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QUALITATIVE AND NUMERICAL STUDY OF BIANCHI IX MODELS

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The qualitative behaviour of trajectories in the Mixmaster universe is studied. The Lyapunov exponents computed directly from the differential equations and from the Poincare map are shown to be different. A detailed discussion of the role of these exponents in analysing the effect of chaos on trajectories is presented.

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1. INTRODUCTION

In the work of Belinskii, Khalatnikov and Lifshitz, or BKL for short, (see[1-3]) the phenomenon of stochasticity was for the first time detected in Einstein's equations. The homogeneous Bianchi IX model, or its diagonal version called the Mixmaster universe, were the prime examples of chaoticity. Subsequent studies on the consequences of chaotic behaviour in general relativity and cosmology were made by a number of authors ([4], [5] and references therein, [6], [7]). The parameters that characterize chaotic behaviour in dynamical systems theory are the Lyapunov exponents. They measure the exponential rate of divergence of trajectories in phase space and there are, in principle, as many exponents as the dimension of phase space. It happens that when the Mixmaster universe is evolved towards the singularity only one parameter is necessary to explain the qualitative and chaotic behaviour. Thus a one dimensional Poincare map can be introduced to describe the discrete dynamics of the model. In the qualitative treatment of Bogoyavleskii and Novikov [8,9,10] this fact is rigorously established since the system can be shown to move towards an attractor that contains three circles of unstable equilibrium points.

One of the aims of this paper is to explore the fact that the Lyapunov exponent computed directly from the equations of motion will not coincide with that obtained from the Poincare map since this map defines a discrete evolution without taking into account the time elapsed during the motion. On approach to the singularity divergence of trajectories will always occur in this model, whether the evolution is discrete or not giving a positive exponent, but the time spent between effective deviations of the orbit tend to become unbounded as mentioned in § 2. Since the equations of motion give the rate of divergence in time, the Lyapunov exponent calculated from them will undergo a steady decrease as discussed in § 3. These results are interpreted in the light of Shaw's studies on the characteristic time (or either the number of iterations by the Poincare map) necessary to loose information about initial conditions. We also examine how the Lyapunov exponent is affected when Kasner initial conditions are given closer to the singularity.

We assume that spacetime can be split as $M=\Sigma\times\mathbb{R}$ where Σ is a 3-D compact manifold without boundary. The line element on M is given by $ds^2= g_{\mu\nu}(\underline{x},t)\, dx^\mu \otimes dx^\nu$, $\underline{x}\in\Sigma$, $t\in\mathbb{R}$, and assumed to be written in a synchronous frame where $g_{00}=1$, $g_{0i}=0$, i = 1,2,3. The approach to the cosmological singularity occurs whenever $\det g_{ij}=0$.

2. HOMOGENEOUS DYNAMICS AND NUMERICAL SIMULATION

We start from Einstein's field equations [12]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.1}$$

where the speed of light is $c = 1$ and G the gravitational constant. For a perfect relativistic fluid the energy-momentum tensor has the form

$$T_{\mu\nu} = -p g_{\mu\nu} + (p+\rho) U_\mu U_\nu \tag{2.2}$$

where p is the pressure, ρ the energy density and U_μ the velocity of the fluid in a comoving frame. The presence of (2.1) will not affect the main results of this paper but when driving noise terms associated to an effective energy-momemtum tensor are taken into account the chaotic behaviour of the model may be suppressed (see Zardecki [7]). In what follows we will assume that $T_{\mu\nu} = 0$.

In a synchronous coordinate frame the spacetime interval is written as

$$ds^2 = dt^2 - g_{ij}(\underline{x}, t)\, dx^i \otimes dx^j . \tag{2.3}$$

For homogeneous models one writes the spatial metrics as

$$g_{ij}(\underline{x}, t) = \eta_{ab}(t)\, e_i^a(\underline{x})\, e_j^b(\underline{x}) \tag{2.4}$$

where η_{ab} has no spatial dependence and $\omega^a = e^a_i dx^i$ is a basis of invariant one-forms determined by the homogneity group of the model. Types VIII and IX have SO(3) symmetry while type I (Kasner) is the simplest model with group $\mathbb{R}^3$. (see Ryan [13] for a discussion of all Bianchi types). Here we restrict ourselves to the case of SO(3) invariance and diagonal type IX, or <u>Mixmaster</u>, metric given by

$$\eta_{ab}(t) = \begin{pmatrix} a^2(t) & & \\ & b^2(t) & \\ & & c^2(t) \end{pmatrix} \tag{2.5}$$

and line element

$$ds^2 = dt^2 - \eta_{ab}\, \omega^a \otimes \omega^b .$$

Nondiagonal type IX models will present the same qualitative behaviour.

