - \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n} L_{-n}(L_n\mathcal{A}_2 - L_1\mathcal{A}_{n+1} - \frac{n^2-2n-2}{n+1}\mathcal{A}_{n+2}) \tag{5.2}$$

where

$$\mathcal{A}_n = L_1L_{n-1} + \frac{(n-1)(n+1)}{n} L_n \tag{5.3}$$

is defined for $n \geq 2$. The computation is partly identical to that given in Ref. [15].

Next we note that if $D = 26$ and a state $|\psi\rangle$ satisfies

$$\mathcal{A}_p|\psi\rangle = 0 \tag{5.4}$$

for all $p \geq 2$, it follows from (5.1) and (5.2) that $\mathcal{K}|\psi\rangle$ is annihilated by all L_n with positive n (remember that L_1 and L_2 generate all L_n). As a consequence, $\mathcal{K}|\psi\rangle$ is an element of $\mathbf{V}^{(0)}$.

For the rest of this section, we set

$$|\psi\rangle = \mathcal{S}_{-n}|\chi\rangle \in \mathbf{V}^{(n)} \tag{5.5}$$

for a given n, where $\mathcal{S}_{-n}$ represents the ansatz (4.6). The equation for the polynomials $a^i_{(n)}$ thus reads

$$\mathcal{K}|\psi\rangle = 0 \tag{5.6}$$

and has to be valid for all $|\chi\rangle \in \mathbf{V}^{(0)}$. Now suppose that (5.4) holds for all $|\chi\rangle \in \mathbf{V}^{(0)}$. Since $\mathcal{K}|\psi\rangle \in \mathbf{V}^{(0)}$, as stated above,

$$\mathcal{L}^{(n)}_i\mathcal{K}|\psi\rangle = 0. \tag{5.7}$$

Inserting (4.6), one finds

$$\mathcal{M}_{ij}^{(n)}(L_0)\mathcal{K}^{(n)j}{}_k(L_0)a_{(n)}^k(L_0)|\chi\rangle = 0. \tag{5.8}$$

Since this has to be true for *all* $|\chi\rangle \in \mathbf{V}^{(0)}$ - especially for eigenvectors of L_0 corresponding to generic eigenvalues h - we conclude

$$\mathcal{K}_{ij}^{(n)}(h)a_{(n)}^j(h) = 0, \tag{5.9}$$

where

$$\mathcal{K}_{ij}^{(n)}(h) = \mathcal{M}_{ik}^{(n)}(h)\mathcal{K}^{(n)k}{}_j(h). \tag{5.10}$$

Since $\mathcal{M}_{ij}^{(n)}(h)$ is generically invertible, one arrives at equation (4.12). Note that in the generic sense, $\mathcal{M}_{ij}^{(n)}$ plays the role of a metric which may serve to raise and lower indices. The matrix (5.10) may also be defined as

$$\mathcal{K}_{ij}^{(n)}(h) = \langle\chi|\mathcal{L}_i^{(n)}\mathcal{K}\mathcal{L}_{-j}^{(n)}|\chi\rangle \tag{5.11}$$

where $|\chi\rangle$ satisfies (A.26) and $\langle\chi|\chi\rangle = 1$. Hence

$$\mathcal{K}_{ij}^{(n)}(h) = \mathcal{K}_{ji}^{(n)}(h) \tag{5.12}$$

for all h.

What we have shown so far is that, if (5.4) is valid for all $|\chi\rangle \in \mathbf{V}^{(0)}$, the $a_{(n)}^i(h)$ solve equation (4.12). Since (5.4) is a generic equation, it is equivalent to

$$\mathcal{L}_i^{(n-p)}\mathcal{A}_p|\psi\rangle = 0 \tag{5.13}$$

for $2 \le p \le n$. These are in fact $\sum_{p=2}^n \ell_{n-p}$ ordinary linear equations for the ℓ_n unknowns $a_{(n)}^i$.

Studying the algebra of the operators $\mathcal{A}_p$, one discovers another set of operators $\mathcal{B}_p$ which in some respect are more convenient. From (5.3) it follows

$$L_m = \frac{m}{(m-1)(m+1)}\mathcal{A}_m - \frac{m}{(m-1)(m+1)}L_1 L_{m-1}. \tag{5.14}$$

Again replacing L_{m-1} by means of (5.3) and iterating the procedure, one finds an expression for L_m in terms of $\mathcal{A}_p$ ($p \le m$) and powers of L_1. Defining

$$\mathcal{B}_m = L_m + \frac{2m(-)^m}{(m+1)!}L_1^m \tag{5.15}$$

for $m \geq 2$, the expression reads

$$\mathcal{B}_m = \sum_{p=2}^{m} (-)^{m-p} \frac{m}{(m+1)!} \frac{p!}{p-1} L_1^{m-p} \mathcal{A}_p. \tag{5.16}$$

Clearly, this relation may be inverted to express $\mathcal{A}_m$ in terms of all $\mathcal{B}_p$ $(p \leq m)$. Thus, one may substitute $\mathcal{B}_p$ for $\mathcal{A}_p$ in the equations (5.4) and (5.13). It is worth noting that, upon the use of the Virasoro algebra, one finds that the commutators of the $\mathcal{B}$'s close in the sense $(2 \leq p < m)$

$$[\mathcal{B}_p, \mathcal{B}_m] = \sum_{q=p+1}^{p+m} c_{pmq} L_1^{p+m-q} \mathcal{B}_q \tag{5.17}$$

where c_{pmq} are non-zero real numbers. From this it follows that (5.4) - as it stands or with $\mathcal{B}_p$ instead of $\mathcal{A}_p$ - is already valid for all p if it holds for $p = 2, 3$ and 4. All higher $\mathcal{B}_p$ or $\mathcal{A}_p$ may then be expressed in terms of the lower ones by (5.17). This observation might be useful for practical calculations of the higher $\mathcal{S}_{-n}$.

The existence proof runs as follows. Consider the set of operators $\mathcal{L}_i^{(n-p)} \mathcal{B}_p$ $(2 \leq p \leq n)$. They lie in the ℓ_n-dimensional space spanned by the $\mathcal{L}_i^{(n)}$. Although they are larger in number than ℓ_n, they actually span only an $(\ell_n - 1)$-dimensional subspace. This can be seen by noting that it is not possible to find real numbers $\lambda^i_{(n-p)}$ such that

$$\sum_{p=2}^{n} \lambda^i_{(n-p)} \mathcal{L}_i^{(n-p)} \mathcal{B}_p = L_n. \tag{5.18}$$

Hence, (5.13) only provides a set of $\ell_n - 1$ linear equations for ℓ_n unknowns. Since the $\mathcal{K}^{(n)i}{}_j(h)$ are of degree one in h, the $a^i_{(n)}(h)$ may be chosen as polynomials of degree $\leq \ell_n - 1$.

It is actually not difficult to prove the uniqueness of solutions to (5.13), which is equivalent to show that the $\ell_n - 1$ equations mentioned above are linearly independent. Imposing (5.4) - now using the $\mathcal{A}_p$ - and requiring the single additional equation

$$L_n |\psi\rangle = 0, \tag{5.19}$$

one may check that the only solution is the trivial one, $a^i_{(n)}(h) \equiv 0$. To do this, one takes into account the definition (5.3) of the $\mathcal{A}_p$ and again uses the Virasoro algebra to show successively $L_{n-1}|\psi\rangle = L_{n-2}|\psi\rangle = \cdots = 0$. Ending up with $|\psi\rangle \in \mathbf{V}^{(0)}$ for all $|\chi\rangle \in \mathbf{V}^{(0)}$, one concludes $a^i_{(n)} \equiv 0$.

