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RELATIVISTIC THEORY OF THE LAMB SHIFT BASED ON SELF ENERGY *

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ABSTRACT

A study is made to evaluate the Lamb shift to all orders in $(Z\alpha)$ using relativistic Dirac Coulomb wavefunctions and without resorting to the dipole approximation. Use is made of the angular integrals and spins sums performed elsewhere exactly. A regularization procedure is given that makes the sum over the positive and negative energy states finite. Finally, the energy shift ΔE_n^{LS} is given in terms of an integral that may be done numerically.

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I. INTRODUCTION

The main goal of the present work is to formulate and investigate the Lamb shift to all orders in $Z\alpha$ using the fully relativistic Dirac-Coulomb wavefunctions and without making the dipole approximation. To our knowledge, such a calculation has been made so far only for the vacuum polarization contribution to the Lamb shift by Wichmann and Kroll[1] and more recently by Barut and Kraus[2], using an entirely different method, but not for the Bethe term. Recently, we have used the Dirac-Coulomb wavefunctions in a precision calculation of spontaneous emission[3]. There are considerable technical problems to solve involving the full Dirac spin calculation, angular integrations and complicated radial integrations which one has to collect and treat once and for all. In most studies of the Lamb shift the nonrelativistic formula due to Bethe[4] is used for low energies and the result is matched to the solutions of the high energy perturbation theory graphs[5].

We have also used, on the conceptual level, a formalism of Quantum electrodynamics based on self energy[6-8] which does not involve the second quantized fields. In principle, the method is nonperturbative, but in practice we solve the equations by iteration. The theory makes fewer assumptions than the standard second quantized theories and its results should be compared directly with experiment, which we have done in the case of spontaneous emission with a high precision agreement[3] and for the influence of cavities[9,10] on some radiative processes.

Another fundamental problem is the finiteness of the theory, the problem of regularization or renormalization. Our study of the self energy integrals using Dirac-Coulomb wavefunctions is such that there are neither infrared divergences nor divergences coming from the dipole approximation. Moreover, we have developed a Mellin transform method, also used in treating the vacuum polarization contribution[2], to correctly define the sum over the positive and negative energy continuum states in the Coulomb problem. Unphysical spurious poles in the inverse Mellin transform plane which give rise to ultraviolet divergences are naturally eliminated. The result of sum-

ming over the continuum states (which corresponds in perturbation theory to the loop integrals) is reduced to a sum of residues at the physical poles in the Mellin plane. The contribution of the first pole, for example, consists of the sum of two terms having the coefficients $\alpha(Z\alpha)^5$ and $\alpha(Z\alpha)^4$ showing up explicitly in the final result.

In this sense we prove the finiteness of the theory. No cutoffs need to be introduced and the mass renormalization does not seem to be necessary.

The paper is organized as follows. In section II we present a brief review of the nonpertubative approach to the radiative corrections based on self energy. We reproduce Bethe's nonrelativistic formula for the Lamb shift, by making the dipole approximation, in the first part of section III. Our relativistic formula is then analysed in some detail in the following subsection and some approximations are made leaving the most important terms for a deeper investigation. We conclude section III by presenting more of the analytical work aiming at demonstrating the finiteness of the theory. Finally, in section IV, we give some concluding remarks and a breif discussion.

II. REVIEW OF THE NONPERTURBATIVE APPROACH

It has recently been shown[6-8] that some of the most important effects in Quantum Electrodynamics can be attributed to the electron's self energy. In this section we present a brief review of this theory in which the electromagnetic field is related to its source, the electron, and which is thus closer in spirit than conventional QED to the intuitively clear classical theory of electrodynamics. The idea is to set up the QED action for an electron in an arbitrary external electromagnetic field, A_μ^{ext}, together with its own self field, A_μ^{self}. Employing Heaviside units, where $\hbar = c = 1$, and taking $dx \equiv d^4x$ we write the action in the following form:

$$W = \int dx \left(\bar{\Psi}(\gamma^\mu i\partial_\mu - m)\Psi + J^\mu A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right), \tag{1}$$

In this expression, the first term is the action of the noninteracting Dirac electron, where $\Psi(x)$ is the electron matter field at the spacetime point $x \equiv (x_0, \mathbf{x})$ and γ_μ stands for the familiar Dirac gamma matrices. The second term describes the interaction of the electron with the radiation field. In this term, the electron current is given by:

$$J^\mu = -e\bar{\Psi}\gamma^\mu\Psi \tag{2}$$

and the total electromagnetic field is given by:

$$A_\mu = A_\mu^e + A_\mu^s \tag{3}$$

with the superscripts e and s standing for *external* and *self*, respectively. Here A_μ^e is treated as a given nondynamical function. In other words, the external field is due to sources that are far away from the region of the dynamics under consideration. The last term in equation (1) is what we call the self energy, where the electromagnetic field tensor $F_{\mu\nu} = A^s_{\nu,\mu} - A^s_{\mu,\nu}$ satisfies the Maxwell equations:

$$F^{\mu\nu}{}_{,\nu} = J^\mu \tag{4}$$

We use equation (4) in order to put equation (1), after a single integration by parts has been performed on the last term, into the following form:

$$W = \int dx \left(\bar{\Psi}[\gamma^\mu(i\partial_\mu - eA_\mu^e) - m]\Psi + \frac{1}{2}J^\mu A_\mu^s \right). \tag{5}$$

In the next step, we complete the elimination of A_μ^s from the action by inserting into (5) the solution of the wave equation[11]:

$$\Box A_\mu^s = J_\mu = -e\bar{\Psi}\gamma_\mu\Psi, \tag{6}$$

namely:

$$A_\mu^s(x) = -e\int dy D_{\mu\nu}(x-y)\bar{\Psi}(y)\gamma^\nu\Psi(y). \tag{7}$$

In (7), $D_{\mu\nu}(x-y)$ is the causal Green's function in the covariant gauge $A^{\mu}{}_{,\mu} = 0$, which we take as :

$$D_{\mu\nu}(x-y) = -g_{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik.(x-y)}}{k^2}. \tag{8}$$

Thus equation (5) now becomes :

$$W = \int dx\bar{\Psi}(x)[\gamma^{\mu}(i\partial_{\mu} - eA^{e}_{\mu}) - m]\Psi(x)$$

$$- \frac{e^2}{2}\int dxdy\bar{\Psi}(x)\gamma^{\mu}\Psi(x)\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik.(x-y)}}{k^2}\bar{\Psi}(y)\gamma_{\mu}\Psi(y)$$

$$= W_0 + W_1. \tag{9}$$

Nex, we introduce the following Fourier expansion for the electron matter field Ψ in the time variable:

$$\Psi(x) = \sum\!\!\!\!\!\!\int_n \psi_n(\mathbf{x})e^{-iE_n x_0} \tag{10}$$

The Fourier coefficients in this expansion will be given an interpretation at a later stage. We now substitute for the $\Psi's$ in equation (9) from (10) and carry out the time and time-like integrations over k_0, y_0, and x_0, in this order for convenience. After all this has been done the linear kinetic energy piece of the total action becomes:

$$W_0 = 2\pi\sum\!\!\!\!\!\!\int_n d^3x\bar{\psi}_n(\mathbf{x})(\gamma^0 E_n - \gamma.\mathbf{p} - e\mathbf{A}^e - m)\psi_n(\mathbf{x}), \tag{11a}$$

while the nonlinear self energy piece will read:

$$W_1 = -2\pi\frac{e^2}{2}\sum\!\!\!\!\!\!\int_{nmrs}\delta(E_n - E_m + E_r - E_s)\int d^3x\bar{\psi}_n(\mathbf{x})\gamma^{\mu}\psi_m(\mathbf{x})$$

$$\times\int d^3y\bar{\psi}_r(\mathbf{y})\gamma_{\mu}\psi_s(\mathbf{y})\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})}}{2}\left\{\frac{i\pi}{k}[\delta(E_r - E_s + k)\right.$$

$$\left.+\delta(E_r - E_s - k)] + \frac{\mathbf{P}}{2k}\left(\frac{1}{E_r - E_s - k} - \frac{1}{E_r - E_s + k}\right)\right\}, \tag{11b}$$

Here **P** stands for the principal value integral and $\sum\!\!\!\!\!\!\int$ implies a sum over the discrete part and an integration over the continuum part of the system's spectrum. In carrying out the k_0 -integration, the contour is closed in the upper half plane for $y_0 > x_0$ where it encloses the simple pole at $k_0 = -k$, $(k \equiv |k|)$, and in the lower half plane for the case $y_0 < x_0$ where it encloses the pole at $k_0 = +k$; θ-functions are used in order to distinguish between the two cases. The y_0-integrations turn out to be simply Fourier transforms of the θ-functions which give rise to the principal value integrals and the δ-functions in (11b).

Now , the δ-function, $\delta(E_n - E_m + E_r - E_s)$, can be satisfied by the two choices[2]:

(1) $n = m$ and simultaneously $r = s$.

(2) $n = s$ and simultaneously $r = m$.

With this, W_1 becomes:

$$W_1 = -2\pi\frac{e^2}{2}\sum\!\!\!\!\!\!\int_{ns}\int d^3x\bar{\psi}_n(\mathbf{x})\gamma^{\mu}\psi_n(\mathbf{x})\int d^3y\bar{\psi}_s(\mathbf{y})\gamma_{\mu}\psi_s(\mathbf{y})$$

$$\times\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})}\left\{\frac{i\pi}{2k}[\delta(k)+\delta(-k)] + \frac{\mathbf{P}}{2k}[-\frac{1}{k} - \frac{1}{k}]\right\}$$

$$- 2\pi\frac{e^2}{2}\sum\!\!\!\!\!\!\int_{ns}\int d^3x\bar{\psi}_n(\mathbf{x})\gamma^{\mu}\psi_s(\mathbf{x})\int d^3y\bar{\psi}_s(\mathbf{y})\gamma_{\mu}\psi_n(\mathbf{y})$$

$$\times\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})}\left\{\frac{i\pi}{2k}[\delta(E_s - E_n + k) + \delta(E_s - E_n - k)]\right.$$

$$\left.+\frac{\mathbf{P}}{2k}[\frac{1}{E_s - E_n - k} - \frac{1}{E_s - E_n + k}]\right\}, \tag{12}$$

In equation (12) the term proportional to $\delta(k) + \delta(-k) = 2\delta(k)$ will not contribute as a result of the integration over k.