The spatial components of (2.1) give the evolution equations [12]

$$\begin{aligned} (\ln a^2)'' &= (b^2 - c^2)^2 - a^4 . \\ (\ln b^2)'' &= (a^2 - c^2)^2 - b^4 \\ (\ln c^2)'' &= (a^2 - b^2)^2 - c^4 \end{aligned} \tag{2.6}$$

where the prime denotes derivatives with respect to time τ given by

$$d\tau = (abc)^{-1}\, dt . \tag{2.7}$$

if we substitute (2.6) into the temporal component R_{00} of (2.1) given by $\frac{1}{2}(\ln a + \ln b + \ln c)'' = (\ln a)'(\ln b)' + (\ln a)'(\ln c)' + (\ln b)'(\ln c)'$

we arrive at

$$(\ln a)'(\ln b)'+(\ln a)'(\ln c)'+(\ln b)'(\ln c)'=\frac{1}{4}(a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2). \tag{2.8}$$

In order to study the evolution (2.6) we introduce the notation

$$x=(\ln a^2)' \qquad y=(\ln b^2)' \qquad z=(\ln c^2)' \tag{2.9}$$

and write the evolution equations as a first order system

$$\begin{aligned} x' &= (b^2-c^2)^2-a^4 & a' &= ax/2 \\ y' &= (a^2-c^2)^2-b^4 & b' &= by/2 \\ z' &= (a^2-b^2)-c^4 & c' &= cz/2 . \end{aligned} \tag{2.10}$$

Equation (2.8) becomes

$$xy+xz+yz-a^4-b^4-c^4+2a^2b^2+2a^2c^2+2b^2c^2=0. \tag{2.11}$$

We can now use standard methods of numerical analysis to find the behaviour of trajectories that evolve according to (2.10). Whenever the initial conditions are given so as to satisfy (2.11) then such relation is preserved during the evolution (see Zardecki [7] for a presentation of different kinds of initial conditions for this system). We chose the Hamming predictor-corrector algorithm together with a 4 th - order Runge-Kutta as a starter [14]. Several sets of initial conditions were run on a VAX 11/780 and on a PC equipped whith a floating point coprocessor. The qualitative results obtained in these computers were identical and since a PC is accessible to everyone we here report only on calculations performed on it.

The system was iterated typically 2.10^5 times while the time step chosen was rouglhly 5.10^{-4} (double precision always). Kasner initial conditions at $\tau = \tau_0$ were imposed:

$$a = \exp(\tau_0 p_1) \qquad b = \exp(\tau_0 p_2) \qquad c = \exp(\tau_0 p_3) \tag{2.12}$$

where the numbers p_1 , p_2 , p_3 satisfy [12]

$$\sum_{i=1}^{3} p_i = 1 = \sum_{i=1}^{3} p_i^2 \tag{2.13}$$

and thus parametric expressions can be introduced :

$$p_1(u) = \frac{-u}{1+u+u^2} \qquad p_2(u) = \frac{1+u}{1+u+u^2} \qquad p_3(u) = \frac{u(1+u)}{1+u+u^2} , \tag{2.14}$$

with $u > 1$ [12]. Kasner initial conditions are chosen because they are intimately related to type IX dynamics as will be seen bellow. From FIG.1 we see that the evolution towards the singularity, $\tau \to -\infty$, (or $t \to 0$) is comprised of a complicated pattern of oscillations (the same pattern is also found when other classes of initial conditions are imposed [7]). The typical behaviour is composed of a series of <u>Kasner eras</u> in which two components of the metric oscillate while the third decreases. Another era starts when the decreasing component develops oscillatory motion and there is an infinite succession of eras on approach to the singularity. The most important feature of this mechanism is the shift from one era to another and we will call it a <u>bounce</u> [6]. Notice that the evolution of the metric can be decomposed into a series of turning points where the trajectories change direction: increasing scales invert and become decreasing & vice versa. Between consecutive turning points the type IX behaviour can be approximated by a Kasner metric

$$\eta_{ab}(t) = \begin{pmatrix} t^{2p_l(u)} & & \\ & t^{2p_m(u)} & \\ & & t^{2p_n(u)} \end{pmatrix} \tag{2.15}$$

for some $u > 1$, where l, m, n is a permutation of 1, 2,3 and, from (2.7), (2.12), (2.13), $\tau = \ln t$ is the logarithmic time. Thus we arrive at the well known BKL approximation : given a Kasner initial condition the subsequent evolution is given by a certain series of Kasner configurations. In terms of the Kasner parameter the evolution is thus encoded in a sequence

$$u_0 \longrightarrow u_1 \longrightarrow \cdots \longrightarrow u_k \longrightarrow \cdots$$