Thus, we have proven the existence of gauge transformations defined by $\mathcal{S}_{-n}$ at each level n if $D = 26$. The solutions $a^i_{(n)}(h)$ arising from the equations (5.4) are unique (up to common factors). We have *not* proven that these are the only solutions satisfying (5.6). Nevertheless, a lot of playing around with (5.4) and expressions like (5.1) and (5.2) has convinced ourselves that the systems (5.4) and (5.6) - as generic equations for the $a^i_{(n)}(h)$ - are equivalent. It also seems to be true that (5.6) together with (5.19) leads to $a^i_{(n)} \equiv 0$, which would again prove the equivalence. However, we have no rigorous proof but only the definite statement that everything works up to $n = 7$.

In the following we assume that our conjecture holds. In other words this means that the rank of the matrix $\mathcal{K}^{(n)i}{}_j(h)$ is equal to $\ell_n - 1$ except possibly for a finite number of values of h. These critical values may be found by computing the characteristic polynomial

$$\det(\lambda - \mathcal{K}^{(n)}(h)) = \lambda p^{(n)}(h) + \cdots + \lambda^{\ell_n}. \tag{5.20}$$

for $D = 26$. $\lambda = 0$ is always an eigenvalue. The zeros of the polynomial $p^{(n)}(h)$ are those values where a second eigenvector with eigenvalue zero *might* arise (in the next section we will conjecture that this is not the case).

Let us at the end of this section mention an interesting property of the $a^i_{(n)}$. Evaluating the equations (5.13) explicitly, one gets

$$A^{j(n-p)}_{(n)i} \mathcal{M}^{(n)}_{jk}(L_0) a^k_{(n)}(L_0)|\chi\rangle = 0 \tag{5.21}$$

for $2 \leq p \leq n$, where

$$\mathcal{L}^{(n-p)}_i \mathcal{A}_p = \mathcal{L}^{(n)}_j A^{j(n-p)}_{(n)i}. \tag{5.22}$$

The $\mathcal{A}_p$ contain only L_m with positive m, hence A does not depend on L_0. Since $\mathcal{M}^{(n)}_{jk}$ is generically non-singular, (5.21) may be read as an equation for $a^{(n)}_j = \mathcal{M}^{(n)}_{jk} a^k_{(n)}$, the "covariant" components of $a^{(n)}$. The arguments given above apply to this interpretation as well. One finds a solution $b^{(n)}_j$ which is independent of L_0. Thus, the generic solution $a^j_{(n)}(L_0)$ satisfies

$$a^{(n)}_j(h) \equiv \mathcal{M}^{(n)}_{jk}(h) a^k_{(n)}(h) = q^{(n)}(h) b^{(n)}_j \tag{5.23}$$

for some polynomial $q^{(n)}(h)$. The real numbers $b^{(n)}_j$ are a solution for the equation adjoint to (4.12),

$$b^{(n)}_j \mathcal{K}^{(n)j}{}_i(h) = 0. \tag{5.24}$$

In order to compute $a^i_{(n)}$ explicitly, one may solve (5.24) for *some* non-critical value of h to find $b^{(n)}_i$. Then

$$a^i_{(n)}(h) = q^{(n)}(h) \mathcal{M}^{ij}_{(n)}(h) b^{(n)}_j, \tag{5.25}$$

where $q^{(n)}(h)$ is a polynomial with minimal degree that eats up all singularities of $\mathcal{M}^{ij}_{(n)}(h)b^{(n)}_j$ (cf. Equ. (A.31)).

6 Existence of the Virasoro Gauge

Once having an infinite sequence of gauge transformations $\mathcal{S}_{-n}$ and the conjecture that they are essentially unique, the question arises whether these may actually be used to obtain the Virasoro conditions as a special gauge-fixing. Neglecting the possibility of non-generic states, the answer is trivial: In the generic sense, the decomposition (4.23) *is* the solution of the field equation (4.3). However, we must be a little bit more careful in order not to oversee the non-generic states that lie in the intersections $\mathbf{V}^{(n)} \cap \mathbf{V}^{(m)}$.

Let $|\psi\rangle$ be any state that contains only Verma level contributions up to some positive n, i.e. $|\psi\rangle \in \mathbf{W}^{(n)}$. Then $|\psi\rangle$ is of the form (A.32). The existence of the gauge conditions is ensured if $\mathcal{K}|\psi\rangle = 0$ implies

$$|\psi\rangle = |\xi\rangle + \sum_{p=1}^{n} \mathcal{S}_{-p}|\chi_p\rangle \tag{6.1}$$

for $|\xi\rangle$ and $|\chi_p\rangle$ in $\mathbf{V}^{(0)}$. Note that from

$$\mathcal{K}|\psi\rangle \equiv 2(L_0 - 1)|\chi_0\rangle + \sum_{p=1}^{n} \mathcal{L}^{(p)}_{-j}\mathcal{K}^{(p)j}{}_i(L_0)|\chi^i_p\rangle = 0 \tag{6.2}$$

one cannot straightforwardly conclude that all the expressions at the different levels vanish separately. Multiplying by $\mathcal{L}^{(n)}_i$, we find

$$\mathcal{L}^{(n)}_i\mathcal{K}|\psi\rangle \equiv \mathcal{K}^{(n)}_{ij}(L_0)|\chi^j_n\rangle = 0. \tag{6.3}$$

If there exists some $|\chi\rangle \in \mathbf{V}^{(0)}$ such that

$$\mathcal{M}^{(n)}_{ij}(L_0)(|\chi^j_n\rangle - a^j_{(n)}(L_0)|\chi\rangle) = 0, \tag{6.4}$$

one finds, taking into account the definition (A.27) of $\mathcal{M}^{(n)}$ that

$$\mathcal{L}^{(n)}_{-i}|\chi^i_n\rangle - \mathcal{S}_{-n}|\chi\rangle \in \mathbf{W}^{(n-1)}. \tag{6.5}$$

Identifying $|\chi\rangle$ with $|\chi_n\rangle$ in (6.1), one has already confirmed the highest term. The remaining state

$$|\psi'\rangle = |\psi\rangle - \mathcal{S}_{-n}|\chi\rangle \tag{6.6}$$

which is an element of $\mathbf{W}^{(n-1)}$, is treated analogously. By induction, one arrives at (6.1). Thus, in order to complete the argument, one has to deduce (6.4) from (6.3). This is the second point where we have not been able to give a rigorous proof.

The conjecture we are forced to make simply states that the rank of the matrices $\mathcal{K}^{(n)i}{}_j(h)$ is $\ell_n - 1$ for *all* values h - including the zeros of the polynomial $p^{(n)}(h)$ in (5.20). Thus, at the critical values, $\mathcal{K}^{(n)i}{}_j(h)$ becomes non-diagonalizable since the eigenvalue zero appears twice in the characteristic polynomial. This situation is illustrated by (4.17) for the case $n = 2$. The critical values of h seem to be just those where two generic eigenvectors coincide.