The common procedure of minimizing the total action, consisting of the two pieces given by equations (11), with respect to ψ_n will result in a Dirac equation for

our electron that is nonlinear and in which $\psi_n(x)$ plays the role of the electron's wavefunction. This can be seen without even having to go through the actual derivation. In this approach we do not deal with this nonlinear equation directly, instead we shall study the piece of the total action giving rise to the nonlinearity and which, it turns out, contains the radiative corrections in the Coulomb problem. To begin with, the exact solutions to the nonlinear equations when put back into the total action will make it assume its minimum value ; namely zero. Therefore, if we write the solution of such an equation as $\psi_n = \psi_n^c + \delta\psi_n$ corresponding to the energy eigenvalue $E_n = E_n^c + \delta E_n$, where ψ_n^c belongs to the complete set of solutions to the Dirac equation in the appropriate external field A_μ^c, and E_n^c is its eigenenergy, then in a first iteration of the action we can drop the correction $\delta\psi_n$. When this is done, along with the demand that the total action vanishes in this first iteration, the nonlinear piece, W_0, will contribute a factor $2\pi\sum_n \delta E_n$ and W_1 will be evaluated in terms of the set of functions $\{\psi_n^c(\mathbf{x})\}$. Finally we get:

$$W_1^{(1)} = -2\pi \sum_n \delta E_n, \tag{13}$$

where the superscript on $W_1^{(1)}$ is added to indicate that we are considering a first iteration of the action. From (12) and (13) we immediately identify the shift in the nth energy level as a sum of three terms having the following physical interpretations. (From here on we shall drop the superscript c on ψ_n).

(1) Vacuum Polarization:

$$\Delta E_n^{VP} = -\frac{e^2}{2}\sum_s\!\!\!\!\!\!\int d^3x\bar{\psi}_n(\mathbf{x})\gamma^\mu\psi_n(\mathbf{x})\mathbf{P}\left(\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})}}{k^2}\right)\int d^3y\bar{\psi}_s(\mathbf{y})\gamma_\mu\psi_s(\mathbf{y}) \tag{14}$$

(2) Spontaneous Emission and Absorption:

$$\Delta E_n^{SE} = \frac{e^2}{2}\sum_s\!\!\!\!\!\!\int d^3x\bar{\psi}_n(\mathbf{x})\gamma^\mu\psi_s(\mathbf{x})\int d^3y\bar{\psi}_s(\mathbf{y})\gamma_\mu\psi_n(\mathbf{y})$$

$$\times\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})}\left\{\frac{i\pi}{2k}[\delta(E_s-E_n+k)+\delta(E_s-E_n-k)]\right\} \tag{15}$$

(3) The Lamb-Shift:

$$\Delta E_n^{LS} = \frac{e^2}{2}\sum_s\!\!\!\!\!\!\int\bar{\psi}_n(\mathbf{x})\gamma^\mu\psi_s(\mathbf{x})\int d^3y\bar{\psi}_s(\mathbf{y})\gamma_\mu\psi_n(\mathbf{y})$$

$$\times\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})}}{2k}\mathbf{P}\left(\frac{1}{E_s-E_n-k}-\frac{1}{E_s-E_n+k}\right) \tag{16}$$

The vacuum polarization term has been treated elsewhere[2] and the spontaneous emission term has been the subject of a paper[3] by the present authors as well. In the next section we study in detail the last item in the list above.

III. THE LAMB SHIFT

(a) The Dipole Limit:

Our starting point in the relativistic calculation of the Lamb shift based on self energy is going to be equation (16) of the previous section. Here,too,we make a little digression in order to show that Bethe's formula[4] can in fact be obtained from (16) in the dipole approximation (DA). Before we do that though, one should remember that in (16) full relativistic Dirac-Coulomb wavefunctions are to be used. In any nonrelativistic limit that involves the use of nonrelativistic wavefunctions, an overall factor of 2 must be included[3] in order to compensate for the electron's spin which is relativistic in origin. Moreover, if one is willing to accept the interpretation[12] of the first term as resulting from virtual transitions to the negative energy levels, then one should drop it in any nonrelativistic approximation. In the DA, the exponential factor in (16) is approximated by unity. This translates physically into taking the radiation wavelength

to be much greater than the atomic dimensions, or $k|\mathbf{x}-\mathbf{y}| \ll 1$. In the next step, we make the expansion:

$$\int d^3x\bar{\psi}_n\gamma^\mu\psi_s \int d^3x'\bar{\psi}_s\gamma_\mu\psi_n = \left(\int \psi_n^\dagger\psi_s d^3x\right)\left(\int \psi_s^\dagger\psi_n d^3x'\right) - \left(\int \psi_n^\dagger\alpha\psi_s d^3x\right)\cdot\left(\int \psi_s^\dagger\alpha\psi_n d^3x'\right)$$

The first term on the right vanishes as a result of the orthogonality of the wavefunctions, while the second term is immediately recognized as the scalar product of the expectation values of the velocity operator ($\mathbf{v} = c\alpha, c = 1$) between the states n and s. Therefore:

$$\int \bar{\psi}_n\gamma^\mu\psi_s d^3x \int \bar{\psi}_s\gamma_\mu\psi_n d^3x' = -\mathbf{v}_{ns}\cdot\mathbf{v}_{sn}.$$

On the other hand, the Heisenberg equations of motion give: $\mathbf{v}_{ns} = i\mathbf{p}_{ns}/m$ and $\mathbf{v}_{sn} = i\mathbf{p}_{sn}/m$. Putting all of this back into equation (16), we get:

$$\Delta E_n^{LS}(DA) \approx -\frac{\alpha}{8\pi^2m^2}\sum_s\!\!\!\!\!\!\int_0^\infty kdkd\Omega_k\frac{\mathbf{p}_{ns}\cdot\mathbf{p}_{sn}}{\omega+k}, \tag{17}$$

where $\omega \equiv E_s - E_n$. With the polarization taken care of as usual, the angular integration in (17) gives $\frac{8\pi}{3}$. If we also multiply by the factor of 2 mentioned above, which comes out automatically in the relativistic calculation, we arrive at:

$$\Delta E_n^{LS}(DA) \approx -\frac{2}{3\pi}\alpha\frac{1}{m^2}\int_0^\infty kdk\sum_s\!\!\!\!\!\!\int\frac{\mathbf{p}_{ns}\cdot\mathbf{p}_{sn}}{E_s - E_n + k}. \tag{18}$$

This is precisely Bethe's result, provided that some finite cut off λ is introduced as usual to replace the infinite upper limit on the integration over k. We now move on to present our derivation for the relativistic Lamb-shift formula.

(b) The Relativistic Calculation:

As has been done in the spontaneous emission case[3], we begin by expanding the exponential factor in equation (16) in terms of partial waves and carrying out the angular k–integration. The result of doing so is:

$$\Delta E_n^{LS} = 2\alpha\sum_s\!\!\!\!\!\!\int\sum_{\tilde{\ell}\tilde{m}}\int d^3x Y_{\tilde{\ell}\tilde{m}}(\hat{r})\bar{\psi}_n(\mathbf{x})\gamma^\mu\psi_s(\mathbf{x})\int d^3x' Y^*_{\tilde{\ell}\tilde{m}}(\hat{r}')\bar{\psi}_s(\mathbf{x}')\gamma_\mu\psi_n(\mathbf{x}') \times \mathrm{P}\{\frac{k}{\omega-k} - \frac{k}{\omega+k}\}\, j_{\tilde{\ell}}(kr)\, j_{\tilde{\ell}}(kr')\, dk$$

$$= 2\alpha\sum_s\!\!\!\!\!\!\int\sum_{\tilde{\ell}\tilde{m}}\int_0^\infty V_{\tilde{\ell}}(r,r')drdr'\, Y_{\tilde{\ell}\tilde{m}}(\hat{r})\,\bar{\psi}_n(\mathbf{x})\,\gamma^\mu\,\psi_s(\mathbf{x})\,do \times \int Y^*_{\tilde{\ell}\tilde{m}}(\hat{r}')\,\bar{\psi}_s(\mathbf{x}')\,\gamma_\mu\,\psi_n(\mathbf{x}')\,do' \tag{19}$$

where $\mathbf{x} \equiv (r,\theta,\phi)$, $\mathbf{x}' \equiv (r',\theta',\phi')$, $\hat{r} \equiv \mathbf{x}/r$, $\hat{r}' \equiv \mathbf{x}'/r'$, $do \equiv sin\theta d\theta d\phi$, and the effective potential (see Appendix B) is given by:

$$V_{\tilde{\ell}}(r,r') = (rr')^2\int_0^\infty\{\frac{k}{\omega-k} - \frac{k}{\omega+k}\}\, j_{\tilde{\ell}}(kr)\, j_{\tilde{\ell}}(kr')\, dk$$

$$= -\frac{i\pi\omega}{2}(rr')^2\, h_{\tilde{\ell}}^{(2)}(\omega r_>)\{\, h_{\tilde{\ell}}^{(1)}(\omega r_<) + \, h_{\tilde{\ell}}^{(2)}(\omega r_<)\,\} \tag{20}$$

In (20), the $h_{\tilde{\ell}}^{(1)}$ and $h_{\tilde{\ell}}^{(2)}$ are the spherical Hankel functions of the first and second kinds, respectively. In terms of form factors similar to those of reference [3], the Lamb shift formula becomes:

$$\Delta E_n^{LS} = 2\alpha\sum_s\!\!\!\!\!\!\int\sum_{\tilde{\ell}\tilde{m}}\int_0^\infty V_{\tilde{\ell}}(r,r')drdr'\{{}_nT_s^0(\mathbf{x})\;{}_nT_s^{0\dagger}(\mathbf{x}') - \;{}_n\mathbf{T}_s(\mathbf{x}).\;{}_n\mathbf{T}_s^\dagger(\mathbf{x}')\}$$

where:

$${}_nT_s^0(\mathbf{x}) = [\frac{(2J_n+1)(2J_s+1)}{4\pi}]^{\frac{1}{2}}\,\{W_{ns}^{\tilde{\ell}\tilde{m}}\;\; Q_1(r) + W_{n's}^{\tilde{\ell}\tilde{m}}\;\; Q_2(r)\}$$

and:

$${}_n\mathbf{T}_s(\mathbf{x}) = [\frac{(2J_n+1)(2J_s+1)}{4\pi}]^{\frac{1}{2}}\,\{\mathbf{K}_{ns}^{\tilde{\ell}\tilde{m}}\;\; Q_3(r) - \mathbf{K}_{n's}^{\tilde{\ell}\tilde{m}}\;\; Q_4(r)\}$$

and where:

$$\begin{aligned} Q_1(r) &= g_n(r)\; g_s(r) \\ Q_2(r) &= f_n(r)\; f_s(r) \\ Q_3(r) &= g_n(r)\; f_s(r) \\ Q_4(r) &= f_n(r)\; g_s(r) \end{aligned} \tag{21}$$

Note that the subscript s on the final continuum radial wavefunctions is understood to stand for the Dirac quantum number κ as well as the energy E and that the angular matrix elements have the same definitions as in reference [3] and which we shall have occasion to quote from there as the need for them arises. So we have to evaluate a total number of eight integrals of the form:

$$\lambda_{ij} = \int_0^\infty V_{\tilde{\ell}}(r,r')\ \ Q_i(r)\ \ Q_j(r')\ \ dr dr' \tag{22}$$

where $(i,j) = (1,2)$ or $(3,4)$. As an example we show here the general structure of the first of such integrals, namely:

$$\lambda_{11} = -i\pi\omega \int_0^\infty r^2\ \ h_{\tilde{\ell}}^{(1)}(\omega r)\ \ g_n(r)\ \ g_s(r)\ \Phi(r)\ dr,$$

where:

$$\Phi(r) = \int_0^r r'^2\ \ h_{\tilde{\ell}}^{(1)}(\omega r')\ \ g_n(r')\ \ g_s(r')\ dr' \ +\ c.c \tag{23}$$

and with $c.c$ standing for complex conjugate. The strategy for carrying out the exact full derivation is straightforward but extremely complicated. It is easy to see that each of the $\lambda'_{ij}s$ consists of a sum of eight terms plus their complex conjugates and that each of these terms contains, in turn, two radial integrations. This brings the total number of terms ΔE_n^{LS} will ultimately consist of to:

$$8 \times 8 \times 2 \times 2 = 256!$$

We shall study the most important of these terms here; namely the ones that give the largest contribution to the energy level shift. It will be sufficient for our purposes here to make the following two approximations:

(a) Typically the radial part of the smaller component of a Dirac-Coulomb wavefunction of the discrete spectrum scales with respect to its larger component counterpart according to:

$$f_n \approx \sqrt{\frac{1-\varepsilon_n}{1+\varepsilon_n}}\ g_n \approx \ \frac{Z\alpha}{4},$$

where $\varepsilon_n = E_n/m$, Z is the charge of the nucleous, and α is the fine structure constant. This is a sufficiently good approximation for the $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ states of the Hydrogen isoelectronic sequence which we shall be considering as a first example elsewhere[13].