One can summarize this by writing

$$u_{k+1} = S\, u_k \tag{2.16}$$

where S plays the role of a discrete evolution operator. A detailed discussion of this map can be found in Barrow [5] and also in [2,3]. Here we only mention that there is a special set of transitions contained in (2.16) that are responsible for the chaotic behaviour of the model : they correspond to the bounce already mentioned above. By using the variable $x = 1/u$ these transformations can be expressed by

$$\begin{aligned} T : (0,1) &\longrightarrow (0,1) \qquad\qquad k \in Z^{+} \\ x &\longmapsto \frac{1}{x} - k \;, \quad \frac{1}{k} \leq x \leq \frac{1}{k-1} \end{aligned} \tag{2.17}$$

where from now on T will be called Poincare map for the Mixmaster universe. We call either x or u the Kasner parameter.

A better setting for investigating this model is the qualitative treatment of Bogoyavlenskii and Novikov [8, 9]. Their work allow us to use current terminology of dynamical systems theory when discussing the evolution towards the singularity. The main point here is to notice that the singular region can be defined in phase space in terms of the field variables as $\det \eta_{ab} = 0$ i.e.

$$abc = 0 \tag{2.18}$$

This condition is implicit in (2.15) since $\det \eta_{ab} \to 0$ as $t \to 0$, or $\tau \to -\infty$. Since the physical variables cannot satisfy (2.18) one has to attach to the physical phase space a boundary Γ defined precisely by $abc = 0$ that is, we identify Γ with the singularity. The next step is to extend the equations of motion to Γ and to use the theorem on continuity of solutions with respect to initial conditions to approximate the evolution near the singularity (since $\dim\Gamma$ is always less than the dimension of the physical space, the equations on Γ are simpler). To implement this program one has to change the scale of time and make a change of phase space coordinates in order to remove a highly degenerate equilibrium point in the boundary. This change of coordinates serves to amplify, like a microscope, the fine details of the dynamics near the equilibrium (see Arnold [15], p.9, on resolution of degenerate points). In our simulations this also amplified the numerical errors rendering the system in this parametrization far more difficult to handle than (2.10). From a mathematical point of view one can show that, in the direction of contraction i.e. $\det \eta_{ab} \to 0$, the system converges to an attractor contained in the boundary which is composed of three circles of unstable equilibrium points and the separatrices joining them. Thus a generic trajectory necessarily falls close to some circle and then its motion is approximated by a sequence of separatrices (NOTE 1). One can prove that a trajectory close to any of these separatrices has the form (2.15) and so we recover the notion of evolution as a sequence of Kasner configurations. The motion back and forth between some pair of circles corresponds to the small oscillations of a pair of metric components similar to FIG.1. Eventually this motion is shifted to the oscillation between some other pair of circles and this corresponds to a bounce. The Kasner parameter is here defined on the circles and and transitions (2.17) can be recovered [9]. This provides a rigorous justification for the fact that the Poincare map for this system is one dimensional. Finally it is important to remark that on the separatrix it will take an infinite time to go from a circle to another (recall the example of the pendulum in NOTE 1). Consequently the attractor of this system has a distinctive feature since the closer we are from the singularity the longer the metric stays in a given Kasner configuration or, in other words, the longer it takes for trajectories to be diverted from a given direction.

3. EFFECT OF CHAOS ON TRAJECTORIES

In this section we deal with general nonlinear dynamical systems given by a differential equation

$$\dot{x}(t) = F(x(t)) \tag{3.1}$$

where $x \in \mathbb{R}^m$ is a set of coordinates describing the system and F is a C^2 map. The solution trajectory, or orbit, for (3.1) is denoted by

$$x(t) = f^t(x(0)) \tag{3.2}$$

for a given initial condition x(0). Numerical treatment always involve a discretization of (3.1) and one has to choose a sufficiently small $\bar{t}$ in order to construct a discrete trajectory $x(n\bar{t}) = (f^{\bar{t}})^n(x(0))$ or, recursively,