Taking this staement for granted, one concludes by means of elementary linear algebra that there exist polynomials $B^{(n)i}{}_j(h)$ and real numbers $f^j_{(n)}$ such that

$$f^j_{(n)} b^{(n)}_j = 1 \tag{6.7}$$

and

$$\mathcal{K}^{(n)i}{}_j(h) B^{(n)j}{}_k(h) = \delta^i_k - f^i_{(n)} b^{(n)}_k. \tag{6.8}$$

Thus, from

$$\mathcal{K}^{(n)i}{}_j(L_0)|\xi_i\rangle = 0 \tag{6.9}$$

for $|\xi_i\rangle \in \mathbf{V}^{(0)}$, it follows upon multiplication by $B^{(n)j}{}_i(L_0)$ that

$$|\xi_i\rangle = b^{(n)}_i |\tilde{\chi}\rangle \tag{6.10}$$

with

$$|\tilde{\chi}\rangle = f^i_{(n)}|\xi_i\rangle. \tag{6.11}$$

Now consider an equation of the type (6.3)

$$\mathcal{K}^{(n)}_{ij}(L_0)|\chi^j\rangle \equiv \mathcal{K}^{(n)k}{}_i(L_0)\mathcal{M}^{(n)}_{kj}(L_0)|\chi^j\rangle = 0, \tag{6.12}$$

where one has made use of (5.12). Denoting

$$|\xi_k\rangle = \mathcal{M}^{(n)}_{kj}(L_0)|\chi^j\rangle, \tag{6.13}$$

one arrives at (6.9) which implies via (6.10)

$$\mathcal{M}^{(n)}_{kj}(L_0)|\chi^j\rangle = b^{(n)}_k|\tilde{\chi}\rangle \tag{6.14}$$

for some $|\tilde{\chi}\rangle \in \mathbf{V}^{(0)}$. Since the polynomial $q^{(n)}(h)$ appearing in (5.23) is non-vanishing, one may always find a $|\chi\rangle \in \mathbf{V}^{(0)}$ such that

$$|\tilde{\chi}\rangle = q^{(n)}(L_0)|\chi\rangle. \tag{6.15}$$

Hence, using (5.23),

$$\mathcal{M}^{(n)}_{kj}(L_0)(|\chi^j\rangle - a^j_{(n)}|\chi\rangle) = 0 \tag{6.16}$$

which is just the statement (6.4).

We should summarize now in a simpler notation what we have proven under the assumption that the rank of $\mathcal{K}^{(n)i}{}_j(h)$ equals $\ell_n - 1$ for all h. Let $n \geq 1$ and $|v\rangle \in \mathbf{V}^{(n)}$. Then $\mathcal{K}|v\rangle \in \mathbf{W}^{(n-1)}$ if and only if there exists a $|\chi\rangle \in \mathbf{V}^{(0)}$ such that $|v\rangle - \mathcal{S}_{-n}|\chi\rangle \in \mathbf{W}^{(n-1)}$. In this formulation, the state $|v\rangle$ is just the highest term in the decomposition (A.37), and the statement that $\mathcal{K}|v\rangle$ lies in $\mathbf{W}^{(n-1)}$ is identical to the equation (6.3).

Applying this result to each mass level of a given state satisfying the field equation (4.3), one arrives at the infinite decomposition (4.23).

There is another algorithm which yields the Virasoro gauge conditions. It has been checked for $n \leq 4$, and its general validity would prove the conjecture made above. Take any $|\psi\rangle \in \mathbf{W}^{(n)}$, subject to the field equation (4.3). Trying to find a gauge in which $L_n|\psi\rangle = 0$, one obtains the equation

$$\delta(L_n|\psi\rangle) \equiv L_n\mathcal{S}_{-n}|\chi_n\rangle \equiv f_n(L_0)|\chi_n\rangle = -L_n|\psi\rangle \tag{6.17}$$

where $f_n(L_0)$ is a non-vanishing polynomial. Since $L_n|\psi\rangle$ lies in $\mathbf{V}^{(0)}$, this equation can be solved for $|\chi_n\rangle \in \mathbf{V}^{(0)}$. Now we conjecture that from

$$\mathcal{K}|\psi\rangle = L_n|\psi\rangle = L_{n-1}|\psi\rangle = \cdots = L_{p+1}|\psi\rangle = 0 \tag{6.18}$$

it follows by pure application of the Virasoro algebra (A.22) that $L_p|\psi\rangle$ is annihilated by all L_m ($m \geq 1$), hence an element of $\mathbf{V}^{(0)}$. Then the equation

$$\delta(L_p|\psi\rangle) \equiv L_p\mathcal{S}_{-p}|\chi_p\rangle \equiv f_p(L_0)|\chi_p\rangle = -L_p|\psi\rangle \tag{6.19}$$

is solvable for some $|\chi_p\rangle \in \mathbf{V}^{(0)}$. Iterating this process, one ends up with the gauge condition (A.16).

7 Gauge Transformations with Unconstrained Parameter

Instead of performing an infinite sequence of gauge transformations

$$\delta|\psi\rangle = \sum_{p=1}^{\infty} \mathcal{S}_{-p}|\chi_p\rangle \tag{7.1}$$

with parameters $|\chi_p\rangle \in \mathbf{V}^{(0)}$, one might try to formulate (7.1) in terms of a simple operator

$$\delta|\psi\rangle = \mathcal{S}|\Lambda\rangle \tag{7.2}$$

with an unconstrained parameter $|\Lambda\rangle$. In the following, we will construct an operator $\mathcal{S}$ satisfying

$$\mathcal{K}\mathcal{S} = 0 \tag{7.3}$$

and acting upon the different Verma levels according to

$$\mathcal{S} : \mathbf{V}^{(n)} \to \mathbf{V}^{(n+1)} \tag{7.4}$$

for $n \geq 0$. Repeated application of $\mathcal{S}$ should give (with $|\chi\rangle \in \mathbf{V}^{(0)}$)

$$\mathcal{S}^p|\chi\rangle = \mathcal{S}_{-p}|\xi_p\rangle \tag{7.5}$$

for some $|\xi_p\rangle \in \mathbf{V}^{(0)}$. Thus, $|\Lambda\rangle$ in (7.2) may be chosen as

$$|\Lambda\rangle = \sum_{p=0}^{\infty} \mathcal{S}^p|\eta_p\rangle \tag{7.6}$$

for appropriate elements $|\eta_p\rangle$ of $\mathbf{V}^{(0)}$ in order to achieve (7.1). Clearly, the gauge transformations in the form (7.2) are highly redundant because $|\Lambda\rangle$ is completely arbitrary.

Making the ansatz (using $\mathcal{L}^{(0)} = 1$ by convention)

$$\mathcal{S} = \sum_{p=0}^{\infty} \mathcal{L}_{-i}^{(p+1)} s_{(p+1,p)}{}^{ij}(L_0)\mathcal{L}_j^{(p)} \tag{7.7}$$

and requiring that $\mathcal{K}\mathcal{S}$ annihilates a general element of $\mathbf{V}^{(n)}$, one finds the recursion formula

$$s_{(n+1,n)}{}^{\ell j}(h)\mathcal{M}_{jk}^{(n)}(h) = a_{(n+1)}^{\ell}(h)c_k^{(n)}(h) - \sum_{p=0}^{n-1} s_{(p+1,p)}{}^{ij}(h-p+n)C_{(n+1)}^{\ell}{}^{(p+1,p,n)}{}_{ijk}(h) \tag{7.8}$$

where $c_k^{(n)}(h)$ is arbitrary and the C's arise from the identity

$$\mathcal{L}_{-i}^{(p+1)}\mathcal{L}_j^{(p)}\mathcal{L}_{-k}^{(n)}|\chi\rangle = \mathcal{L}_{-\ell}^{(n+1)}C_{(n+1)}^{\ell}{}^{(p+1,p,n)}{}_{ijk}(L_0)|\chi\rangle \tag{7.9}$$

for $|\chi\rangle \in \mathbf{V}^{(0)}$. Solving (7.8) with respect to $s_{(n+1,n)}{}^{\ell j}$, one has to multiply by the inverse $\mathcal{M}_{(n)}^{ki}(h)$ and finds operators $\mathcal{S}$ which satisfy (7.3) but are in general nonlocal.