(b) According to Bethe[4], the important values of $\omega = E_s - E_n$ for virtual transitions contributing to the Lamb shift are of the order of magnitude of the ground state binding energy or Z^2Ry, where Ry is one Rydberg. The continuum is also much larger in size and has a much bigger density of states than does the discrete part of the spectrum. This implies that the biggest contribution to the real energy shift will come from final states in the continuum. We shall therefore replace the sum-integral in our formulas by an integration over the positive and negative continuum energies. Also, since r and r' can effectively be thought of as radial coordinates of an observation point[19] and a source point, respectively, and since both points lie within the dimensions of an atom $\approx 1\mathring{A}$, we can take:

$$\omega r\ ,\ \omega r' \approx 10^{-3} \ll 1 \tag{24}$$

Consequently, we replace the spherical Bessel and Hankel functions in the effective potential by their limiting forms as the arguments tend to zero:

$$V_{\tilde{\ell}}(r,r') \approx \frac{\pi}{2(2\tilde{\ell}+1)}\ r_< r_>\ (\frac{r_<}{r_>})^{\tilde{\ell}}\ , \tag{25}$$

where $r_<$ ($r_>$) is the smaller (larger) of r and r' (see Appendix B). This is reminiscent of the dipole approximation but is not quite equivalent to it. In the DA the exponentials are dropped altogether and the effect of retardation is thus lost completely, which we do not do here.

When the above two limits are taken, the Lamb shift formula becomes:

$$\Delta E_n^{LS} \approx \frac{\alpha}{2\pi}\ (2J_n+1)\ \int dE(2J_s+1) \sum_{\tilde{\ell}\tilde{m}} \int V_{\tilde{\ell}}(r,r')\ dr dr'$$
$$\times \{|W_{ns}^{\tilde{\ell}\tilde{m}}|^2\ Q_1(r)\ Q_1(r')\ -\ |\mathbf{K}_{ns'}^{\tilde{\ell}\tilde{m}}|^2\ Q_3(r)\ Q_3(r')\}$$

$$= \frac{\alpha}{4} \int dE \sum_{\tilde{\ell}\tilde{m}} \frac{(2J_n+1)(2J_s+1)}{2\tilde{\ell}+1} \{ |W_{ns}^{\tilde{\ell}\tilde{m}}|^2 \, R_1^{\tilde{\ell}} \; - \; |\mathbf{K}_{ns'}^{\tilde{\ell}\tilde{m}}|^2 \, R_2^{\tilde{\ell}} \} \quad (26)$$

where the radial matrix elements are now defined by:

$$R_1^{\tilde{\ell}} = \int_0^\infty r_<^{\tilde{\ell}+2} \, r_>^{-\tilde{\ell}+1} \, Q_1(r) Q_1(r') \, dr dr'$$

$$R_2^{\tilde{\ell}} = \int_0^\infty r_<^{\tilde{\ell}+2} \, r_>^{-\tilde{\ell}+1} \, Q_3(r) \, Q_3(r') \, dr dr', \quad (27)$$

and the angular matrix elements, as quoted from reference [3], are given by:

$$W_{ns}^{\tilde{\ell}\tilde{m}} = (-1)^{M_n - \frac{1}{2}} \sqrt{(2\tilde{\ell}+1)(2\ell_n+1)(2\ell_s+1)} \begin{pmatrix} \ell_n & \ell_s & \tilde{\ell} \\ 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} J_n & J_s & \tilde{\ell} \\ -M_n & M_s & \tilde{m} \end{pmatrix} \begin{Bmatrix} J_n & J_s & \tilde{\ell} \\ \ell_s & \ell_n & \frac{1}{2} \end{Bmatrix} \quad (28a)$$

The range of $\tilde{\ell}$ in this expression is restricted to the values given by:

$$|\ell_n - \ell_s| < \tilde{\ell} < \ell_n + \ell_s \text{ such that } \ell_n + \tilde{\ell} + \ell_s = \text{an even integer.} \quad (28b)$$

Moreover, we have:

$$\mathbf{K}_{ns'}^{\tilde{\ell}\tilde{m}} = (-1)^{J_n + J_s - M_s + \frac{1}{2}} \sqrt{(2\tilde{\ell}+1)(2\ell_n+1)(2\ell_s'+1)} \begin{pmatrix} \ell_n & \ell_s' & \tilde{\ell} \\ 0 & 0 & 0 \end{pmatrix} \times \{(a_1 - a_2)\hat{i} - i(a_1 + a_2)\hat{j} + (b_1 + b_2)\hat{k}\} \quad (29a)$$

where:

$$a_1 = \begin{pmatrix} \ell_n & J_n & \frac{1}{2} \\ M_n - \frac{1}{2} & -M_n & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \ell_s' & J_s & \frac{1}{2} \\ M_s + \frac{1}{2} & -M_s & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \ell_n & \ell_s' & \tilde{\ell} \\ -M_n + \frac{1}{2} & M_s + \frac{1}{2} & \tilde{m} \end{pmatrix}$$

$$a_2 = \begin{pmatrix} \ell_n & J_n & \frac{1}{2} \\ M_n + \frac{1}{2} & -M_n & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \ell_s' & J_s & \frac{1}{2} \\ M_s - \frac{1}{2} & -M_s & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \ell_n & \ell_s' & \tilde{\ell} \\ -M_n - \frac{1}{2} & M_s - \frac{1}{2} & \tilde{m} \end{pmatrix}$$

$$b_1 = \begin{pmatrix} \ell_n & J_n & \frac{1}{2} \\ M_n - \frac{1}{2} & -M_n & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \ell_s' & J_s & \frac{1}{2} \\ M_s - \frac{1}{2} & -M_s & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \ell_n & \ell_s' & \tilde{\ell} \\ -M_n + \frac{1}{2} & M_s - \frac{1}{2} & \tilde{m} \end{pmatrix}$$

$$b_2 = \begin{pmatrix} \ell_n & J_n & \frac{1}{2} \\ M_n + \frac{1}{2} & -M_n & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \ell_s' & J_s & \frac{1}{2} \\ M_s + \frac{1}{2} & -M_s & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \ell_n & \ell_s' & \tilde{\ell} \\ -M_n - \frac{1}{2} & M_s + \frac{1}{2} & \tilde{m} \end{pmatrix}$$

(29b)

and with the range of $\tilde{\ell}$ defined by:

$$|\ell_n - \ell_s'| < \tilde{\ell} < \ell_n + \ell_s' \text{ such that } \ell_n + \tilde{\ell} + \ell_s' = \text{an even integer} \quad (29c)$$

Next we evaluate the radial matrix elements exactly and study them in some detail. Putting the wavefunctions explicitly into $R_1^{\tilde{\ell}}$, it becomes:

$$R_1^{\tilde{\ell}} = \int_0^\infty g_n(r) \, g_s(r) \, r^{-\tilde{\ell}+1} \, dr \int_0^r g_n(r') \, g_s(r') \, r'^{\tilde{\ell}+2} \, dr' + \int_0^\infty g_n(r') \, g_s(r') \, r'^{-\tilde{\ell}+1} \, dr' \int_0^{r'} g_n(r) \, g_s(r) \, r^{\tilde{\ell}+2} \, dr$$

$$= 2 \int_0^\infty g_n(r) \, g_s(r) \, r^{-\tilde{\ell}+1} \, \Phi(r) dr$$

It can easily be shown that the relevant radial integration contained in $\Phi(r)$, for final states in the continuum, is of the form[14]:

$$\int_0^r r'^{\mu-1} \, e^{-ap_n r'} \, dr' = (ap_n)^{-\mu} \gamma(\mu, ap_n r)$$

$$= \frac{r^\mu}{\mu} {}_1F_1(1, \mu+1; ap_n r) \quad (30)$$

where $\gamma(\mu, ap_n r)$ is the incomplete gamma function and ${}_1F_1$ is a confluent hypergeometric function. With a little hindsight, $\Phi(r)$, in this case, can finally be put into the following compact form ($\varepsilon = E/m$ and $\varepsilon_n = E_n/m$):

$$\Phi(r) = CU_n[(1+\varepsilon_n)(1+\varepsilon)]^{\frac{1}{2}} (2p_n)^{\gamma_n + \gamma - 1} \, y^\gamma \frac{\Gamma(2\gamma+1)}{|\Gamma(\gamma + i\nu)|^2} \times \left\{ \int_0^1 dt \wedge (t) \left(\sum_{n_1=0}^{n_r-1} \Pi_{n_1} \, G_{n_1}(r,t) - \sum_{n_2=0}^{n_r} \Pi_{n_2} \, G_{n_2}(r,t) \right) + c.c \right\} \quad (31a)$$