$$x_{n+1} = f^{\bar{t}}(x_n), \quad x_n = x(n\bar{t}). \tag{3.3}$$

This procedure was used in the last section to obtain FIG. 1 where, typically, $\bar{t} = 5 \cdot 10^{-4}$. Another method to discretize dynamical systems was introduced by Poincare [16]. It consists of a clever way of reducing the dimensionality of the system and at the same time substituting the orbit (3.2) by the iteration of a map. The idea is to place an (m-k)-dimensional, $1 \leq k < m$, cross-section $\sigma \subset \mathbb{R}^m$ so that σ is infinitely intersected by the solution trajectories. The discrete sequence of intersections induces a mapping of σ into itself, the Poincare map. In conclusion, the study of the differentiable dynamical (3.1) can be reduced to the analysis of a discrete-time mapping

$$f : A \subset \mathbb{R}^m \to B \subset \mathbb{R}^m \tag{3.4}$$

where f can either be $f^{\bar{t}}$ as in (3.3) or the Poincare map defined on some cross-section σ .

In the case of a dynamical system given by the discrete map (3.4) let

$$D(x) = d_x f \tag{3.5}$$

be the Jacobian matrix of partial derivatives. Writing

$$D_x^n = D(f^{n-1}(x)) \cdots D(f(x))\, D(x) \tag{3.6}$$

then the largest Lyapunov exponent can be obtained in a computer experiment by implementing the expression [17]

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \ln \| D_x^n v \| \tag{3.7}$$

for almost any vector $v \in \mathbb{R}^n$. Expanding $f(z) \sim f(x) + D(x)(z-x)$ or $v_1 = D(x)\, v$, with $v_1 = f(z) - f(x)$, $v = z - x$, then we see that $D(x)$ is the operator that envolves the separation vector between pairs of trajectories starting at x and z. Formula (3.7) gives an asymptotic measure of the average rate of local exponential divergence of orbits. Systems with $\lambda > 0$ exhibit unstability of nearby trajectories and are called <u>chaotic</u>. Due to unavoidable fluctuations of the initial direction v we always find the largest exponent in computer calculations involving (3.7) (see Eckmann -Ruelle [17]).

Several methods are available to finding numerically the maximal Lyapunov exponent of dynamical systems [17,18,19]. One way to circumvent the overflow due to the increase of the norm $d_n = \| D_x^n v \|$ is to choose a sufficiently small time Δt (typically $\Delta t = .2$) and to redefine d_n every Δt units. The method by Bennetin et al[18] (see also the book [20]) used in our calculations does precisely this and since it computes the L.E. (Lyapunov exponent) per unit time we denote it from now on as λ_t (NOTE 2).

As an example we consider the differential equations studied by Lorenz; $x' = -\sigma x + \sigma y$, $y' = -xz + rx - y$, $z' = xy - bz$, $\sigma = 10$, $b = 8/3$, $r = 28$ (see Lorenz [21]). One finds that asymptotic time evolution of this system gives a L.E. $\lambda_t \sim 1.22$ (NOTE 3). This result coincides with that found by Shaw[11] to within 2% where he used eq. (3.11). He showed that λ_t has a predictive value and below we will explore his interpretation of the L.E. and examine its consequence in the context of the Mixmaster universe.

A characteristic time $\hat{t}$ can be defined for dynamical systems which measures how long one should evolve the system in order that a certain amount of information about the initial conditions be lost. More precisely, one defines this information by $H_{IN} = -\log_2 \Delta$ where Δ is the accuracy given in phase space by the size of a cell containing the initial conditions. For one-dimensional systems one has [11]

$$\hat{t} = \frac{H_{IN}}{\lambda_t} \quad . \tag{3.9a}$$

From an analog computer simulation of the Lorenz differential equations Shaw found that indeed, after envolving a set of initial conditions contained in a cell 1.3×10^{-4} units wide, the trajectories dispersed and lost their identity after an evolution time $t \sim 11$ sec. . Of course we can also study this system by introducing an appropriate cross-section in phase space and finding its Poincare map. The L.E. λ_P for this map will <u>not</u> coincide with the exponent λ_t calculated directly from the differential equations. This is because the map by itself does not take into account the time elapsed between consecutive piercings of the cross-section. Instead of a characteristic time λ_P is related to the number n of iterations of the Poincare map necessary to erase information about the initial conditions. A formula similar to (3.9a) also holds in this case [11]

$$n = \frac{H_{IN}}{\lambda_P} \quad . \tag{3.9b}$$

The attractor of the Mixmaster universe presents some uncommon features and next we intend to discuss how this affects the determination and interpretation of the L.