However, there are some hints that the arbitrary functions $s_{(1,0)}(h)$ and $c_i^{(n)}(h)$ may be adjusted in such a way that all $s_{(n+1,n)}{}^{ij}(h)$ become polynomial. In order to get a local action of $\mathcal{S}$ only upon $\mathbf{W}^{(n)}$, one just has to solve (7.8) for any choice of the arbitrary functions up to the n-th order and multiply the result by the overall denominator. In order to get more insight into the structure of $\mathcal{S}$, we introduce the non-local operators [13]

$$\begin{aligned} \Pi^{(0)} &= 0 \\ \Pi^{(n)} &= 1 - \mathcal{L}_{-i}^{(n)} \mathcal{M}_{(n)}^{ij}(L_0) \mathcal{L}_j^{(n)} \end{aligned} \tag{7.10}$$

for $n \geq 1$, and

$$\mathcal{P}^{(n)} = \Pi^{(n)} \Pi^{(n+1)} \Pi^{(n+2)} \dots . \tag{7.11}$$

Denoting further

$$c_i^{(n)}(h) = \mathcal{M}_{ij}^{(n)}(h) c_{(n)}^j(h), \tag{7.12}$$

$$\mathcal{C}_n \equiv c_{(n)}^i(L_0) \mathcal{L}_i^{(n)}, \tag{7.13}$$

a closed expression for $\mathcal{S}$ - representing the general solution of (7.8) - is given by

$$\mathcal{S} = \sum_{n=0}^{\infty} \mathcal{S}_{-n-1} \mathcal{C}_n (1 - \Pi^{(n)}) \mathcal{P}^{(n+1)}, \tag{7.14}$$

where $c_{(0)}$ is to be identified with $s_{(1,0)}$. This equation is easily confirmed by using (7.10) to compute the action of $\mathcal{S}$ upon $\mathbf{V}^{(n)}$

$$\mathcal{S} \mathcal{L}_{-i}^{(n)} |\chi\rangle = \mathcal{S}_{-n-1} \mathcal{C}_n \mathcal{L}_{-i}^{(n)} |\chi\rangle. \tag{7.15}$$

Iterative application of $\mathcal{S}$ upon an element of $\mathbf{V}^{(0)}$ gives

$$\mathcal{S}^p |\chi\rangle = \mathcal{S}_{-p} g_p(L_0) |\chi\rangle, \tag{7.16}$$

where

$$g_p(h) = f_{p-1}(h) f_{p-2}(h) \cdots f_2(h) f_1(h) c_{(0)}(h) \tag{7.17}$$

and

$$f_n(h) = c_{(n)}^i(h) \mathcal{M}_{ij}^{(n)}(h) a_{(n)}^j(h) \equiv c_j^{(n)}(h) a_{(n)}^j(h). \tag{7.18}$$

Thus, if $c_{(0)}$ and all f_n are non-trivial polynomials, we recover (7.5) with

$$|\xi_p\rangle = g_p(L_0) |\chi\rangle.$$

Choosing $s_{(1,0)}(h) = 1$ and setting all subsequent $c_i^{(n)}(h)$ equal to zero, the result for $\mathcal{S}$ is just the operator $L_{-1}P$, where P is the projection onto $\mathbf{V}^{(0)}$. This can be seen by comparing (7.14) with the equation $P = \mathcal{P}^{(1)}$, which was given in Ref. [13]. Clearly, this solution is neither local nor realizes (7.5) non-trivially. This was to be expected, because setting all $c_i^{(n)}$ equal to zero, no detailled information about $\mathcal{K}$ enters in $\mathcal{S}$.

In order to obtain a local solution for $\mathcal{S}$ which realizes all gauge transformations (7.5) occurring in the present formulation of free bosonic string field theory, we have to choose the $c_j^{(n)}$ in such a way that all singularities arising from the matrices $\mathcal{M}^{ij}_{(n)}(h)$ cancel, thereby keeping the $f_n(h)$ non-trivial. From (7.8) or (7.15) one concludes that the $c_i^{(n)}(h)$ have to be polynomials as well.

At the first three levels it is possible to adjust the arbitrary functions such that no L_0 appears explicitly. The result is unique up to a constant factor:

$$\mathcal{S} = L_{-1} - \frac{1}{3}L_{-2}L_1 - \frac{1}{13}L_{-3}(L_2 - \frac{2}{3}L_1^2) + \cdots, \tag{7.19}$$

where the dots denote higher order terms. Inserting this expression into the equations for the next level, one inevitably encounters polynomials in L_0. Apparently, *any* local solution for $\mathcal{S}$ up to the level $\mathcal{L}_{-i}^{(p+1)}\mathcal{L}_j^{(p)}$ admits a family of local solutions at the next higher level. This has been checked up to $\mathcal{L}_{-i}^{(4)}\mathcal{L}_j^{(3)}$. The calculation of some solution for $\mathcal{S}$ to arbitrarily high levels is then straightforward but very tedious.

The general solution contains infinitely many arbitrary polynomials in L_0. This can also be seen by noting that, along with $\mathcal{S}$, also all operators $\mathcal{S}' = \mathcal{S}\mathcal{T}$ do the job if the non-trivial realization of (7.5) remains valid - and this is the case for various choices of $\mathcal{T}$.

If one prefers, however, to have an even more explicit expression for the unification of the gauge transformations into one closed formula, one may proceed as follows. Let

$$|\Lambda\rangle = \sum_{n=0}^{\infty} |\Lambda\rangle_n \tag{7.20}$$

be the (unique) decomposition of an arbitrary state with respect to the mass level operator (cf. Equs. (A.11) and (A.12)). Then define an operator $\mathcal{R}$ on Fock-space by its action upon the n-th mass level,

$$\mathcal{R}|\Lambda\rangle_n = r_n(L_0)P|\Lambda\rangle_n, \tag{7.21}$$

where the r_n are polynomials which cancel the singularities contained in the relevant part of P at the n-th mass level. They are essentially the overall denominator of the products $\Pi^{(1)}\Pi^{(2)}\dots\Pi^{(n)}$. Although $\mathcal{R}$ may not be expressed as a sum over products of Virasoro operators, it acts locally upon the space-time fields in $|\Lambda\rangle$, leaves the mass level invariant and puts any state to Verma level zero,

$$\mathcal{R} : \mathbf{F} \to \mathbf{V}^{(0)}. \tag{7.22}$$

Of course, $\mathcal{R}$ is no longer a projector.

As a consequence of (7.22), any of the operators $S_{-n}\mathcal{R}$ generates a gauge transformation of the type (4.7). Defining the operator

$$\hat{S} = \sum_{n=1}^{\infty} S_{-n}\mathcal{R}L_n, \tag{7.23}$$

we again recover *all* transformations (4.7) by setting

$$\delta|\psi\rangle = \hat{S}|\Lambda\rangle \tag{7.24}$$

for unconstrained $|\Lambda\rangle$. Note that the L_n at the right hand side of (7.23) may be replaced by any local operator $u_i^{(n)}(L_0)\mathcal{L}_i^{(n)}$ (or likewise by $u_i^{(n-1)}(L_0)\mathcal{L}_i^{(n-1)}$ in order to make the level zero part of $|\Lambda\rangle$ contribute to the variation of $|\psi\rangle$). This construction is essentially a straightforward generalization of (3.14) to all space-time fields contained in a general Fock-state. Since it makes explicit reference to the mass level decomposition (whereas the expression (4.22) for the kinetic operator does not) it seems artificial as compared to (7.7), but it is sufficient to merely reproduce the correct gauge transformation law for each space-time field.

One could think of using $2\mathcal{R}(L_0 - 1)$ as an alternative kinetic operator, which contains only a finite order of space-time derivatives at a definite mass level. Due to the general property

$$\mathcal{R}L_{-n} = 0 \tag{7.25}$$

$(n \geq 1)$, one could always achieve the Virasoro conditions (A.16) as a special choice of gauge. However, the remaining field equation would contain the polynomials $r_n(L_0)$ at the n-th mass level in addition to the factor $L_0 - 1$. The mathematical reason for $\mathcal{R}$ being applicable in the transformation law (7.24) but not serving as kinetic operator is the fact that the restricted function

$$\mathcal{R} : \mathbf{V}^{(0)} \to \mathbf{V}^{(0)} \tag{7.26}$$

is surjective (as long as boundary conditions are omitted) but not injective.