If we further write:

$$r^{-\tilde{\ell}+1}\, g_n(r)\, g_s(r) = CU_n[(1+\varepsilon_n)(1+\varepsilon)]^{\frac{1}{2}}(2p_n)^{\gamma_s+\gamma-1}\, y^{\gamma}\frac{\Gamma(2\gamma+1)}{|\Gamma(\gamma+i\nu)|^2}$$

$$\times\left\{\int_0^1 du\wedge(u)\left(\sum_{n_1'=0}^{n_r-1}\Pi_{n_1'}\,H_{n_1'}(r,u)-\sum_{n_2'=0}^{n_r}\Pi_{n_2'}\,H_{n_2'}(r,u)\right)\;+\;c.c\right\}\quad(31b)$$

where the definitions of all the quantities in equations (31) that have not yet been given are collected in Appendix C, then $R_1^{\tilde{\ell}}$ becomes:

$$R_1^{\tilde{\ell}} = 2(CU_n)^2(1+\varepsilon_n)(1+\varepsilon)(2p_n)^{2\gamma_s+2\gamma-2}\, y^{2\gamma}\left(\frac{\Gamma(2\gamma+1)}{|\Gamma(\gamma+i\nu)|^2}\right)^2$$

$$\times\sum_{n_1,n_1'=0}^{n_r-1}\sum_{n_2,n_2'=0}^{n_r}\sum_{i,j=1}^{2}(-1)^{i+j}\,\frac{\Pi_{n_i}\,\Pi_{n_j'}}{(2p_n)^{n_i+n_j'}}\,\left\{\int_0^1 dt\int_0^1 du\;\wedge(t)\;\wedge(u)\;\;I^{n_in_j'}(t,u)\right.$$

$$\left.+\int_0^1 dt\int_0^1 du\;\wedge(t)\;\wedge^*(u)\;\;J^{n_in_j'}(t,u)\;+\;c.c\right\},\qquad(32a)$$

and similarly:

$$R_2^{\tilde{\ell}} = -2(CU_n)^2(1+\varepsilon_n)(-1+\varepsilon)(2p_n)^{2\gamma_s+2\gamma-2}\, y^{2\gamma}\left(\frac{\Gamma(2\gamma+1)}{|\Gamma(\gamma+i\nu)|^2}\right)^2$$

$$\times\sum_{n_1,n_1'=0}^{n_r-1}\sum_{n_2,n_2'=0}^{n_r}\sum_{i,j=1}^{2}(-1)^{i+j}\,\frac{\Pi_{n_i}\,\Pi_{n_j'}}{(2p_n)^{n_i+n_j'}}\,\left\{\int_0^1 dt\int_0^1 du\;\wedge(t)\;\wedge(u)\;\;I^{n_in_j'}(t,u)\right.$$

$$\left.-\int_0^1 dt\int_0^1 du\;\wedge(t)\;\wedge^*(u)\;\;J^{n_in_j'}(t,u)\;+\;c.c\right\},\qquad(32b)$$

and where[14]:

$$I^{n_in_j'}(t,u) = \int_0^\infty G_{n_i}(r,t)\;H_{n_j'}(r,u)\;dr$$

$$=\frac{1}{\mu_i}\frac{\Gamma(\mu_i+\nu_j)}{[(a+b)p_n]^{\mu_i+\nu_j}}\;{}_2F_1(1,\mu_i+\nu_j,\mu_i+1;\frac{a}{a+b}),\qquad(33a)$$

and:

$$J^{n_in_j'}(t,u) = \int_0^\infty G_{n_i}(r,t)\;H^*_{n_j'}(r,u)\;dr$$

$$=\frac{1}{\mu_i}\frac{\Gamma(\mu_i+\nu_j)}{[(a+b^*)p_n]^{\mu_i+\nu_j}}\;{}_2F_1(1,\mu_i+\nu_j,\mu_i+1;\frac{a}{a+b^*}).\qquad(33b)$$

For the sake of convenience in carrying out the Mellin transformation, to be introduced shortly, we isolate the $y-$ dependence in $R_1^{\tilde{\ell}}$ and $R_2^{\tilde{\ell}}$ and insert into them the explicit expression for the constant C from Appendix A. Thus (32a) and (32b) take the form:

$$R_1^{\tilde{\ell}} = \frac{U_n^2}{2\pi(2p_n)^3}(1+\varepsilon_n)(1+\varepsilon)\,\frac{m}{p}\,\frac{e^{\pi\nu}}{|\Gamma(\gamma+i\nu)|^2}\sum_{n_1,n_1'=0}^{n_r-1}\sum_{n_2,n_2'=0}^{n_r}\sum_{i,j=1}^{2}\frac{(-1)^{i+j}}{\mu_i}$$

$$\times\frac{\Pi_{n_i}\,\Pi_{n_j'}}{(2p_n)^{n_i+n_j'}}\,\left\{\int_0^1 dt\int_0^1 du\;\vee(t)\;\vee^*(u)\;\;f_1(y)\right.$$

$$\left.+\left(\frac{i\nu m/E-\kappa}{\gamma+i\nu}\right)\int_0^1 dt\int_0^1 du\;\vee(t)\;\vee(u)\;f_2(y)\;+\;c.c\right\},\quad(34a)$$

and:

$$R_2^{\tilde{\ell}} = \frac{U_n^2}{2\pi(2p_n)^3}(1+\varepsilon_n)(-1+\varepsilon)\frac{m}{p}\,\frac{e^{\pi\nu}}{|\Gamma(\gamma+i\nu)|^2}\sum_{n_1,n_1'=0}^{n_r-1}\sum_{n_2,n_2'=0}^{n_r}\sum_{i,j=1}^{2}\frac{(-1)^{i+j}}{\mu_i}$$

$$\times\frac{\Pi_{n_i}\,\Pi_{n_j'}}{(2p_n)^{n_i+n_j'}}\,\left\{\int_0^1 dt\int_0^1 du\;\vee(t)\;\vee^*(u)\;f_1(y)\right.$$

$$\left.-\left(\frac{i\nu m/E-\kappa}{\gamma+i\nu}\right)\int_0^1 dt\int_0^1 du\;\vee(t)\;\vee(u)\;f_2(y)+\;c.c\right\},\quad(34b)$$

where:

$$f_1(y) = y^{2\gamma}\;\;\Gamma(\mu_i+\nu_j)(\frac{2}{a+b^*})^{\mu_i+\nu_j}\;{}_2F_1(1,\mu_i+\nu_j,\mu_i+1;\frac{a}{a+b^*})\qquad(35a)$$

$$f_2(y) = y^{2\gamma}\;\;\Gamma(\mu_i+\nu_j)(\frac{2}{a+b})^{\mu_i+\nu_j}\;{}_2F_1(1,\mu_i+\nu_j,\mu_i+1;\frac{a}{a+b})\qquad(35b)$$

Note that all the t and u dependence is contained in the $\vee$'s as well as in the functions f_1 and f_2 through the quantities a and b the definitions of which we give in Appendix C. We devote the next subsection to the evaluation of the integral over the infinite set of continuous energy levels using the Mellin transform technique.

(c) **Finiteness of the Energy Integral:**

The sum (integration) over the set of final states in the continuum will now be regularized using the Mellin transform technique. We shall show that the integral is well-behaved at both of its infrared and ultraviolet ends. The advantage of using this method is that it enables us to write the final finite result in terms of the sum of the residues at the physical singularities in the $s-$plane, where s is the Mellin variable to be introduced below. In brief, the procedure consists of isolating part of the $y-$dependence, Mellin transforming it and finally carrying out the inverse transformation. The Mellin transform of $f(y)$ is given by:

$$\tilde{f}(s) = \int_0^\infty y^{s-1}\, f(y)\, dy, \tag{36}$$

where y is the energy variable $y = p/p_n$ and we transform the functions $f_1(y)$ and $f_2(y)$. With suitable substitutions in equation (D4) of Appendix D, the Mellin transforms of f_1 and f_2 will, respectively, be given by:

$$\tilde{f}_1(s) = [i(1-t-u)]^{-(2\gamma+s)}\Gamma(2\gamma+s)\Gamma(\mu_i+\nu_j-2\gamma-s)$$

$$\times\ F_1(2\gamma+s, \mu_i+\nu_j-2\gamma-s, 2\gamma+s, \mu_i+1; \frac{1}{2}, \frac{1}{2}\frac{1-2t}{1-t-u}) \tag{37a}$$

$$\tilde{f}_2(s) = [i(u-t)]^{-(2\gamma+s)}\Gamma(2\gamma+s)\Gamma(\mu_i+\nu_j-2\gamma-s)$$

$$\times\ F_1(2\gamma+s, \mu_i+\nu_j-2\gamma-s, 2\gamma+s, \mu_i+1; \frac{1}{2}, \frac{1}{2}\frac{1-2t}{u-t}) \tag{37b}$$

where F_1 is an Appell hypergeometric function[14,15] of two variables. The region of convergence of $F_1(\alpha, \beta, \beta', \gamma; x_1, x_2)$ is defined by:

$$|x_1| < 1 \qquad \& \qquad |x_2| < 1$$

from which follows that the Appell hypergeometric functions in equations (37a) and (37b) are divergent along the lines defined by:

$$t+u = 1 \qquad \& \qquad t-u = 0,$$

respectively. We shall show below, however, that this difficulty can and will be removed as a byproduct of our treatment of the spurious singularities in the s-plane. With the radial matrix elements written in terms of the inverse Mellin transforms of $\tilde{f}_1$ and $\tilde{f}_2$ our Lamb shift formula now becomes:

$$\Delta E_n^{LS} = \frac{\alpha}{8\pi}\sum_{\tilde{\ell}\tilde{m}} \frac{(2J_n+1)(2J_s+1)}{2\tilde{\ell}+1}\frac{U_n^2}{(2p_n)^3}(1+\varepsilon_n)$$

$$\times \sum_{n_1,n_1'=0}^{n_r-1}\ \sum_{n_2,n_2'=0}^{n_r}\ \sum_{i,j=1}^{2} \frac{(-1)^{i+j}}{\mu_i}\,\frac{\Pi_{n_i}\,\Pi_{n_j'}}{(2p_n)^{n_i+n_j'}}\int dE\,\frac{m}{p}\,\frac{e^{\pi\nu}}{|\Gamma(\gamma+i\nu)|^2}$$

$$\times \Big\{\Big((1+\varepsilon)\,|W_{ns}^{\tilde{\ell}\tilde{m}}|^2 \;-\; (-1+\varepsilon)\,|\mathbf{K}_{ns'}^{\tilde{\ell}\tilde{m}}|^2\Big)\;J^{(1)}$$

$$+\Big(\frac{i\nu m/E-\kappa}{\gamma+i\nu}\Big)\Big((1+\varepsilon)\,|W_{ns}^{\tilde{\ell}\tilde{m}}|^2 \;+\; (-1+\varepsilon)\,|\mathbf{K}_{ns'}^{\tilde{\ell}\tilde{m}}|^2\Big)\;J^{(2)} + c.c\Big\} \tag{38}$$

where:

$$J^{(1)} = \int_0^1 dt \int_0^1 du\, \vee(t)\, \vee^*(u)\, M^{-1}(\tilde{f}_1) \tag{39a}$$

$$J^{(2)} = \int_0^1 dt \int_0^1 du\, \vee(t)\, \vee(u)\, M^{-1}(\tilde{f}_2) \tag{39b}$$

Here $M^{-1}(\tilde{f}_i)$ is the inverse Mellin transform of $\tilde{f}_i$ in a suggestive notation. In equation (38) the integration over the continuum of energy levels is understood to run from $-\infty$ to $-m$ and from m to ∞. Also, the squared angular matrix elements, $|W_{ns}^{\tilde{\ell}\tilde{m}}|^2$ and $|\mathbf{K}_{ns'}^{\tilde{\ell}\tilde{m}}|^2$, are understood to be summed over the total magnetic quantum numbers M_n, and M_s.