E.. One knows [5,6] that the Poincare map for this model is given by (2.16) and it remains the problem of determining what is the cross-section σ . In the qualitative theory [9] an obvious choice is the subset of the attractor given by the three circles of equilibrium points. We remark that the usual approaches to this model do not seem to provide us with a clear way of identifying the cross section. Since the Poincare map is one-dimensional we can directly compute the L.E. for T by using the formula [5,11]

$$\lambda_P = \int_0^1 \log_2 |T'(x)| \, \mu(x) \, dx \tag{3.10}$$

where μ is a measure preserved [22] by T . Since [2,5] this measure is $\mu(x) = ((1+x)\ln 2)^{-1}$ one finds $\lambda_P \sim 3.42$. Thus it is necessary $n = H_{IN}/\lambda_P$ iterations of T to erase the information $H_{IN} = -\log_2 \Delta$ about the initial conditions defined with precision Δ . Although these statements do not contain any explicit reference to time it is possible to incorporate in (3.10) the time spent between piercings of the cross-section: following Shaw [11] we write

$$\lambda_t = \int_0^1 \frac{1}{\theta(x)} \log_2 |T'(x)| \, \mu(x) \, dx \tag{3.11}$$

where $\theta(x)$ gives the time between a bounce parametrized by x and the next during the evolution. Our simulations have in fact confirmed the predictions of formula (3.11) since as the singularity is approached $\theta(x)$ becomes unbounded, as discussed at the end for the last section, and consequently the exponent λ_t must undergo a steady decrease. We did not however use (3.11) since T and μ are objects defined on the boundary Γ while θ has clearly to be evaluated away from it. We made direct use of the differential equations (2.6) and the method of Benettin et al to obtain FIG.2. Notice that there is an average stabilization around .63 after about -50 units of τ-time: afterwards λ_t will tend to decrease. Of course this does not mean that chaos is suppressed since λ_t measures the rate of divergence per unit time. An analogous situation could be observed in the well known billiards where the trajectory is diverted after each colision against the boundary wall. If the edges of the billiards were expanding the same effect would be observed. In fact the Mixmaster universe can also be understood as a relativistic point bouncing inside an expanding triangular wall. Refering back to FIG. 2 if we evolve the system up to $\tau \sim -56$ (the stabilization time) then we can make two trajectories starting as close as 3.3×10^{-10} units apart to loose their identity. If we want to evolve the system further towards the singularity, say $\tau << -56$, then formula (3.9) gives only a lower bound for the amount of information that can be erased.

In this paper Kasner initial conditions where used throughout since they are intimately related to the evolution of the model. As discussed in the last section the motion towards the singularity is only approximately given as a sequence of Kasner metrics and this approximation gets we ask what happens to λ_t when the initial condition is given closer to the singularity: our results are reported in FIG. 3 where the evolution is considered up to the stabilization time. The explanation for the increase of λ_t when the distance from the singularity decreases can be found in the fact that in the region closer to the boundary the transitions from one Kasner configuration to another will be more abrupt (the potential in the r.h.s. of(2.6)gets sharper and increases or, in the qualitative language,the deviation of trajectories near the separatrices is more pronounced). The result is that there is a bigger contribution to the rate of divergence expressed by λ_t and this accounts for the increase of the exponent. These facts do not alter what has been said about the interpretation of the L.E.. One has just to state whether the exponent applies to the Poincare map or to time evolution and also one should specify how far from the singularity the motion starts. Then a meaningful estimate is obtained of how long the trajectories have to be evolved in order to become totally distinct from each other.

4. DISCUSSION

The interpretation of the Lyapunov exponent for the Mixmaster universe has been given in terms of Shaw's studies of chaos in one-dimensional dynamical systems. A little known property of this cosmological model has been emphazised: whenever the L.E. measures the rate of divergence in time we observe that its value is subject to a slow decrease but it never reaches zero. A characteristic time can be introduced which gives a lower bound to the amount of information about the initial conditions that can be erased. On the other hand the L.E. computed from the Poincaré map can be understood as if the time evolution has been factored out and one obtains a constant positive value for it.

