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RESUMO

Um novo modelo de elementos finitos, cuja formulação incorpora automaticamente a busca da direção de upwind mais apropriada para a construção das funções peso do método de Petrov-Galerkin, é apresentado. Mostra-se também que modificando-se essas funções, apenas para os elementos adjacentes as fronteiras onde ocorrem fenômenos típicos de camada limite, consegue-se eliminar eficazmente as oscilações numéricas que normalmente aparecem na vizinhança dessas camadas.

ABSTRACT

A new Petrov-Galerkin Finite Element Model which automatically incorporates the search for the appropriate upwind direction is presented. It is also shown that altering the Petrov-Galerkin weighting functions associated with elements adjacent to downwind boundaries effectively eliminates numerical oscillations normally obtained near boundary layers.

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1. INTRODUCTION

When applied to convection-dominated flow problems standard Galerkin methods generate unstable finite element approximations. To overcome this deficiency many others numerical methods has been proposed. Among them a very successful one is the SUPG derived by Hughes et all [1]. For regular problems this method works well but it presents spurious localized oscillations in regions of high gradients. To circumvent this difficult Hughes et all [2] proposed the DC2 method adding to SUPG a discontinuity-capturing term. In the absence of source terms this method is able to predict boundary layers and presents better results than SUPG. A different situation occurs when non-homogeneous or transient problems are considered. For these problems the method derived by Do Carmo et all [3] presents higher stability properties. Internal and boundary layers are accurately approximated with this new model which reproduces the DC2 when steady-state problems with no source terms are considered. As shown by Galeão et all [4] this method called CAU presents a sistematic procedure to obtain the appropriate upwind direction and associated Petrov-Galerkin weighting function.

In this paper we first review the fundamental aspects of this method. Then we present a procedure to isolate the boundary layer singularity from the regular part of the domain solution. Finally, steady as well as transient test problems are numerically solved in order to demonstrate the performance of these methods.

2. CONVECTION-DIFFUSION PROBLEMS

Let Ω be a bounded region in $\mathbb{R}^n$ ($n \geq 2$) with a piecewise boundary Γ and unit outward normal n. Given a velocity vector field $u(x,t)$; ($x \in \Omega$; $t \in [0,T)$), the mathematical model for the transport problem we are going to consider is described by the *transient advection-diffusion equation*:

$$\dot{\phi} + \underline{u} \cdot \nabla \phi + \mathrm{div}\,(-\underline{\underline{K}} \nabla \phi) = f \qquad \text{in } \Omega\,, \tag{1a}$$

where: $\underline{K}(\underline{x})$ is the diffusivity tensor; the source term $f(\underline{x},t)$ is a specified function of position x and time t, and $\nabla\phi$ is the gradient of the unknown scalar field $\phi(x,t)$. The notations: div ( ) stands for the divergence operator; $(\cdot)$ denotes vector scalar product, and $(\dot{\ })$ is the time derivative.

Dirichlet and Newman boundary conditions are simultaneously considered, and given respectively by:

$$\phi(\underset{\sim}{x},t) = g(\underset{\sim}{x},t) \quad ; \quad \underset{\sim}{x} \in \Gamma_g \tag{1b}$$

$$-\underset{\approx}{K}\nabla\phi\cdot\underset{\sim}{n} = q(\underset{\sim}{x},t) \quad ; \quad \underset{\sim}{x} \in \Gamma_q \tag{1c}$$

where: $\Gamma_g \cap \Gamma_q = \emptyset$ and $\Gamma_g \cup \Gamma_q = \Gamma$ (1d)

Finally at the initial time the condition

$$\phi(\underset{\sim}{x},0) = \phi_0(\underset{\sim}{x}) \quad , \quad \underset{\sim}{x} \in \Omega \tag{1e}$$

completes the *initial boundary value problem*. The function $\phi(x,t)$ which satisfies eqs. (1a-e) is then the classical solution of the convection-diffusion problem under consideration.