Next we study $J^{(1)}$ and $J^{(2)}$ in some detail. In what follows $\tau \equiv \mu_i + \nu_j$ and the subscript on μ will be temporarily dropped, for convenience. With the help of the definitions collected in Appendix C, $J^{(1)}$ becomes:

$$J^{(1)} = \frac{1}{2\pi i}\int_{-i\infty+C_s}^{i\infty+C_s} ds\; i^{-(2\gamma+s)} y^{-s}\Gamma(2\gamma+s)\Gamma(\tau-2\gamma-s)$$

$$\times \int_0^1 dt \int_0^1 du\; (1-t-u)^{-2\gamma-s} F_1(2\gamma+s, \tau-2\gamma-s, 2\gamma+s, \mu+1; x, y)$$

$$\times t^{\gamma+i\nu}(1-t)^{\gamma-i\nu-1} u^{\gamma-i\nu}(1-u)^{\gamma+i\nu-1},$$

where $x = \frac{1}{2}$ and $y = \frac{1}{2}\frac{1-2t}{1-t-u}$. We shall make use now of the integral representation[16] of the Appell hypergeometric function:

$$F_1(\alpha,\beta,\beta',\gamma;x,y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 dv v^{\alpha-1}(1-v)^{\gamma-\alpha-1}(1-vx)^{-\beta}(1-vy)^{-\beta'}$$

$$Re(\alpha) > 0, \quad Re(\gamma-\alpha) > 0$$

in order to make possible a factorization of the t and u dependences. Hence $J^{(1)}$ becomes:

$$J^{(1)} = \frac{1}{2\pi i}\int_{-i\infty+C_s}^{i\infty+C_s} ds\; i^{-(2\gamma+s)} y^{-s}\;\frac{\Gamma(\mu+1)\Gamma(\tau-2\gamma-s)}{\Gamma(\mu+1-2\gamma-s)}$$

$$\times \int_0^1 dt \int_0^1 du \int_0^1 dv\; t^{\gamma+i\nu}\;(1-t)^{\gamma-i\nu-1}\;u^{\gamma-i\nu}\;(1-u)^{\gamma+i\nu-1}\;v^{2\gamma+s-1}$$

$$\times (1-v)^{\mu-2\gamma-s}\;(1-\frac{v}{2})^{-\tau+2\gamma+s}\;\left(1-\frac{v}{2}\;\frac{1-2t}{1-t-u}\right)^{-2\gamma-s}\;(1-t-u)^{-2\gamma-s}$$

The last two factors in the previous expression can be written as:

$$\left(1-\frac{v}{2}\frac{1-2t}{1-t-u}\right)^{-2\gamma-s}(1-t-u)^{-2\gamma-s} = \left(1-t-u-\frac{v}{2}(1-2t)\right)^{-2\gamma-s}$$

$$= (1-u)^{-2\gamma-s}\left(1+\frac{vt-t-v/2}{1-u}\right)^{-2\gamma-s}$$

We complete the factorization by employing, twice, the inverse Mellin transform of the beta function[16], namely:

$$(1+x)^{-\lambda} = \frac{1}{2\pi i}\int_{-i\infty+C_s}^{i\infty+C_s} dz\; x^{-z}B(z,\lambda-z), \quad 0 < Re(z) < Re(\lambda) \qquad (40)$$

After this has been done the t, u and $v-$ integrations factor out and can easily be evaluated in terms of gamma functions. We therefore write $J^{(1)}$ in the following form:

$$J^{(1)} = \frac{1}{2\pi i}\int_{-i\infty+C_s}^{i\infty+C_s} ds\; i^{-(2\gamma+s)}\; y^{-s}\frac{\Gamma(\mu+1)\Gamma(\tau-2\gamma-s)}{\Gamma(\mu+1-2\gamma-s)\Gamma(2\gamma+s)}$$

$$\times \frac{1}{2\pi i}\int_{-i\infty+C_z}^{i\infty+C_z} dz\; (-1)^{-z}\Gamma(2\gamma+s-z)\frac{1}{2\pi i}\int_{-i\infty+C_w}^{i\infty+C_w} dw\; 2^w\Gamma(w)\Gamma(z-w)$$

$$\times \int_0^1 dt\; t^{\gamma+i\nu+w-s}(1-t)^{\gamma-i\nu-1}\int_0^1 du\; u^{\gamma-i\nu}(1-u)^{-\gamma+i\nu-s+z-1}$$

$$\times \int_0^1 dv v^{2\gamma+s-w-1}(1-v)^{\mu-2\gamma-s+w-z}(1-\frac{v}{2})^{-\tau+2\gamma+s} \qquad (41)$$

where:

$$\text{The t}- \text{ integration} = \frac{\Gamma(\gamma+i\nu+w-z+1)\Gamma(\gamma-i\nu)}{\Gamma(2\gamma+w-z+1)},$$

$$\text{The u}- \text{ integration} = \frac{\Gamma(\gamma-i\nu+1)\Gamma(z+i\nu-\gamma-s)}{\Gamma(z+1-s)},$$

and:

$$\text{The v}- \text{ integration} = \frac{\Gamma(2\gamma+s-w)\Gamma(\mu-2\gamma-s+w-z+1)}{\Gamma(\mu-z+1)}$$

$$\times\; {}_2F_1(\tau-2\gamma-s, 2\gamma+s-w, \mu-z+1; \frac{1}{2})$$

We also make the following transformation which will prove to be useful for eliminating the factor $e^{\pi\nu}$ from our Lamb shift formula. If we let $z \to z - i\nu$, which obviously does not involve any change in the contour of integration, then:

$$(-1)^{-z} \to (-1)^{-z}(-1)^{i\nu} = (-1)^{-z}e^{i\pi(i\nu)} = (-1)^{-z}e^{-\pi\nu}$$

Finally, $J^{(1)}$ can be put into the following compact form:

$$J^{(1)} = \Gamma(\mu+1)\Gamma(\gamma-i\nu)\Gamma(\gamma-i\nu+1)\;e^{-\pi\nu}\;\frac{1}{2\pi i}\int_{-i\infty+C_s}^{i\infty+C_s} ds\; K(s)$$

$$\times \frac{1}{2\pi i}\int_{-i\infty+C_z}^{i\infty+C_z} dz\; L(s,z)\;\frac{1}{2\pi i}\int_{-i\infty+C_w}^{i\infty+C_w} dw\; M(s,z,w)$$

where:

$$K(s) = y^{-s}\ \frac{\Gamma(\tau - 2\gamma - s)}{\Gamma(2\gamma + s)\Gamma(\mu + 1 - 2\gamma - s)}$$

$$L(s,z) = (-1)^{-s}\ \frac{\Gamma(s + 2\gamma + i\nu - z)\Gamma(-s - \gamma + z)}{\Gamma(-s + 1 - i\nu + z)\Gamma(\mu + 1 + i\nu - z)}\ i^{-(2\gamma+s)}$$

$$M(s,z,w) = 2^w \frac{\Gamma(-s + \mu - 2\gamma + i\nu - z + 1 + w)\Gamma(\gamma + 2i\nu + 1 - z + w)\Gamma(s + 2\gamma - w)}{\Gamma(2\gamma + i\nu + 1 - z + w)}$$

$$\times\ \Gamma(w)\Gamma(-i\nu + z - w)\,{}_2F_1(\tau - 2\gamma - s, 2\gamma + s - w, \mu - z + i\nu + 1; \frac{1}{2})$$

A similar procedure also yields:

$$J^{(2)} = \Gamma(\mu + 1)\Gamma(\gamma - i\nu)\Gamma(\gamma - i\nu)\ e^{-\pi\nu}\ \frac{1}{2\pi i}\int_{-i\infty+C_s}^{i\infty+C_s} ds\ K(s)$$

$$\times\ \frac{1}{2\pi i}\int_{-i\infty+C_s}^{i\infty+C_s} dz\ (-\gamma - s + z)\ L(s,z)\ \frac{1}{2\pi i}\int_{-i\infty+C_w}^{i\infty+C_w} dw\ M(s,z,w)$$

In order to study the infra-red and ultraviolet behavior of the energy shift, we are going to make the change of variable: $x = E/p$. In terms of the new variable, we find that:

$$\int_m^\infty dE \to \int_1^\infty \frac{p dx}{x^2 - 1}$$

$$y \to \frac{N_n/Z\alpha}{\sqrt{x^2 - 1}}$$

$$E/m \to \frac{x}{\sqrt{x^2 - 1}}$$

$$\nu \to Z\alpha x \qquad (42)$$

and the energy (or x)- dependent part of equation (38) becomes:

$$A = m\int_1^\infty \frac{dx}{x^2 - 1}\ \frac{e^{\pi Z\alpha x}}{|\Gamma(\gamma + iZ\alpha x)|^2}\ \left\{\left[(1 + \frac{x}{\sqrt{x^2 - 1}})|W_{ns}^{\tilde{l}\tilde{m}}|^2\right.\right.$$

$$\left. -(-1 + \frac{x}{\sqrt{x^2 - 1}})|\mathbf{K}_{ns'}^{\tilde{l}\tilde{m}}|^2\right] J^{(1)}(x) + (\frac{iZ\alpha\sqrt{x^2 - 1} - \kappa}{\gamma + iZ\alpha x})$$

$$\left. \times\ \left[(1 + \frac{x}{\sqrt{x^2 - 1}})|W_{ns}^{\tilde{l}\tilde{m}}|^2 + (-1 + \frac{x}{\sqrt{x^2 - 1}})|\mathbf{K}_{ns'}^{\tilde{l}\tilde{m}}|^2\right] J^{(2)}(x) + c.c\right\} \qquad (43)$$

together with a similar expression, except of course for the limits of integration, that will take care of the contribution from the negative energy continuum. A can be greatly simplified by making the following two observations:

(a) The first observation concerns the dependence of A upon the Dirac quantum number κ. This dependence is explicit in the presence of κ in the second term and is implicit elsewhere only through $\gamma = \sqrt{|\kappa|^2 - (Z\alpha)^2}$. Since ultimately we are going to sum our final result for the Lamb shift over this quantum number, the contributions from $+\kappa$, and $-\kappa$ of the term proportional to κ will cancel out.

(b) The second simplifying remark is about the $x-$ dependence. We adopt the interpretation of the negative energy solutions to the Dirac equation as the scattering states of the positron. Thus, for an electron, the Coulomb potential is given by (+ for positive energy):

$$V_+ = \frac{Ze_1e_2}{r} = -\frac{Ze^2}{r} = -\frac{4\pi Z\alpha}{r},$$

while for a positron, we have:

$$V_- = \frac{4\pi Z\alpha}{r}$$

This formally amounts to letting $\alpha \to -\alpha$ in the electron's potential. As a direct consequence of this symmetry, while the positive energy (electron) solutions have $\nu_+ = Z\alpha E/p \equiv \nu$, the negative energy (positron) solutions have a ν given by:

$$\nu_- = (-Z\alpha)(-E)/p = Z\alpha E/p = \nu$$

Therefore, $J^{(1)}(x)$ and $J^{(2)}(x)$ in the positive energy part of A are exactly the same as their counterparts belonging to the negative energy piece. Consequently, contributions from $+x$ and $-x$ of all terms in A that are proportional to x will likewise cancel.