Since the Mixmaster model is intimately related to the BKL paradigm of a construction of a general solution to Einstein's equations near the singularity [3] one is led to ask whether the positivity of the L.E. has any relevance when the homogeneyty condition is dropped. Briefly this paradigm says that the behaviour of a general nonhomogeneous spatial metric tensor $g_{ij}(\underline{x},t)$ in its journey towards the singularity can be simulated by the following model. At each $\underline{x}$ in a Cauchy hypersurface Σ the metric behaves as a Bianchi IX model. In other words a Bianchi IX universe attached at each $\underline{x}\in\Sigma$ gives a good qualitative approximation to the evolution. Let us suppose for a moment that this paradigm is true for a wide class of solutions to Einstein equations. Since each type IX model has a positive L.E. we see that pairs of trajectories are allowed to diverge in the nonhomogeneous phase space and so they become totally distinct from each other. Thus the possibility arises that Kasner transitions be lost during the evolution. This kind of argument serves as an anticipation to the problems one is likely to face when trying to give a rigorous interpretation to the BKL studies. In the homogeneous case a similar situation is found but there it can shown that the model has a compact attractor in its boundary and Kasner behaviour is guaranteed. Only by showing a similar result in the nonhomogeneous case it is possible to justify the dynamical behaviour implied by BKL paradigm.

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Note 1 (p.9): Separatrices are solution trajectories that converge to an equilibrium point. An elementary example is the pendulum with an infinite number of equilibrium points and separatrices joining then. An initial condition sufficiently close to one separatrix will have an evolution that can be approximated by a sequence of separatrices.

Note 2 (p.13): Bennetin et al [18] have suggested an operational definition to (3.7):

$$\lambda = \frac{1}{N\Delta t} \sum_{n=1}^{N} \ln \frac{\| D(x)\, v_n \|}{\| v_n \|}$$

where $\| v_n \| << 1$ and $N >> 1$.

We start from a vector v_1, and evolve it with the operator D (x) during time Δt (around .2 in our simulations). Then the first term of the summation may be calculated. The second one is calculated similarly by choosing the vector v_2 (with $\|v_2\| = \|v_1\|$) along the direction of $D_x v_1$, and so on.

Note 3 (p. 13): The unit for λ_t is here bit/sec since Shaw uses log base 2.

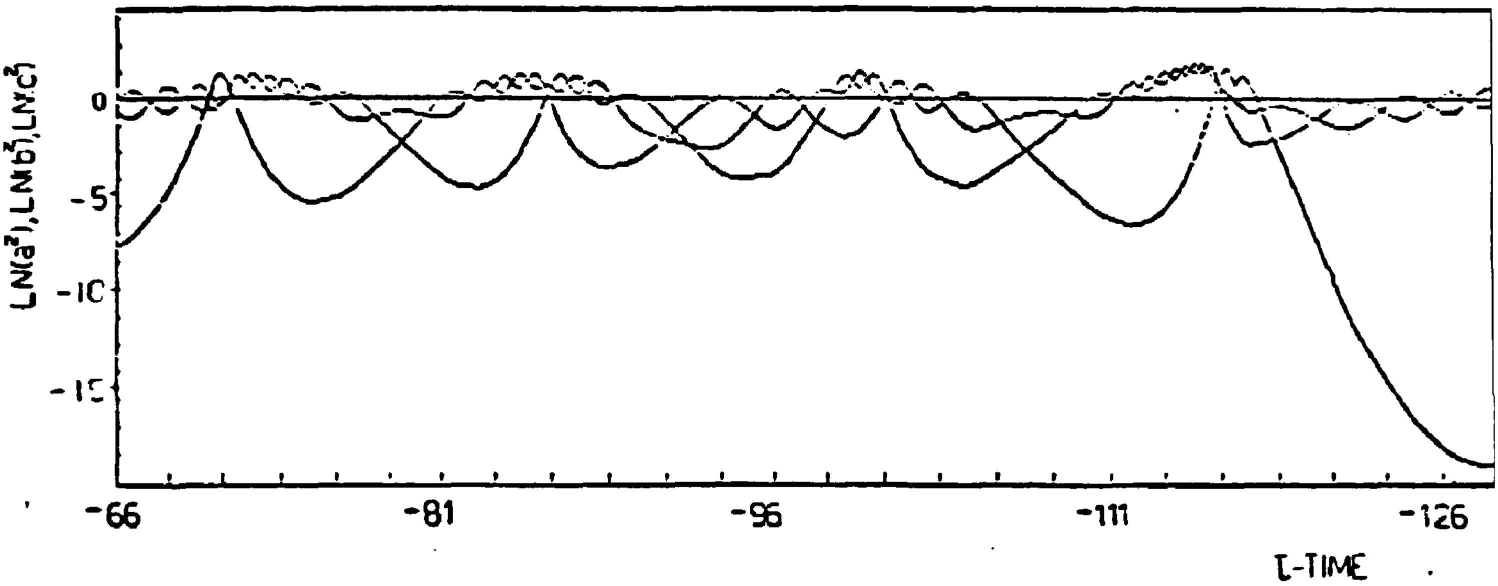

FIG. 1: Evolution of the components $x = \ln a^2$, $y = \ln b^2$, $z = \ln c^2$ from Kasner initial condition given by $u_0 = 8\pi$, $\tau_0 = -6$ We show the portion between $\tau = -66$ and -126. The motion is comprised of eras in which while two components oscillate the third has quasi-monotonic behaviour.