3. WEAK FORMULATION

Let us consider now the set S of all kinematically admissible functions, and the space V of admissible variations, which we designate respectively by:

$$S \equiv \{\psi(\underset{\sim}{x},t);\ \text{for each } t \in [0,T):\ \psi \in H_1(\Omega);\ \psi\big|_{\Gamma_g} = g\} \ , \tag{2a}$$

$$V \equiv \{\theta;\ \theta \in H_1(\Omega);\ \theta\big|_{\Gamma_g} \equiv 0\} \ , \tag{2b}$$

where: $H_1(\Omega) \equiv \{\eta;\ \eta \in L_2(\Omega);\ (\underset{\sim}{\nabla}\eta)_i \in L_2(\Omega);\ i=1,2,3\}$, (2c)

and $L_2(\Omega)$ is the well known space of squared-integrable functions with inner product

$$\langle\psi,\theta\rangle_\Omega = \int_\Omega \psi\theta \, d\Omega \ . \tag{2d}$$

Defining:

$$a(\phi,\theta) = \langle\dot{\phi}+\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\phi,\theta\rangle_\Omega + \langle\underset{\approx}{K}\underset{\sim}{\nabla}\phi,\underset{\sim}{\nabla}\theta\rangle_\Omega \tag{3a}$$

$$\ell(\theta) = \langle f,\theta\rangle_\Omega + \langle q,\theta\rangle_\Gamma \ ; \ \langle q,\theta\rangle_\Gamma = \int_{\Gamma_q} q\theta \, d\Gamma \ , \tag{3b}$$

the function $\phi \in S$ such that at the initial time

$$\langle\phi-\phi_0,\theta\rangle_\Omega = 0 \ , \tag{3c}$$

and for each time $t \in [0,T)$ and for all $\theta \in V$ satisfies

$$a(\phi,\theta) - \ell(\theta) = 0 \tag{3d}$$

is the *weak solution* of problem (1a-e).

4. APPROXIMATE SOLUTION: PETROV-GALERKIN F.E.M.

Suppose a finite element partition τ^h, consisting of N_e elements such that

$$\bar{\Omega} = \bigcup_{e=1}^{N_e} \bar{\Omega}_e \qquad \text{and} \qquad \bigcap_{e=1}^{N_e} \Omega_e = \emptyset \, , \tag{4}$$

was chosen. Then, a class of *Upwind Petrov-Galerkin* approximations of problem (3a-d) can be constructed requiring the approximate solution $\phi^h \in S^h$ to satisfy

$$a(\phi^h,\theta^h) - \ell(\theta^h) = - \sum_{e=1}^{N_e} \langle \dot{\phi}^h + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\phi^h + \operatorname{div}(-\underset{\approx}{K}\underset{\sim}{\nabla}\phi^h) - f, \omega^h \rangle_{\Omega_e} \; ; \; \forall\theta^h \in V^h \tag{5}$$

where: $S^h \equiv \{\phi^h \in C^0(\Omega);\ \phi^h|_{\Omega_e} \in P^k;\ \forall\Omega_e \in \tau^h;\ \phi^h|_{\Gamma_g} = g\}$ (6a)

$$V^h \equiv \{\theta^h \in C^0(\Omega);\ \theta^h|_{\Omega_e} \in P^k;\ \forall\Omega_e \in \tau^h;\ \theta^h|_{\Gamma_g} = 0\} \, , \tag{6b}$$

and P^k is the space of polynomials of degree $\leq k$.

Consequently using this formulation the space of weighting functions consists of elements

$$\eta^h = \theta^h + \omega^h \; ; \; \theta^h \in V^h \; . \tag{7}$$

Of course for different choices of the weighting function ω^h, different Petrov-Galerkin's approximations are generated. For $\omega^h=0$ the solution of (5) degenerates in the classical Galerkin approximation. Discontinuous weighting functions proposed by Hughes et all [1,2] give rise to SUPG and DC2 methods.