With the aforementioned two remarks taken into account, A finally reduces to:

$$A = 2m\int_1^\infty \frac{dx}{x^2-1}\,\frac{e^{\pi Z\alpha x}}{|\Gamma(\gamma+iZ\alpha x)|^2}\,\left\{\left(|W^{\tilde{\ell}\tilde{m}}_{ns}|^2+|\mathbf{K}^{\tilde{\ell}\tilde{m}}_{ns'}|^2\right)J^{(1)}(x)\right.$$
$$\left.+\frac{iZ\alpha\sqrt{x^2-1}}{\gamma+iZ\alpha x}\left(|W^{\tilde{\ell}\tilde{m}}_{ns}|^2-|\mathbf{K}^{\tilde{\ell}\tilde{m}}_{ns'}|^2\right)J^{(2)}(x)+c.c\right\} \qquad (44)$$

where the integration is now only over positive x. More conveniently, we let:

$$J^{(i)}(x)=\Gamma(\mu+1)\Gamma(\gamma-iZ\alpha x)e^{-\pi Z\alpha x}\,\frac{1}{2\pi i}\int_{-i\infty+C_s}^{i\infty+C_s} ds\,K(s)\,\tilde{J}^{(i)}(x,s), \qquad i=1,2 \quad (45)$$

and write A in the form:

$$A = 2m\,\Gamma(\mu+1)\,\frac{1}{2\pi i}\int_{-i\infty+C_s}^{i\infty+C_s} ds\,\left(\frac{Z\alpha}{N_n}\right)^s\,\frac{\Gamma(\tau-2\gamma-s)}{\Gamma(2\gamma+s)\Gamma(\mu+1-2\gamma-s)}$$
$$\times\int_1^\infty dx\,(x^2-1)^{\frac{s}{2}-1}\,\frac{\Gamma(\gamma-iZ\alpha x+1)}{\Gamma(\gamma+iZ\alpha x)}\,\left\{\left(|W^{\tilde{\ell}\tilde{m}}_{ns}|^2+|\mathbf{K}^{\tilde{\ell}\tilde{m}}_{ns'}|^2\right)\tilde{J}^{(1)}(s,x)\right.$$
$$\left.+\frac{iZ\alpha\sqrt{x^2-1}}{\gamma+iZ\alpha x}\left(|W^{\tilde{\ell}\tilde{m}}_{ns}|^2-|\mathbf{K}^{\tilde{\ell}\tilde{m}}_{ns'}|^2\right)\tilde{J}^{(2)}(s,x)+c.c\right\} \qquad (46)$$

In equation (46) we distinguish between two kinds of singularities in the s- plane:

(a) The fixed (spurious) poles coming from the end points of integration[17] over t, u and v at $t, u, v = 1$ in equation (41) and which occur at $s = 0, 1, 2, \cdots$ and $s = 1, 2, \cdots$ in $J^{(1)}$ and $J^{(2)}$, respectively. These singularities can be removed by 1;0coducing a convergence factor $e^{-\epsilon p r}$ in the radial integrals of equations (30) and (33). The net result of including this factor is to replace a and b there by $1+iy(1-2t-i\epsilon)$ and $1+iy(1-2u-i\epsilon)$, respectively. As a byproduct

of this prescription, the divergences in the Appell hypergeometric functions no longer occur. Finally, the contour of integration over the Mellin variable can be deformed in such a way as to make the final result independent of ϵ.

(b) The only other kind of singularity we have to worry about is physical. As is well known, the gamma function is singular at all points that make its argument equal to zero or a negative integer. Thus, the function $\Gamma(\tau-2\gamma-s)$ has poles at:

$$s=\mu_i+\nu_j-2\gamma+r\equiv s_r, \qquad r=0,1,2,\cdots \qquad (47)$$

contributing the residue:

$$Res(s=s_r)=\frac{(-1)^r}{r!} \qquad (48)$$

at each pole.

As a function of the sum of the residues at the physical poles, A now becomes:

$$A = 2m\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\,\left(\frac{Z\alpha}{N_n}\right)^{s_r}\,\frac{\Gamma(\mu+1)}{\Gamma(\tau+r)\Gamma(1-\nu-r)}$$
$$\times\int_1^\infty dx\,(x^2-1)^{\frac{s_r}{2}-1}\,\frac{\Gamma(\gamma-iZ\alpha x+1)}{\Gamma(\gamma+iZ\alpha x)}\,\left\{\left(|W^{\tilde{\ell}\tilde{m}}_{ns}|^2+|\mathbf{K}^{\tilde{\ell}\tilde{m}}_{ns'}|^2\right)\tilde{J}^{(1)}(s_r,x)\right.$$
$$\left.+\frac{iZ\alpha\sqrt{x^2-1}}{\gamma+iZ\alpha x}\left(|W^{\tilde{\ell}\tilde{m}}_{ns}|^2-|\mathbf{K}^{\tilde{\ell}\tilde{m}}_{ns'}|^2\right)\tilde{J}^{(2)}(s_r,x)+c.c\right\} \qquad (49)$$

When the final form of the $x-$ dependence, equation (49), is put back into equation (38) the shift in the nth energy level to all orders in $(Z\alpha)$ will be given by:

$$\Delta E_n^{LS} = \frac{\alpha}{4\pi} \sum_{\tilde{\ell}\tilde{m}} \frac{(2J_n+1)(2J_s+1)}{2\tilde{\ell}+1} \frac{U_n^2}{(2p_n)^3}(1+\epsilon_n)$$

$$\times \sum_{n_1,n_1'=0}^{n_r-1} \sum_{n_2,n_2'=0}^{n_r} \sum_{i,j=1}^{2} \frac{(-1)^{i+j}}{\mu_i} \frac{\Pi_{n_i}\,\Pi_{n_j'}}{(2p_n)^{n_i+n_j'}} m \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{Z\alpha}{N_n}\right)^{4r}$$

$$\times \frac{\Gamma(\mu_i+1)}{\Gamma(\mu_i+\nu_j+r)\Gamma(1-\nu_j-r)} \int_1^\infty dx\,(x^2-1)^{\frac{\mu}{2}-1} \frac{\Gamma(\gamma-iZ\alpha x+1)}{\Gamma(\gamma+iZ\alpha x)}$$

$$\times \left\{ \left(|W_{ns}^{\tilde{\ell}\tilde{m}}|^2 + |\mathrm{K}_{ns'}^{\tilde{\ell}\tilde{m}}|^2\right) \tilde{J}^{(1)}(x) + \frac{iZ\alpha\sqrt{x^2-1}}{\gamma+iZ\alpha x} \left(|W_{ns}^{\tilde{\ell}\tilde{m}}|^2 - |\mathrm{K}_{ns'}^{\tilde{\ell}\tilde{m}}|^2\right) \tilde{J}^{(2)}(x) + c.c \right\} \tag{50}$$

We also state, in what follows, the main results of the calculation, via contour integration, of the quantities $\tilde{J}^{(i)}(x), i = 1$ and 2, introduced above in equations (45) and (46). Most of the details of this calculation can be found in Appendix E.

$$\tilde{J}^{(1)}(x) = -\sum_{n=0}^{\infty} \frac{(-1)^{-(\frac{3}{2}\tau+\frac{1}{2}r+i\nu)}}{n!} \frac{\Gamma(\gamma+i\nu+n)}{\Gamma(2\gamma+1+n)\Gamma(\mu-\tau+1-r-n)} W^{(1)} \tag{51}$$

where:

$$W^{(1)} = -\sum_{m=0}^{m_1} 2^{\tau-\gamma-i\nu-1+r+n-m}\,\Pi_{k=0}^{n-1}(k-\gamma-1-i\nu+n-m) \frac{(-1)^m}{m!}$$

$$\times \frac{\Gamma(\mu-\gamma-\tau-i\nu-r-m)\Gamma(\tau-\gamma-1-i\nu+r+n-m)}{\Gamma(\gamma-i\nu-m)}$$

$$\times \Gamma^2(\gamma+1+i\nu-n+m)$$

$$\times {}_2F_1(-r,\gamma+1+i\nu-r-n+m,\mu-\tau-n+1;\frac{1}{2})$$

$$-\sum_{m=0}^{m_2} 2^{2\tau-\mu-1+2r+n-m}\,\Pi_{k=0}^{n-1}(k-\tau+\mu+1-r-n+m)$$

$$\times \Gamma(\tau+\gamma-\mu+i\nu+r-m)$$

$$\times \frac{(-1)^m}{m!} \frac{\Gamma(2\tau-\mu-1+2r+n-m)\Gamma^2(-\tau+\mu+1-r-n+m)}{\Gamma(\tau+2\gamma-\mu+r-m)}$$

$$\times {}_2F_1(-r,-\tau+\mu+1-r-n+m,\mu-\tau-r-n+1;\frac{1}{2})$$

$$-\sum_{m=0}^{\infty} \frac{2^{\tau+r+m+1}}{(m!)^2} \left(\sum_{k=1}^{m}\frac{1}{k}-\mathrm{C}\right) \Pi_{k=1}^{n-1}(k-m)$$

$$\times \frac{\Gamma(\tau+r+m)\Gamma(-\tau+\mu+1-r-n+m)\Gamma(\gamma+1+i\nu-n+m)}{\Gamma(2\gamma+1-n+m)}$$

$$\times {}_2F_1(-r,-m,-\tau+\mu-r-n+1;\frac{1}{2}) \tag{52}$$

and where m_1 = the integer $< -\gamma-1+n$ and m_2 = the integer $< \tau-\mu-1+r+n$ and $\mathrm{C} = 0.577215\cdots$ is the Euler constant, customarily denoted by γ. Moreover:

$$\tilde{J}^{(2)}(x) = -\sum_{n=0}^{\infty} \frac{(-1)^{-(\frac{3}{2}\tau+\frac{1}{2}r+i\nu)}}{n!} \frac{\Gamma(\gamma+i\nu+n+1)}{\Gamma(2\gamma+1+n)\Gamma(\mu-\tau+1-r-n)} W^{(2)} \tag{53}$$

where:

$$W^{(2)} = W^{(1)} \tag{54}$$

By reference to Appendix C, we can immediately show that the lowest pole in the $s-$ plane (r=0) occurs at $s_0 = 2\gamma_n+1 \approx 3$ ($Z\alpha \ll 1$). Poles corresponding to $r > 0$,

therefore, occur at points along the real $s-$ axis to the right of $s = 3$. Consequently, $\frac{s_r}{2} - 1$ is positive at all of the physical poles causing the $x-$ integrand in equation (49) to vanish as $x \to 1$ (the ultraviolet end). On the other hand, as $x \to \infty$ (the infrared limit), the integrand in equation (49) can easily be shown, with the help of equations (51) and (52), to go roughly as:

$$\approx x^{s_r-2} \frac{\Gamma(-iZ\alpha x)\Gamma(2iZ\alpha x)}{\Gamma(iZ\alpha x)}$$

$$\approx x^{s_r-3} e^{-2\pi Z\alpha x}, \tag{55}$$

apart from some constant multiplicative factors. For the first few poles in the $s-$ plane, the result expressed in equation (55) tends asymptotically to zero. Thus we have shown that the energy integral, or the sum over the continuum of energy states, in our Lamb shift formula suffers from no infrared or ultraviolet divergences. So no cutoffs need to be introduced and the mass renormalization is not necessary either.