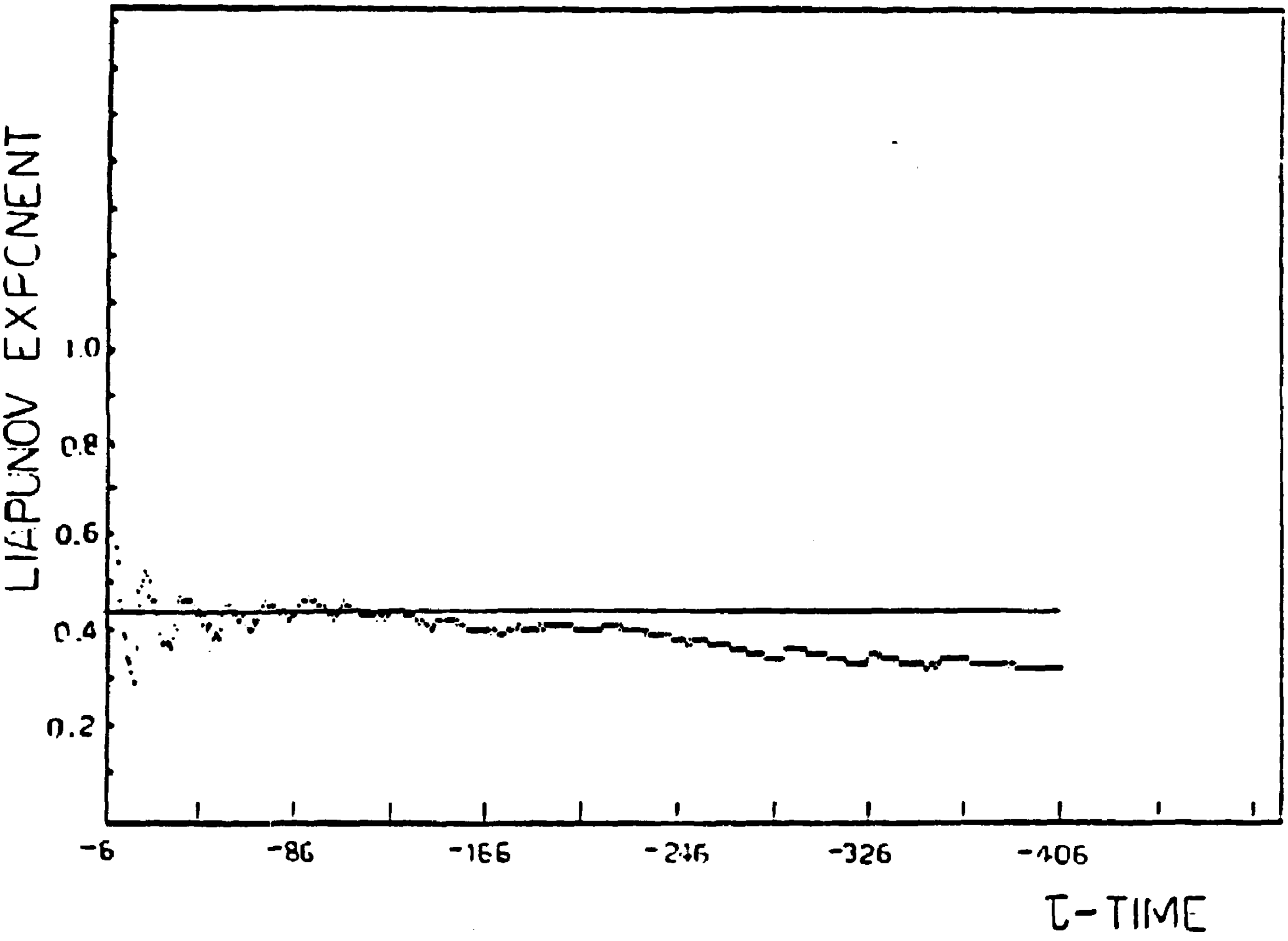

FIG. 2: Lyapunov exponent for a pair of trajectories given by initial conditions $u_1 = 8\pi$ and $u_2 = 8\pi + .001$. After stabilization around $\lambda \sim .44$ (or .63 if log base 2 is used) the exponent will start a steady decrease. A stabilization time $\bar{\tau} = -56$ can be naturally defined for this system.