4.1. CAU Method

In our approach, the weighting function ω^h is defined as

$$\omega^h = \underset{\sim}{u}^h\cdot\underset{\sim}{\nabla}\theta^h \tag{8}$$

where u^h is a *Consistent Approximate Upwind* direction, which depends continuously on the approximate solution ϕ^h, in the sense that:

$$\text{as } \phi^h \underset{h\to 0}{\to} \phi \ (\textit{exact solution}) \Longrightarrow \underset{\sim}{u}^h \underset{h\to 0}{\to} \underset{\sim}{u} \ (\textit{streamline direction}). \tag{9}$$

As shown in reference [4], if $\underset{\sim}{u}^h$ is constructed as

$$\underset{\sim}{u}^h = \alpha\underset{\sim}{v}^h + \beta(\underset{\sim}{u} - \underset{\sim}{v}^h) \, , \tag{10}$$

where for each ϕ^h the vector field $\underset{\sim}{v}^h$ is such that:

i) it satisfies in each element the discretized version of the advection-diffusion equation (1a), that is,

$$\dot{\phi}^h + \underset{\sim}{v}^h\cdot\underset{\sim}{\nabla}\phi^h + \operatorname{div}(-\underset{\approx}{K}\underset{\sim}{\nabla}\phi^h) - f = 0 \quad \text{in } \Omega_e \quad (e=1,2,\ldots,N_e); \tag{11}$$

ii) in the $L_2(\Omega_e)$ norm it is the closest vector field to the real transport velocity field $\underset{\sim}{u}$, that is,

$$\| \underset{\sim}{v}^h - \underset{\sim}{u} \|^2_{0,\Omega_e} = \langle \underset{\sim}{v}^h - \underset{\sim}{u}, \underset{\sim}{v}^h - \underset{\sim}{u} \rangle_{\Omega_e} \quad \textit{is a minimum} \quad (e=1,2\ldots N_e); \tag{12}$$

then condition (9) is attained.

It is not difficult to show that conditions (11) and (12) imply that in each element

$$\underset{\sim}{v}^h = \underset{\sim}{u} - \frac{[\dot{\phi}^h + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\phi^h + \mathrm{div}(-\underset{\approx}{K}\underset{\sim}{\nabla}\phi^h) - f]}{|\underset{\sim}{\nabla}\phi^h|^2}\, \underset{\sim}{\nabla}\phi^h \tag{13a}$$

when $\quad |\underset{\sim}{\nabla}\phi^h| = (\underset{\sim}{\nabla}\phi^h \cdot \underset{\sim}{\nabla}\phi^h)^{1/2} > 0 \quad ,$ (13b)

and $\quad \underset{\sim}{v}^h = \underset{\sim}{u} \quad \text{if} \quad |\underset{\sim}{\nabla}\phi^h| \equiv 0.$ (13c)

Using equations (8), (10) and (13) it follows that, in our model, the weighting function ω^h is given by

$$\omega^h = \alpha \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\theta^h + (\beta-\alpha)\, \frac{[\dot{\phi}^h + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\phi^h + \mathrm{div}(-\underset{\approx}{K}\underset{\sim}{\nabla}\phi^h) - f]}{|\underset{\sim}{\nabla}\phi^h|^2}\, \underset{\sim}{\nabla}\phi^h \cdot \underset{\sim}{\nabla}\theta^h \quad , \tag{14}$$

where α and β are the *upwind functions*. For their determination the same approach suggested in reference [2] will be adopted here.

5. ISOLATION OF NUMERICAL BOUNDARY LAYERS DISTURBANCES

The numerical examples presented in references [3] and [4] show that for convection-dominated problems governed by non-homogeneous or transient advection-diffusion equation the CAU model outlined in the preceding section exhibits better accuracy and stability properties when compared with SUPG or DC2 methods. Globally, the approximate solutions given by these three methods compare well with the corresponding exact solutions. But near boundary layers the localized oscillations tipically produced by SUPG are not removed by DC2 method which sometimes gave larger oscillations. This did not occur with CAU, but some oscillations were still present, and the purpose of this section is to derive a methodology in order to isolate the numerical perturbation caused by boundary layers on the regular part of the domain solution.

To this end the following definitions are introduced;

the sets: $\Gamma^- \equiv \{\underset{\sim}{x} \in \Gamma;\ \underset{\sim}{u}\cdot\underset{\sim}{n}<0\};\quad \Gamma^+ = \Gamma - \Gamma^-$

$$\mu \equiv \{\Omega_e;\ \Omega_e \cap \Gamma^+ \neq \emptyset;\ e=1,2,\ldots N_e\}$$

$$\chi \equiv \{\underset{\sim}{x} \in \mu;\ \underset{\sim}{x} \notin \Gamma^+\} \tag{15a-c}$$

and the function: $Q : \Omega \cup \Gamma \to \mathbb{R}$, (16)