Having established the forms (7.2) or (7.24) of the gauge symmetry, we are left with a single operator $\mathcal{S}$ (or $\hat{S}$) instead of L_{-1} and L_{-2} (and hence all L_{-n}) as in the usual gauge-covariant approach. Choosing $|\psi\rangle$ itself as gauge parameter, we find that the action functional admits "global" symmetries

$$\delta|\psi\rangle = \mathcal{S}|\psi\rangle \tag{7.27}$$

(or $\hat{\mathcal{S}}$ instead pf $\mathcal{S}$), replacing the freedom of reparametrizations on the world sheet

$$\delta|\psi\rangle = L_{-n}|\psi\rangle \tag{7.28}$$

in the gauge-covariant formulations. The physical interpretation of these symmetries is not yet clarified.

We finally remark that the redundancy in the gauge transformation law (7.2) resp. (7.24) gives rise to "ghosts for ghosts" in the BRST quantization formalism, a feature which is well-known for string theories.

8 The Residual Gauge Freedom

Once having specified the Virasoro gauge condition for a solution to (4.3), the remaining gauge freedom is contained in the set of all states

$$|\psi\rangle \equiv \sum_{p=1}^{\infty} \mathcal{S}_{-p}|\chi_p\rangle \in \mathbf{V}^{(0)} \tag{8.1}$$

where $|\chi_p\rangle \in \mathbf{V}^{(0)}$. Considering only a certain mass level n, the series (8.1) stops at $p = n$. Applying $\mathcal{L}_i^{(n)}$, one finds

$$\mathcal{M}_{ij}^{(n)}(L_0)a_{(n)}^{j}(L_0)|\chi_n\rangle = 0, \tag{8.2}$$

thus - using (5.23) -

$$q^{(n)}(L_0)|\chi_n\rangle = 0, \tag{8.3}$$

which implies

$$q^{(n)}(L_0 - n)\mathcal{S}_{-n}|\chi_n\rangle = 0. \tag{8.4}$$

Equation (8.3) is necessary but not sufficient for $|\chi_n\rangle$ to be contained in a state of the type (8.1). An example of such a state is given by the equations (4.24) and (4.25). The polynomials $q^{(n)}(h)$ are contained as factors in the Kac-determinants (this follows from (5.25)). Thus, only part of the zeros of det $\mathcal{M}^{(n)}$ is actually responsible for the existence of a residual gauge symmetry.

This occurrence of a non-physical left-over in the covariant approach to string theories is not only known in principle but has also been worked out in detail: A general physical state is the sum of a DDF-state - carrying the true physical degrees of freedom - and a physical spurious state [2].

9 Closed Strings

In the theory of oriented closed strings, the number of degrees of freedom is doubled as compared to the open strings. Due to the existence of two independent sets of oscillator variables α_n^μ, $\bar{\alpha}_n^\mu$ (subject to the single relation $p^\mu = \alpha_0^\mu = \bar{\alpha}_0^\mu$), the constraints are realized by means of two sets of Virasoro operators L_n, $\bar{L}_n$ in the closed-string Fock-space. Different type operators commute with each other. The conditions defining the physical states are

$$(L_0 + \bar{L}_0 - 2)|\psi\rangle = 0 \tag{9.1}$$

$$L_n|\psi\rangle = \bar{L}_n|\psi\rangle = 0 \tag{9.2}$$

for all $n \geq 1$, supplemented by

$$(L_0 - \bar{L}_0)|\psi\rangle = 0 \tag{9.3}$$

which can also be written as

$$(N - \bar{N})|\psi\rangle = 0, \tag{9.4}$$

where N and $\bar{N}$ are the respective mass level operators (cf. Equ. (A.10)). Our task is again to find an action functional which produces the above conditions as field equations in a special gauge. In order to achieve (9.4), we define a projection operator Q onto the set of states satisfying (9.4). Expanding an arbitrary state with respect to the two types of mass level operators,

$$|\psi\rangle = \sum_{n,m=0}^{\infty} |\psi\rangle_{nm}, \tag{9.5}$$

where

$$N|\psi\rangle_{nm} = n|\psi\rangle_{nm}, \qquad \bar{N}|\psi\rangle_{nm} = m|\psi\rangle_{nm}, \tag{9.6}$$

the action of Q is given by

$$Q|\psi\rangle = \sum_{n=0}^{\infty} |\psi\rangle_{nn}. \tag{9.7}$$

Q acts locally on the field components (it just puts some of them to zero and leaves the others unchanged) and is hermitian

$$Q^\dagger = Q. \tag{9.8}$$

Although we have not gone so far into the details of the closed string case, a candidate for an appropriate action is

$$S = -(\psi|\mathcal{H}Q|\psi), \tag{9.9}$$

where

$$\mathcal{H} = L_0 + \bar{L}_0 - 2 - \sum_{n=1}^{\infty} \frac{1}{n} L_{-n} L_n - \sum_{n=1}^{\infty} \frac{1}{n} \bar{L}_{-n} \bar{L}_n, \tag{9.10}$$

$$\mathcal{H}^\dagger = \mathcal{H} \tag{9.11}$$

and

$$[\mathcal{H}, Q] = 0. \tag{9.12}$$

It admits the gauge symmetry

$$\delta|\psi\rangle = (1 - Q)|\Lambda\rangle \equiv \sum_{n,m=0,n\neq m}^{\infty} |\Lambda\rangle_{nm} \tag{9.13}$$

which may be used to achieve(9.4) as gauge-fixing condition. One may, alternatively, omit the Q in (9.9) and insert (9.4) by hand.

Moreover, setting $D = 26$, the action (9.9) is invariant under the two types of symmetries

$$\delta|\psi\rangle = \mathcal{S}_{-m}|\chi\rangle \tag{9.14}$$

and

$$\delta|\psi\rangle = \bar{\mathcal{S}}_{-m}|\chi\rangle \tag{9.15}$$

where

$$L_p|\chi\rangle = \bar{L}_p|\chi\rangle = 0 \tag{9.16}$$

for all $p \geq 1$. In order not to destroy (9.4), one must choose $|\chi\rangle$'s of the correct mass levels, e.g. for (9.14)

$$\delta|\psi\rangle_{nn} = \mathcal{S}_{-m}|\chi\rangle_{n-m,n}. \tag{9.17}$$

A typical state $|\chi\rangle_{0,1}$, subject to (9.16), is of the form

$$|\chi\rangle_{0,1} = -i\bar{\alpha}^{\mu}_{-1}\chi_\mu(x)|0\rangle \tag{9.18}$$

with

$$\partial_\mu \chi^\mu = 0. \tag{9.19}$$

Applied to

$$|\psi\rangle_{1,1} = \alpha^{\mu}_{-1}\bar{\alpha}^{\nu}_{-1}k_{\mu\nu}(x)|0\rangle, \qquad (9.20)$$

it gives rise to the gauge transformation law $\delta|\psi\rangle_{1,1} = L_{-1}|\chi\rangle_{0,1}$ or, written in components,

$$\delta k_{\mu\nu} = \partial_{\mu}\chi_{\nu}. \qquad (9.21)$$

Of course, the role of operators with and without bar may be interchanged in (9.17) and (9.18), leading to the additional gauge symmetry at the mass-level $|\psi\rangle_{1,1}$

$$\delta k_{\mu\nu} = \partial_{\nu}\Lambda_{\mu}, \qquad (9.22)$$

with

$$\partial_{\mu}\Lambda^{\mu} = 0. \qquad (9.23)$$

Decomposing the space-time field $k_{\mu\nu}(x)$ contained in (9.20) into its symmetric traceless, pure trace and antisymmetric parts, the transformation laws (9.21) and (9.22) split into the symmetries of the well-known graviton, dilaton and antisymmetric tensor field at the first excited mass level of the closed string.