IV. DISCUSSION AND CONCLUSIONS

Within the framework of the new formalism of Quantum Electrodynamics based on self energy, we have derived a formula for the atomic energy level shift, the Lamb shift, applicable to the Hydrogen isoelectronic sequence. The fully relativistic Dirac-Coulomb wavefunctions have been used and the spin calculation and all the radial integrals have been evaluated exactly analytically. The final formula we got is an expansion in powers of $(Z\alpha)$ and contains an energy integral which remains to be carried out.

A substantial amount of work still needs to be done before any numbers can be extracted from our Lamb shift formula, equation (50), for comparison with the

conventional theory as well as with experiment. First of all, the energy or $x-$ integral has to be carried out, perhaps numerically. We have shown in the last part of section III of this paper that the energy integral is well-behaved at both of its infrared and ultraviolet ends.

Secondly, for the sake of investigating the ultraviolet behavior of the energy integral, the quantities $\tilde{J}^{(i)}(x)$ have been evaluated via contour integration. Other ways of evaluating these quantities in closed numerical form, instead of the sums over the residues may be possible.

Finally, the values of the angular matrix elements, the $W^{\tilde{\ell}\tilde{m}}_{ns}$'s and $\mathbf{K}^{\tilde{\ell}\tilde{m}}_{ns'}$'s, depend upon the particular states n and s under investigation. Related to this is the sum over the indices $\tilde{\ell}$ and $\tilde{m}$. This one has been shown to be finite in reference [3] for every specific example, the lower and upper limits being given by equations (28b) and (29c).

In conclusion, we draw attention to the following evident feature of our result. The dependence upon $(Z\alpha)$ of our equation shows up explicitly as $\frac{\alpha}{2\pi}(Z\alpha)^{s_r}$ times some integrals that depend upon $(Z\alpha)$, where s_r has been shown to be of the order of magnitude of 3 or larger. The leading term in all the standard calculations of the Lamb shift goes like $\alpha(Z\alpha)^4$. All other corrections[19] go as α times some power of $(Z\alpha)$ that is larger than 4. Thus the expansion in our formula should correspond to that of the standard theory only after the remaining integrals have been carried out.

APPENDIX A

The Dirac-Coulomb Wavefunctions

In spinor language, the solutions to the Dirac equation for an electron in a

Coulomb field are usually denoted by[18]:

$$\psi_n(\mathbf{x}) = \begin{pmatrix} g_n(r)\Omega_n(\hat{r}) \\ if_n(r)\Omega_{n'}(\hat{r}) \end{pmatrix} \tag{A1}$$

We distinguish between the following two cases:

(1) *The discrete spectrum* ($|E_n| < m$):

$$\begin{aligned} g_n(r) &= \sqrt{1+\frac{E_n}{m}}U_n(A_n - B_n) \\ f_n(r) &= -\sqrt{1-\frac{E_n}{m}}U_n(A_n + B_n), \end{aligned} \tag{A2}$$

where:

$$U_n = \frac{(2p_n)^{\frac{3}{2}}}{\Gamma(2\gamma_n+1)}\left[\frac{\Gamma(2\gamma_n+n_r+1)}{4N_n(N_n-\kappa_n)n_r!}\right]^{\frac{1}{2}}$$

$$\begin{aligned} A_n(r) &= n_r F(-n_r+1, 2\gamma_n+1; 2p_n r)e^{-p_n r}(2p_n r)^{\gamma_n-1} \\ B_n(r) &= (N_n-\kappa_n)F(-n_r, 2\gamma_n+1; 2p_n r)e^{-p_n r}(2p_n r)^{\gamma_n-1} \end{aligned} \tag{A3}$$

$$F(-n,b;z) = \sum_{m=0}^{n}\frac{(-n)_m}{(b)_m}\frac{z^m}{m!}, \quad (a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}, \quad (a)_0 \equiv 1. \tag{A4}$$

$$\begin{aligned} p_n &= \frac{Z\alpha m}{N_n} \\ N_n &= [n^2 - 2n_r(|\kappa_n| - \gamma_n)]^{\frac{1}{2}} \\ E_n^2 &= -p_n^2 + m^2 \\ \gamma_n &= [\kappa_n^2 - (Z\alpha)^2]^{\frac{1}{2}} \\ n_r &= n - |\kappa_n| \end{aligned}$$

$$\kappa_n = \begin{cases} -(\ell_n+1), & \text{if } J_n = \ell_n + \frac{1}{2}; \\ \ell_n, & \text{if } J_n = \ell_n - \frac{1}{2}. \end{cases}$$

(2) *The continuous spectrum* ($|E| > m$):

$$\begin{aligned} g(r) &= C\sqrt{1+\frac{E}{m}}\,\{S(r) + S^*(r)\} \\ f(r) &= iC\sqrt{-1+\frac{E}{m}}\,\{S(r) - S^*(r)\}, \end{aligned} \tag{A5}$$

where:

$$\begin{aligned} C &= \frac{1}{2}\sqrt{\frac{m}{\pi p}}e^{\frac{\pi\nu}{2}}\frac{|\Gamma(\gamma+i\nu)|}{\Gamma(2\gamma+1)} \\ S(r) &= \frac{(2pr)^\gamma}{r}e^{-ipr+i\ell}(\gamma+i\nu)F(\gamma+1+i\nu, 2\gamma+1; 2ipr) \end{aligned}$$

$$F(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}dt, \quad Re(b) > Re(a) > 0. \tag{A6}$$

$$\begin{aligned} E^2 &= p^2 + m^2 \\ \nu &= \frac{Z\alpha E}{p} \\ \gamma &= \sqrt{\kappa^2 - (Z\alpha)^2} \\ e^{2i\ell} &= -\frac{\kappa - i(\frac{\nu m}{E})}{\gamma + i\nu} \end{aligned} \tag{A7}$$

Finally, the angular parts of the wavefunctions are given by:

$$\Omega_n = (-1)^{\frac{1}{2}-\ell_n-M_n}\sqrt{2J_n+1}\sum_{m_n\mu_n}\begin{pmatrix} \ell_n & \frac{1}{2} & J_n \\ m_n & \mu_n & -M_n \end{pmatrix}|\ell_n m_n > \chi_{\mu_n}$$

and $\Omega_{n'}$ can be gotten from Ω_n by letting $\ell_n \to \ell_n'$ and $m_n \to m_n'$. χ_{μ_n} is the usual two- component Pauli spinor.

APPENDIX B

The Effective Potential

In this Appendix we carry out the $k-$ integration and define an effective potential by:

$$V_{\tilde{\ell}}(r,r') = (rr')^2 \int_0^\infty \{\frac{k}{\omega-k} - \frac{k}{\omega+k}\}\ j_{\tilde{\ell}}(kr)\ \ j_{\tilde{\ell}}(kr')\ dk$$

$$= 2(rr')^2 \int_0^\infty \frac{k^2}{\omega^2-k^2}\ \ j_{\tilde{\ell}}(kr)\ \ j_{\tilde{\ell}}(kr')\ dk$$

$$= -\pi(rr')^{\frac{1}{2}}\ \lim_{\epsilon\to 0}\ \int_0^\infty \frac{J_{\tilde{\ell}+\frac{1}{2}}(kr)\ \ J_{\tilde{\ell}+\frac{1}{2}}(kr')}{k^2+(i\omega+\epsilon)^2}\ \ kdk$$

$$= -\pi(rr')^{\frac{1}{2}}\ \ I_{\tilde{\ell}+\frac{1}{2}}(i\omega r_<)\ \ K_{\tilde{\ell}+\frac{1}{2}}(i\omega r_>)$$

$$= -i\frac{\pi^2}{2}(rr')^{\frac{1}{2}}\ \ J_{\tilde{\ell}+\frac{1}{2}}(-\omega r_<)\ \ H^{(1)}_{\tilde{\ell}+\frac{1}{2}}(-\omega r_>)$$

$$= -i\pi\omega(rr')^2\ \ j_{\tilde{\ell}}(\omega r_<)\ \ h^{(2)}_{\tilde{\ell}}(\omega r_>)$$

$$= -\frac{i\pi\omega}{2}(rr')^2\ \ h^{(2)}_{\tilde{\ell}}(\omega r_>)\ \ \{h^{(1)}_{\tilde{\ell}}(\omega r_<) + h^{(2)}_{\tilde{\ell}}(\omega r_<)\} \qquad (B1)$$

For small values of the argument compared with unity, the real part of $h^{(2)}_{\tilde{\ell}}$ can be neglected, making:

$$V_{\tilde{\ell}}(r,r') \approx -i\pi\omega(rr')^2\ \ j_{\tilde{\ell}}(\omega r_<)\ \ (-in_{\tilde{\ell}}(\omega r_>))$$

$$\to \frac{\pi}{2(2\tilde{\ell}+1)}\ r_< r_>\ (\frac{r_<}{r_>})^{\tilde{\ell}} \qquad (B2)$$

where $n_{\tilde{\ell}}$ is a spherical Neumann function of order $\tilde{\ell}$.