which is assumed to be continuous and positive. The function Q will be ~~constructed element by element according to:~~

$$Q_e(\underline{x}) = \begin{cases} 0 \quad , \text{ if } \Omega_e \notin \iota \\ \sum_{i=1}^{NPE} Q(\underline{x}_i)\psi_i^e(\underline{x}) \quad , \text{ if } \Omega_e \in \iota \end{cases} \qquad (e=1,2,\ldots N_e) \tag{17}$$

$$Q(\underline{x}_i) = \begin{cases} 0 \quad , \text{ if } \bar{P}_e(\underline{x}_i) \le 1 \text{ or } \underline{x}_i \in \chi \\ [\bar{P}_e(\underline{x}_i)-1] \; , \text{ if } \bar{P}_e(\underline{x}_i) > 1 \text{ and } \underline{x}_i \notin \chi \end{cases} \tag{18}$$

where $Q_e(x)$ is the restriction of Q to element e, $\bar{P}_e(\underline{x}_i)$ is the mean value at node i of the element Peclet numbers and $\psi_i^e(\underline{x})$ is the local interpolation function associated to node i of element e.

Now, if η_i^e denotes the restriction to element e of the global Petrov-Galerkin weighting function corresponding to node i, isolation of boundary layers numerical perturbation over the regular part of domain solution can be achieved using the modified weighting function $\eta_{i,I}^e$ given by:

$$\eta_{i,I}^e = \begin{cases} \eta_i^e \quad , \text{ if } \underline{x}_i \notin \chi \\ \eta_i^e \exp(-Q_e) \quad , \text{ if } \underline{x}_i \in \chi \end{cases} \tag{19}$$

6. NUMERICAL RESULTS

Two test examples concerning convection-dominated problems will be presented. For both the medium is assumed homogeneous and isotropic with a physical diffusivity coefficient $k=10^{-6}$. The results shown in figures 1 and 2 refer to a bi-dimensional (1x1) domain discretized by a (20x20) square mesh. Bi-linear elements were employed with (2x2) quadrature.

6.1. Transient Advection Skew to the Mesh

For this example the velocity field u has components $u_x=1$ and $u_y=-1$. The assumed essential boundary conditions are: $\phi(0,y,t)=\phi(1,y,t)=\phi(x,0,t)=0$, and $\phi(x,1,t)$ is a time-dependent boundary condition propagating from position x=0.2, in the positive x-direction, with a fixed profile and constant velocity v=0.05. Definitions of the function $\phi(x,1,t)$; the source term f(x,y,t), and the exact solution $\phi(x,y,t)$ are given in reference [4]. The initial condition adopted was the steady solution (v=0).

In figure 1 the exact solutions (E) at times t=0.3 and t=0.9 are compared with (CAU) and (SUPG) solutions calculated using the backward difference time-integration scheme, with a time step $\Delta t=0.1$. For the CAU method a maximum of four iterations at each time step was needed for convergence. Results for the DC2 method was not presented as no convergence was achieved. The numerical results obtained with CAU agree well with the exact solution, and no additional difficulties occur due to the transient nature of this solution.

6.2. Steady Advection Skew to the Mesh

For this example the velocity field u has components $u_x=1$ and $u_y=-2$, and the source term $f(x,y)=0$. The assumed Dirichlet boundary conditions are: $\phi(x,0)=0$; $\phi(x,1)=1$; $\phi(1,y)=0$ $(0<y<1)$; $\phi(0,y)=0$ $(0<y<0.6)$; $\phi(0,y)=1$ $(0.8<y<1)$ and for $(0.6<y<0.8)$, a piecewise linear function $\phi(0,y)$ was adopted. Numerical results obtained with (CAU), (SUPG) and (DC2) methods are compared in figure 2 with the