Let us now examine whether the gauge symmetries (9.14) and (9.15) are enough to admit the Virasoro constraints (9.2) as a special gauge condition and the mass-shell condition (9.1) as the remainder of the field equation

$$\mathcal{H}|\psi\rangle = 0 \qquad (9.24)$$

in that special gauge. In the following we assume that the symmetry (9.13) has already been exploited and (9.4) is valid from the outset.

At the mass level $|\psi\rangle_{0,0}$ (the tachyon field) there is nothing to prove, because the physical state condition follows immediately. Next consider the states $|\psi\rangle_{1,1}$, as given by (9.20). Trying to achieve (9.2) for $n = 1$, one sets

$$\begin{aligned} \delta(L_1|\psi\rangle_{1,1}) &\equiv L_1 S_{-1}|\chi\rangle_{0,1} \equiv 2L_0|\chi\rangle_{0,1} = -L_1|\psi\rangle_{1,1} \\ \delta(\bar{L}_1|\psi\rangle_{1,1}) &\equiv \bar{L}_1\bar{S}_{-1}|\Lambda\rangle_{1,0} \equiv 2\bar{L}_0|\Lambda\rangle_{1,0} = -\bar{L}_1|\psi\rangle_{1,1}. \end{aligned} \qquad (9.25)$$

These equations admit a solution for $|\chi\rangle$ and $|\Lambda\rangle$ if and only if the integrability conditions

$$L_1\bar{L}_1|\psi\rangle_{1,1} = 0, \qquad (9.26)$$

or in terms of the component field,

$$\partial^{\mu\nu}k_{\mu\nu} = 0, \qquad (9.27)$$

are valid. Applying L_1 or $\bar{L}_1$ upon (9.24), one gets

$$L_{-1}L_1\bar{L}_1|\psi\rangle_{1,1} = \bar{L}_{-1}L_1\bar{L}_1|\psi\rangle_{1,1} = 0, \tag{9.28}$$

or equivalently

$$\partial_{\rho\mu\nu}k^{\mu\nu} = 0, \tag{9.29}$$

stating that $\partial^{\mu\nu}k_{\mu\nu}$ is not strictly zero but only a constant, and this is the best one can do locally. In order to deduce (9.27) from (9.29), one has to impose an appropriate boundary condition, e.g. that $k_{\mu\nu}$ vanishes at spatial infinity (no matter how fast).This is in contrast to the open string case, where the existence of the gauge parameters necessary to achieve the Virasoro constraints was guaranteed by means of pure differential manipulations alone. The reason for this new feature is that the two types of Virasoro operators on y combine into equations like (9.28), which are absent in the open string case. However, once having imposed the necessary boundary condition, the correct gauge fixing is possible.

At the second mass level $|\psi\rangle_{2,2}$, we meet an analogous situation. The integrability conditions necessary to achieve

$$L_2|\psi\rangle_{2,2} = \bar{L}_2|\psi\rangle_{2,2} = 0 \tag{9.30}$$

turn out to be

$$\bar{L}_1L_2|\psi\rangle_{2,2} = \bar{L}_2L_1|\psi\rangle_{2,2} = \bar{L}_2L_2|\psi\rangle_{2,2} = 0. \tag{9.31}$$

Expanding

$$\begin{aligned} \bar{L}_1L_2|\psi\rangle_{2,2} &= \bar{\alpha}^{\mu}_{-1}a_\mu|0\rangle \\ \bar{L}_2L_1|\psi\rangle_{2,2} &= \alpha^{\mu}_{-1}\bar{a}_\mu|0\rangle \\ \bar{L}_2L_2|\psi\rangle_{2,2} &= b|0\rangle, \end{aligned} \tag{9.32}$$

one concludes from the field equation (9 24) by pure differential manipulations that the combinations $a_\mu + \frac{1}{3}\partial_\mu b$ and $\bar{a}_\mu + \frac{1}{3}\partial_\mu b$ are constant. Again upon appropriate boundary conditions, these constants have to be zero. Assuming $a_\mu = \bar{a}_\mu = -\frac{1}{3}\partial_\mu b$, one concludes by further exploitation of (9.24) that $a_\mu = \bar{a}_\mu = b = 0$. Hence, the gauge condition given by (9.31) is possible. In order to obtain the remaining Virasoro constraints

$$L_1|\psi\rangle_{2,2} = \bar{L}_1|\psi\rangle_{2,2} = 0, \tag{9.33}$$

we have to deduce the integrability conditions

$$L_1^2|\psi\rangle_{2,2} = \bar{L}_1^2|\psi\rangle_{2,2} = L_1\bar{L}_1|\psi\rangle_{2,2} = 0. \tag{9.34}$$

These follow from (9.24), (9.30) and (9.31) without imposing any further boundary condition.

Although we have not checked this mechanism for the higher mass levels, the computations done so far provide a certain evidence that the action (9.9) applies to all space-time fields contained in $|\psi\rangle$. The question whether there is a better one - which works without imposing boundary conditions - remains open.

10 Conclusion

The conjectures made so far in the open string case are summarized by the statement that the matrices $\mathcal{K}^{(n)i}{}_j(h)$ have a one-dimensional kernel for all h. Under the assumption that it holds we have shown that a free theory of open bosonic string fields may be formulated without the use of supplementary fields. The action is local in the string fields and contains space-time derivatives only up to the second order. Let us finally raise the question whether there exist other kinetic operators constructed by the Virasoro operators alone. The most general candidate is

$$\tilde{\mathcal{K}} = 2(L_0 - 1) + \sum_{n=1}^{\infty} \mathcal{L}_{-i}^{(n)} \tilde{k}_{(n)}^{ij}(L_0) \mathcal{L}_j^{(n)} \tag{10.1}$$

with

$$\tilde{k}_{(n)}^{ij} = \tilde{k}_{(n)}^{ji} \tag{10.2}$$

to ensure the hermiticity of $\tilde{\mathcal{K}}$. If $\tilde{\mathcal{K}}$ is to contain at most second order space-time derivatives, the $\tilde{k}_{(n)}^{ij}$ have to be constants, and only the combinations $L_{-n}L_n$ have non-zero coefficients. Thus we are led to the ansatz (4.1) made already in Section 4. In this sense, the solution of the problem as given in (4.22) together with $D = 26$ seems to be the *unique* one.

However, in order to understand the significance of the conjecture in this paper, we may omit the restriction to second-order space-time derivatives and ask for a general criterion for (10.1) to be applicable for string field theory. Following essentially the same arguments as given in Section 6, it turns out that $\tilde{\mathcal{K}}$ admits the Virasoro gauge conditions if and only if the dimension of the kernels

$$\dim \ker \tilde{\mathcal{K}}^{(n)i}{}_j(h) = r_n \tag{10.3}$$

does not depend on h. The matrices $\tilde{\mathcal{K}}^{(n)i}{}_j$ are defined analogously to (4.11). The operator $\mathcal{K}$ suggested in this paper has $r_n = 1$ for all $n \geq 1$. An operator $\tilde{\mathcal{K}}$ with $r_n = 0$ for all $n \geq 1$ would not admit any gauge symmetry at all. However, $r_n = 0$

contradicts (10.1). If, on the other hand, the matrices $\tilde{\mathcal{K}}^{(n)i}{}_j(h)$ were non-singular only for generic h - which is probably the case for (4.1) with $a_n \neq 1/n$ at $n \geq 2$ - one would find zero-norm states corresponding to critical values of h, satisfying the field equations but being neither physical nor gauge. The equality of the dimensions in (10.3) for *all* h is thus necessary to absorb also these exceptional states into the decomposition (4.23).