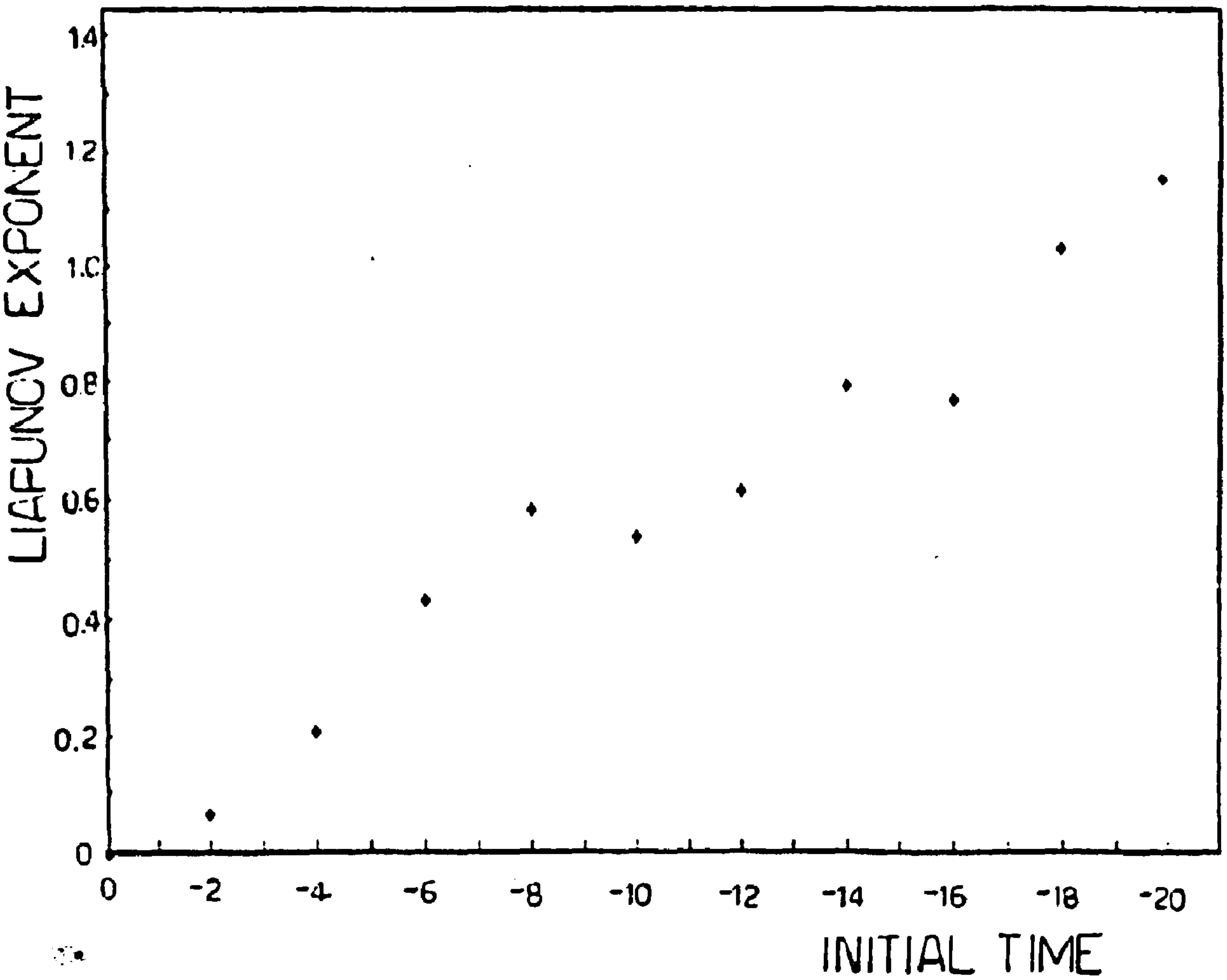

FIG. 3: Lyapunov exponent (natural log) as a function of the initial time τ_0 (2.12). All values were taken at the time of average stabilization after the evolution of 50 units of τ-time from τ_0.