APPENDIX C

Some Definitions

Here we collect, for easy reference, the definitions of some of the quantities introduced in the main body of the calculation, section III. Those quantities are:

$$\mu_i = \gamma_n + \gamma + n_i + \tilde{\ell} + 1$$

$$\nu_i = \gamma_n + \gamma + n'_i - \tilde{\ell}, \qquad i = 1,2$$

$$y = p/p_n$$

$$a = 1 + iy(1-2t)$$

$$b = 1 + iy(1-2t')$$

$$\tau_{ij} = \gamma_n + \gamma_s + n_i + s_j + \tilde{\ell} + 1$$

$$\tau_{kl} = \gamma_n + \gamma_s + n'_k + s'_l - \tilde{\ell}$$

and:

$$\wedge(t) = e^{i\xi} \vee (t)$$

$$\vee(t) = t^{\gamma+i\nu}\ \ (1-t)^{\gamma-1-i\nu}$$

$$\Pi_{n_1} = n_r \frac{(-n_r+1)_{n_1}}{(2\gamma_n+1)_{n_1}}\ \frac{(2p_n)^{n_1}}{n_1!}$$

$$\Pi_{n_2} = (N_n - \kappa_n)\frac{(-n_r)_{n_2}}{(2\gamma_n+1)_{n_2}}\ \frac{(2p_n)^{n_2}}{n_2!}$$

$$G_{n_i}(r,t) = \frac{r^{\mu_i}}{\mu_i}\ e^{-ap_n r}\ {}_1F_1(1,\mu_i+1;ap_n r)$$

$$H_{n'_i(r,t')} = r^{\nu_i-1}\ e^{-bp_n r}, \qquad i = 1,2$$

APPENDIX D

The Mellin Transform

The idea is to express the hypergeometric function in $f(y)$ in series form, trans-

form term by term and sum the series again. So, we take:

$$f(y) = \Gamma(\tau)\, y^{\alpha}(\frac{2}{a+b})^{\tau}\ {}_2F_1(1,\tau,\sigma,\frac{a}{a+b})$$

$$= 2^{\tau}\,\Gamma(\tau)\sum_{m=0}^{\infty}\frac{(1)_m(\tau)_m}{(\sigma)_m}\frac{1}{m!}\,y^{\alpha}\,a^m\,(a+b)^{-(\tau+m)}, \qquad (D1)$$

where the Pochhammer symbols are defined in Appendix A. If we further let:

$$a = 1 + iy(1-2t) \equiv 1 + Ay$$

and:

$$a + b = 2 + 2iy(1 - t - t') \equiv 2(1 + By),$$

and make the change of variable $x = Ay$, then the Mellin transform of $f(y)$ will take the form[14]:

$$\tilde{f}(s) = \frac{\Gamma(\tau)}{A^{\alpha+s}}\sum_{m=0}^{\infty}\frac{(1)_m(\tau)_m}{(\sigma)_m m! 2^m}\int_0^{\infty} x^{\alpha+s-1}(1+x)^m(1+\frac{B}{A}x)^{-(\tau+m)}\,dx$$

$$= \frac{\Gamma(\tau)}{A^{\alpha+s}}\sum_{m=0}^{\infty}\frac{(1)_m(\tau)_m}{(\sigma)_m m! 2^m}\,B(\alpha+s,\tau-\alpha-s)\ {}_2F_1(\tau+m,\alpha+s,\tau;1-\frac{B}{A})$$

$$= \frac{\Gamma(\alpha+s)\Gamma(\tau-\alpha-s)}{A^{\alpha+s}}\sum_{m,n=0}^{\infty}\frac{(1)_m(\tau)_m(\tau+m)_n(\alpha+s)_n}{(\sigma)_m(\tau)_n}\,\frac{x_1^m}{m!}\,\frac{x_2^n}{n!} \qquad (D2)$$

In (D2) $x_1 = \frac{1}{2}$ and $x_2 = 1 - \frac{B}{A}$ and the beta function has been expanded in terms of gamma functions. From the properties of the Pochhammer symbols, we can write:

$$(\tau+m)_n = \frac{\Gamma(\tau+m+n)}{\Gamma(\tau+m)}\cdot\frac{\Gamma(\tau)}{\Gamma(\tau)}$$
$$= \frac{(\tau)_{m+n}}{(\tau)_m}$$

And hence (D2) becomes:

$$\tilde{f}(s) = \frac{\Gamma(\alpha+s)\Gamma(\tau-\alpha-s)}{A^{\alpha+s}}\sum_{m,n=0}^{\infty}\frac{(\tau)_{m+n}(1)_m(\alpha+s)_n}{(\sigma)_m(\tau)_n}\frac{(\frac{1}{2})^m}{m!}\frac{(1-\frac{B}{A})^n}{n!}$$

$$= \frac{\Gamma(\alpha+s)\Gamma(\tau-\alpha-s)}{A^{\alpha+s}}\,F_2(\tau,1,\alpha+s,\sigma,\tau;\frac{1}{2},1-\frac{B}{A})$$

$$= \frac{\Gamma(\alpha+s)\Gamma(\tau-\alpha-s)}{B^{\alpha+s}}\,F_1(\alpha+s,\tau-\alpha-s,\alpha+s,\sigma;\frac{1}{2},\frac{A}{2B}) \qquad (D3)$$

In (D3) F_1 and F_2 are Appell hypergeometric functions[14,15] of two variables.

APPENDIX E

In this Appendix we outline the evaluation of the quantities $\tilde{J}^{(i)}, i = 1,2$ defined by equation (30) which are important at this stage for the discussion of the infrared behaviour of the Lamb shift formula. To begin with, one should note that equation (25) imposes the following conditions upon the contours of integration in the variables z and w:

$$0 < Re(z) = C_z < 2\gamma + s_r = \tau + r \qquad (E1)$$

and:

$$0 < Re(w) = C_w < Re(z) = C_z \qquad (E2)$$

In what follows, we show the details of the calculation in the case of $\tilde{J}^{(1)}$, $\tilde{J}^{(2)}$ will then be written down without any additional difficulty. We begin by identifying the singularities in the z- integrand. These are simple poles coming from the gamma functions $\Gamma(s_r + 2\gamma + i\nu - z)$ and $\Gamma(-s_r - \gamma + z)$ and occur at the following two sets of points:

$$A_1 = \{z = \tau + i\nu + r + n\} \qquad (E3)$$

and:

$$A_2 = \{z = \tau - \gamma + r - n\} \qquad (E4)$$

respectively, where $r, n = 0, 1, 2, \cdots$. The first set of poles , A_1, occur at points along a line parallel to the real z- axis starting with $z_0 = \tau + i\nu + r \approx 5 + i\nu + r$. Succeeding singularities are separated by real intervals of length unity and are all to the right of z_0. The second set is distributed in a similar fashion along the real axis starting with $z_0' = \tau - \gamma + r \approx 4 + r$. In this case, however, all members of the set are to the left of z_0'. With the help of (E1) we can move the contour of the $z-$ integration up to a position between z_0 and z_0' and close it in a large semicircle in the right half plane, thus enclosing all poles in the set A_1 and excluding those in the set A_2. Notice that we have not considered singularities of any of the gamma functions contained in $M(s_r, z, w)$ because those contain w as well and can be made to show up in one variable but not both by the choice of contour. As a sum of the residues at the poles within the contour of integration, $\tilde{J}^{(1)}(x)$ now becomes:

$$\tilde{J}^{(1)}(x) = -\sum_{n=0}^{\infty} \frac{(-1)^{-(\frac{3}{2}\tau+\frac{1}{2}r+i\nu)}}{n!} \frac{\Gamma(\gamma + i\nu + n)}{\Gamma(2\gamma + 1 + n)\Gamma(\mu - \tau + 1 - r - n)} W^{(1)} \quad (E5)$$

where:

$$W^{(1)} = \frac{1}{2\pi i} \int_{-i\infty+C_w}^{i\infty+C_w} dw\ 2^w \frac{\Gamma(w)\Gamma(-\tau + \gamma + 1 + i\nu - r - n + w)}{\Gamma(2\gamma + 1 - \tau - r - n + w)}$$

$$\times \Gamma(-2\tau + \mu + 1 - 2r - n + w)\Gamma^2(\tau + r - w)$$

$$\Pi_{k=0}^{n-1}(\tau + r + k - w)\,{}_2F_1(-r, \tau + r - w, \mu - \tau - r + n + 1; \frac{1}{2})$$

We have written:

$$\Gamma(\tau + r + n - w) = (\tau + r + n - 1 - w)(\tau + r + n - 2 - w)\cdots$$
$$\times (\tau + r - w)\Gamma(\tau + r - w)$$
$$= \Pi_{k=0}^{n-1}(\tau + r + k - w)\Gamma(\tau + r - w)$$

In a similar fashion, we identify the following sets of poles in the $w-$ plane and calculate $W^{(1)}$ in terms of the residues of the integrand at those poles:

(1) $B_1 = \{w = -m\}$ from $\Gamma(w)$,

(2) $B_2 = \{w = \tau - \gamma - 1 - i\nu + r + n - m\}$ from $\Gamma(-\tau + \gamma + 1 + i\nu - r - n + w)$,

(3) $B_3 = \{w = 2\tau - \mu - 1 + 2r + n - m\}$ from $\Gamma(-2\tau + \mu + 1 - 2r - n + w)$,

(4) $B_4 = \{w = \tau + r + m\}$ from $\Gamma(\tau + r - w)$,

where $r, n, m = 0, 1, 2, \cdots$. If we now choose the contour according to the constraint (E2) and close it by a large semicircle in the right half plane, some of the singularities will be excluded. In fact, none of the poles in the first set will contribute to the value of the $w-$ integration. on the other hand, the subsets of B_2 and B_3 that lie within the boundaries of the contour of integration are given respectively by:

$$Re(w) = \tau - \gamma - 1 + r + n - m > \tau + r$$

or:

$$B_2' = \{w = \tau - \gamma - 1 - i\nu + r + n - m | m < -\gamma - 1 + n\}$$

and similarly:

$$B_3' = \{w = 2\tau - \mu - 1 + 2r + n - m | m < \tau - \mu - 1 + r + n\}$$

Finally, all members of the set B_4 are enclosed within the contour and therefore contribute to the value of the integral. The residues of the squared gamma function, $\Gamma^2(x)$, at the negative integers and zero can be calculated by making a Laurent series expansion. Let $|\xi| \ll 1$ and consider:

$$\Gamma^2(-m + \xi) = \left(\frac{\Gamma(1 + \xi)}{\xi(-1 + \xi)(-2 + \xi)\cdots(-m + \xi)}\right)^2$$
$$= \frac{1}{(m!)^2}\left(\frac{1 - C\xi + \cdots}{\xi(1 - \xi)(1 - \xi/2)(1 - \xi/3)\cdots(1 - \xi/m)}\right)^2$$
$$= \frac{1}{(m!)^2}\frac{1}{\xi^2}\left(\frac{1 - C\xi + \cdots}{1 - (\sum_{k=1}^{m}\frac{1}{k})\xi + \cdots}\right)^2$$
$$= \frac{1}{(m!)^2}\frac{1}{\xi^2} + 2\left(\frac{\sum_{k=1}^{m}\frac{1}{k} - C}{(m!)^2}\right)\frac{1}{\xi} + \cdots \quad (E6)$$

where $C = 0.577215\cdots$ is the Euler constant customarily denoted by γ. Thus we can read the residue of $\Gamma^2(z)$ at $z = -m = 0, -1, -2, \cdots$ directly from (E6) as:

$$Res\left(\Gamma^2(z)|_{z=-m}\right) = 2\left(\frac{\sum_{k=1}^{m}\frac{1}{k} - C}{\Gamma^2(m+1)}\right) \qquad (E7)$$

With the help of equation (E7), the $w-$ integration can finally be carried out as a sum of the residues at the various singularities. The final result is stated in equation (36). The only difference between $\tilde{J}^{(1)}$ and $\tilde{J}^{(2)}$ is in the z integration through the presence of the factor $(-\gamma - s_r + z)$. Hence equations (37) and (38) follow immediately from equation (36).

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