Let us finally note that the actions defined by the kinetic operators $\mathcal{K}$ and $\mathcal{H}$ agree with those given by Neveu, Nicolai and West [15] when all supplementary fields are set equal to zero. From this point of view, the symmetries provided by the $\mathcal{S}_{-n}$ are possibly just those of Ref. [15] which leave all supplementary fields invariant. Then the variation of the field should as well be given by

$$\delta|\psi\rangle \equiv \sum_{n=1}^{\infty} \mathcal{S}_{-n}|\chi_n\rangle = \sum_{n=1}^{\infty} L_{-n}|\Lambda_n\rangle \tag{10.4}$$

for appropriately *constrained* parameters $|\Lambda_n\rangle$. On the other hand, one should be able to re-introduce the supplementary fields into the scheme presented here by relaxing the constraints $|\chi_n\rangle \in \mathbf{V}^{(0)}$ in (10.4).

However, the question to what extent it is necessary or convenient restoring the supplementary fields is to be answered at the level of interacting string fields.

A Appendix: Fock-Space, Physical States and Virasoro Operators

We use units in which $c = \hbar = 2\alpha' = 1$. The dynamics of a bosonic open string propagating in D-dimensional Minkowski space with metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,\ldots 1)$ is described by a set of oscillator variables $\alpha_n{}^\mu$ ($n \neq 0$ and integer), the center of mass coordinate x^μ and the total momentum $p^\mu \equiv \alpha_0{}^\mu$. Quantization along the lines of the old covariant approach [1,2] gives rise to the operator algebra

$$\begin{aligned} [\alpha_n{}^\mu, \alpha_m{}^\nu] &= \eta^{\mu\nu} n \delta_{n+m,0} \\ [x^\mu, p_\nu] &= i\delta^\mu_\nu \\ [x^\mu, \alpha_n{}^\nu] &= 0 \qquad \text{for } n \neq 0, \end{aligned} \tag{A.1}$$

and the hermiticity conditions

$$\begin{aligned} (\alpha_n{}^\mu)^\dagger &= \alpha_{-n}{}^\mu \\ (x^\mu)^\dagger &= x^\mu. \end{aligned} \tag{A.2}$$

In the usual Fock representation the above algebra is realized by postulating a formal vacuum state $|0\rangle$, subject to

$$\alpha_n{}^{\mu}|0\rangle = 0 \qquad (A.3)$$

for $n \geq 0$. Since $n = 0$ is included in this condition, some authors would refer to $|0\rangle$ as the "vacuum with respect to momentum $p^{\mu} = 0$". Acting upon $|0\rangle$ by products of the operators $i\alpha_{-n}{}^{\mu}$ ($n \geqslant 1$) and forming linear combinations with real x-dependent coefficients, one arrives at the total Fock-space **F**. The momentum operator is represented by

$$p_{\mu} = -i\frac{\partial}{\partial x^{\mu}} \equiv -i\partial_{\mu}. \qquad (A.4)$$

the action of the $\alpha_n{}^{\mu}$ ($n \geq 1$) is evaluated by means of (A.1) and (A.3). Thus the general element of **F** has the form

$$|\psi\rangle = (\phi(x) - iA_{\mu}(x)\alpha_{-1}{}^{\mu} - iv_{\mu}(x)\alpha_{-2}{}^{\mu} - \frac{1}{2}h_{\mu\nu}(x)\alpha_{-1}{}^{\mu}\alpha_{-1}{}^{\nu} + \cdots)|0\rangle \qquad (A.5)$$

where $h_{\mu\nu} = h_{\nu\mu}$ and the dots denote higher order products of the α's. As far as the open string Fock-space is concerned, it will never be necessary to impose boundary conditions on the component fields ϕ, A_{μ} etc. We merely assume them to be sufficiently differentiable. Formal adjunction leads to

$$\langle\psi| = \langle 0|(\phi(x) + iA_{\mu}(x)\alpha_1{}^{\mu} + +iv_{\mu}(x)\alpha_2{}^{\mu} - \frac{1}{2}h_{\mu\nu}(x)\alpha_1{}^{\mu}\alpha_1{}^{\nu} + \cdots) \qquad (A.6)$$

and

$$\langle\psi|\psi'\rangle = \phi(x)\phi'(x) + A^{\mu}(x)A'_{\mu}(x) + 2v^{\mu}(x)v'_{\mu}(x) + \frac{1}{2}h^{\mu\nu}(x)h'_{\mu\nu}(x) + \cdots \qquad (A.7)$$

where

$$\langle 0|0\rangle = 1 \qquad (A.8)$$

has been used. Note that the formal "scalar product" (A.7) still depends on x^{μ}. In order to have a suitable notation for action functionals, we define

$$(\psi|\psi') = \int d^D x\langle\psi|\psi'\rangle \qquad (A.9)$$

which will in general not be finite.

The mass level operator is defined by

$$N = \sum_{n=1}^{\infty}\alpha_{-n}{}^{\mu}\alpha_{n\mu}. \qquad (A.10)$$

It splits $\mathbf{F}$ into a direct sum such that any $|\psi\rangle \in \mathbf{F}$ may be decomposed uniquely as

$$|\psi\rangle = \sum_{n=0}^{\infty} |\psi\rangle_n \tag{A.11}$$

where

$$N|\psi\rangle_n = n|\psi\rangle_n. \tag{A.12}$$

Any contribution to $|\psi\rangle$ of the form

$$\alpha_{-n_1}{}^{\mu_1} \cdots \alpha_{-n_k}{}^{\mu_k} A_{\mu_1 \cdots \mu_k}(x)|0\rangle$$

has a definite value of N, namely $\sum_{i=1}^{k} n_i$. Note that (A.11) is a formal sequence involving infintely many space-time fields, rather than an actual sum. Thus, any operator whose action upon any definite mass level is finite, is well-defined in the total Fock space. As a consequence, we will never run into convergence problems.

The Virasoro operators are defined by

$$L_n = \frac{1}{2} : \sum_{m=-\infty}^{\infty} \alpha_{n-m}{}^{\mu} \alpha_{m\mu} : = (L_{-n})^{\dagger} \tag{A.13}$$

where the normal ordering with respect to $\alpha_n{}^{\mu}$ $(n \geq 1)$ as annihilation operator is only necessary for L_0, leading to

$$L_0 = \frac{1}{2} p^{\mu} p_{\mu} + N \equiv -\frac{1}{2} \Box + N \tag{A.14}$$

where

$$\Box = \partial^{\mu}\partial_{\mu} = -\partial_{00} + \sum_{i=1}^{D-1} \partial_{ii}. \tag{A.15}$$

The Virasoro operators constitute the quantum version of the classical constraints. They select the physical states by a set of Gupta-Bleuler type conditions (the so-called Virasoro conditions)

$$L_n|\psi\rangle = 0 \tag{A.16}$$

for all $n \geq 1$ and

$$(L_0 - 1)|\psi\rangle = 0. \tag{A.17}$$

As well-known [1,2], the absence of ghosts and a consistent theory of interacting strings is only guaranteed in the critical dimension $D = 26$.

The set of states satisfying only (A.16) is denoted by $\mathbf{V}^{(0)}$. They are also called states of Verma level zero (cf. Ref. [13]). The action of the L_n ($n \geq 1$) upon the lowest mass levels of any state in F is given by

$$L_1|\psi\rangle_1 = -\partial_\mu A^\mu|0\rangle \tag{A.18}$$

$$\begin{aligned} L_1|\psi\rangle_2 &= -2i(v_\mu - \tfrac{1}{2}\partial^\rho h_{\rho\mu})\alpha_{-1}{}^\mu|0\rangle \\ L_2|\psi\rangle_2 &= -2(\partial_\mu v^\mu + \tfrac{1}{4}h^\mu{}_\mu)|0\rangle, \end{aligned} \tag{A.19}$$

$|\psi\rangle_0$ being automatically subject to (A.16), hence an element of $\mathbf{V}^{(0)}$.