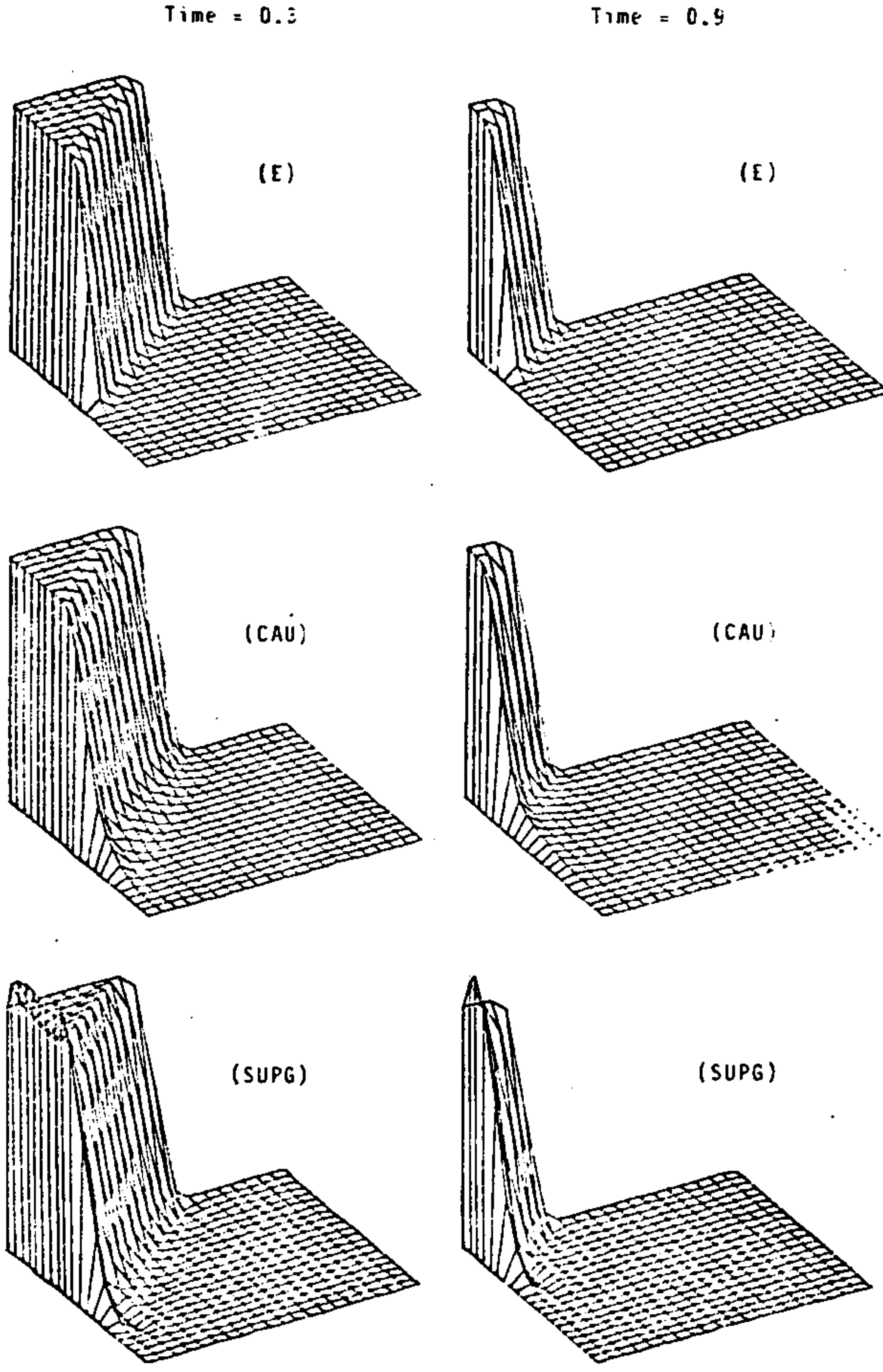

FIGURE 1
Transient advection skew to the mesh

exact solution (E). Because $f(x,y)\equiv 0$ and $\dot{\phi}(x,y)\equiv 0$ CAU and DC2 approximate solutions are exactly the same. This solution still exhibits small oscillations near the boundary layer. But using the procedure presented in section 5 these oscillations are completely removed as shown in the numerical solution labelled (CAUI).