The mass-shell condition (A.17) reads, for the n-th mass level

$$(\Box - 2(n-1))|\psi\rangle_n = 0, \tag{A.20}$$

thus giving rise to a squared mass

$$m_n^2 = 2(n-1) \tag{A.21}$$

for the fields at this level. ϕ is the well-known tachyon and A_μ becomes a massless vector field.

The L_n's satisfy the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12}n(n^2-1)\delta_{n+m,0} \tag{A.22}$$

including the anomalous term which is responsible for the crucial role played by the space-time dimension D.

It is convenient to denote the set of products $L_{n_1}\cdots L_{n_k}$ with $1 \leq n_1 \leq \cdots \leq n_k$ and $\sum_{i=1}^k n_i = n$ collectively by $\mathcal{L}_i^{(n)}$ where the index i ranges from 1 to a certain number ℓ_n. They altogether form a so-called "universal envelopping algebra" (cf. Ref. [12]). The adjoint operators are denoted as

$$\mathcal{L}_{-i}^{(n)} = (\mathcal{L}_i^{(n)})^\dagger \tag{A.23}$$

and constitute an anlogous set of products of Virasoro operators with negative indices. According to the relation

$$[N, \mathcal{L}_{\pm i}^{(n)}] = \mp n\mathcal{L}_{\pm i}^{(n)} \tag{A.24}$$

the $\mathcal{L}_{\pm i}^{(n)}$ change the mass level number by $\mp n$. We adopt the Einstein summation convention for indices of the type i.

As an example, we write down the operators $\mathcal{L}_i^{(3)}$

$$\begin{aligned} \mathcal{L}_1^{(3)} &= L_3 \\ \mathcal{L}_2^{(3)} &= L_1 L_2 \\ \mathcal{L}_3^{(3)} &= L_1^3, \end{aligned} \tag{A.25}$$

and quote the lowest values of ℓ_n

$$(\ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \cdots) = (1, 1, 2, 3, 5, 7, 11, \cdots)$$

where the convention $\ell_0 = 1$ is useful in some formulae.

Given a state $|\chi\rangle$ of Verma level zero which is at the same time an eigenvector of L_0 to a real eigenvector h (a "highest weight vacuum vector"),

$$L_0|\chi\rangle = h|\chi\rangle, \tag{A.26}$$

the linear space spanned by all states of the form $\mathcal{L}_{-i}^{(n)}|\chi\rangle$ is called a Verma-module [24]. By making use of the Virasoro algebra (A.22), one finds

$$\mathcal{L}_i^{(n)} \mathcal{L}_{-j}^{(n)}|\chi\rangle = \mathcal{M}_{ij}^{(n)}(h)|\chi\rangle \tag{A.27}$$

where $\mathcal{M}_{ij}^{(n)}$ is an $\ell_n \times \ell_n$ matrix of polynomials in h (the so-called Shapovalov-matrix). It is generically invertible [25], by which is meant that its determinant (the Kac-determinant) is a nontrivial polynomial and hence admits only a finite number of zeros. Given a value h for which all matrices $\mathcal{M}_{ij}^{(n)}(h)$ are non-singular, the whole Verma-module splits into a direct sum distinguished by the Verma level number n. For the purpose of the work presented here it is convenient to collect all states generated by the action of the $\mathcal{L}_{-i}^{(n)}$ upon any state of Verma level zero. By $\mathbf{V}^{(n)}$ we denote the set of states $\mathcal{L}_{-i}^{(n)}|\chi^i\rangle$ for which $|\chi^i\rangle \in \mathbf{V}^{(0)}$. Due to the existence of zeros in the Kac-determinants, the intersections $\mathbf{V}^{(m)} \cap \mathbf{V}^{(n)}$ are nontrivial. Let $m < n$ and $|\psi\rangle$ be an element of both $\mathbf{V}^{(m)}$ and $\mathbf{V}^{(n)}$,

$$|\psi\rangle = \mathcal{L}_{-i}^{(n)}|\chi^i\rangle = \mathcal{L}_{-j}^{(m)}|\xi^j\rangle \tag{A.28}$$

with $|\chi^i\rangle$ and $|\xi^j\rangle$ elements of $\mathbf{V}^{(0)}$. Acting upon this equation by $\mathcal{L}_k^{(n)}$, one obtains

$$\mathcal{M}_{ki}^{(n)}(L_0)|\chi^i\rangle = 0, \tag{A.29}$$

thus, upon multiplication by the algebraic adjoint of $\mathcal{M}^{(n)}$,

$$\det \mathcal{M}^{(n)}(L_0)|\chi^i\rangle = 0$$

which implies

$$\det \mathcal{M}^{(n)}(L_0 - n)|\psi\rangle = 0. \qquad (A.30)$$

This illustrates that the elements of the intersections $\mathbf{V}^{(n)} \cap \mathbf{V}^{(m)}$ are related to the zeros of the Kac-determinants. One may show further from (A.28) that $|\psi\rangle$ has zero norm. One is sometimes interested in computations involving "generic" states which do not satisfy equations like (A.30). In these cases one may formally compute the inverse matrices

$$\mathcal{M}^{(n)}_{ij}(L_0)^{-1} = \mathcal{M}^{ij}_{(n)}(L_0) \qquad (A.31)$$

whose components are rational functions of L_0. They have been used to construct the projector P onto $\mathbf{V}^{(0)}$ (cf. Refs. [11,13]).

Let $\mathbf{W}^{(n)}$ denote the set of all states $|\psi\rangle$ in $\mathbf{F}$ which are annihilated by the operators $\mathcal{L}^{(n+1)}_i$. Any such element may be written as a sum of lower level states,

$$|\psi\rangle = |\chi_0\rangle + \sum_{p=1}^{n} \mathcal{L}^{(p)}_{-i}|\chi^i_p\rangle, \qquad (A.32)$$

where all $|\chi\rangle$'s are in $\mathbf{V}^{(0)}$. To prove this, one just has to interpret (A.32) as a set of differential equations for the $|\chi\rangle$'s. After multiplication by $\mathcal{L}^{(n)}_k$, one obtains an equation in $\mathbf{V}^{(0)}$ for $|\chi^i_n\rangle$ which - using the fact that $\det \mathcal{M}^{(n)}(L_0)$ is a nontrivial polynomial - admits a solution. The rest follows by induction. Since the $\mathbf{V}^{(p)}$ do not form a direct product, the decomposition (A.32) is not unique, the arbitrariness again being connected with the zeros of the Kac-determinants.

Applying (A.32) separately to the different mass level contributions of a given state $|\psi\rangle$, and using $|\psi\rangle_n \in \mathbf{W}^{(n)}$, one finds that any element of $\mathbf{F}$ has the form

$$|\psi\rangle = |\chi_0\rangle + \sum_{p=1}^{\infty} \mathcal{L}^{(p)}_{-i}|\chi^i_p\rangle \qquad (A.33)$$

with all $|\chi\rangle$'s in $\mathbf{V}^{(0)}$. Usually, the states given by the second expression in (A.33) are called spurious. From the point of view of gauge-covariant string field theory, these states carry only gauge degrees of freedom (cf. Equ. (1.4)). By virtue of the field equations (A.16), (A.17), they are excluded except for certain zero-norm states of the type (A.30). Their occurrence - and thus the ambiguity of the decomposition (A.33) - reflects the fact that upon imposing covariant gauge conditions like (A.16) there is still some gauge freedom left.

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