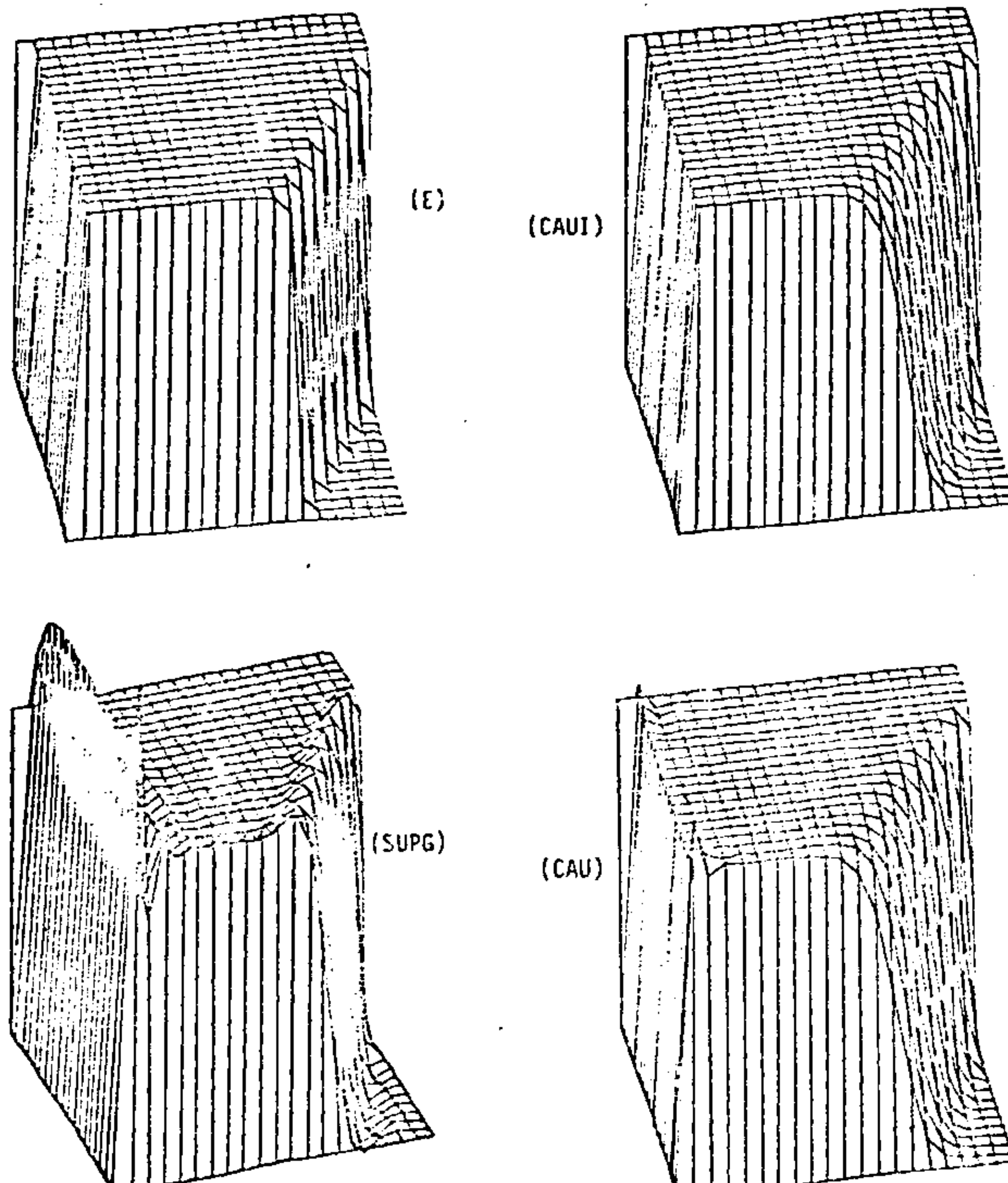

FIGURE 2
Steady advection skew to the mesh

7. CONCLUSIONS

The CAU method presented in this paper shows high stability properties and good accuracy in predicting internal as well as boundary layers. The inherent stability of this method is a consequence of the weighting Petrov-Galerkin function used, which changes continuously with an approximate iterative upwind direction defined in such way as to guarantee that it tends to the streamline direction as the iterative approximate solution approaches the exact one.

The proposed CAUI version of the CAU method effectively eliminates the remaining oscillations near boundary layers, altering the weighting functions associated only with elements adjacent to downwind boundaries where Dirichlet type conditions are prescribed. The additional computational effort is minimum compared with the additional accuracy produced by this procedure.

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